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CONSTRUCTIONS OF THE WEIL REPRESENTATIONS OF CERTAIN SIMPLE LIE ALGEBRAS

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1a. Let G be a simple complex Lie group with the parabolic subgroup P such that the radical N in the Levi decomposition $P = M \cdot N$ is Abelian. An equivalent definition is that $X = P \setminus G$ is an irreducible compact Hermitian symmetric space. Let N_- denote the unipotent subgroup opposite to N, and let $\mathscr{G}$, $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{N}_-$ denote the corresponding Lie algebras; and let $P_X = M_X \cdot N_X$ be the Levi decomposition of the stabilizer of $x \in X$ in G. Henceforth we will identify $\mathfrak{N}$ with $\mathfrak{N}_-^*$.

Definition. $K^*(K_X^*)$ is the cone of the vectors of highest weight in the $M(M_X)$-module $\mathfrak{N}\,(T_x^*X)$, i.e., each vector of the cone is leading with respect to a certain Borel subgroup in $M(M_X)$.

See [1] for more details of the geometry of these cones.

LEMMA 1. A $g \in G$ transforms $K_{X \cdot g}^*$ into K_X^*. ■.

The aim of this note is to embed $\mathscr{G}$ in $\mathscr{D}(K^*)$ – the algebra of regular differential operators on K^*. Thus, we obtain a representation of $\mathscr{G}$ in terms of regular functions on K^*. Roughly speaking, the construction is as follows: G acts in the sections of a certain linear G-bundle E_λ associated with a character $\chi_\lambda\colon P \to C^*$ [for $t \in \operatorname{Cent}\mathfrak{M} \simeq C$ we have $d_{\chi_\lambda}(t) = \lambda \cdot t$]. Expressing a neighborhood of the identity in G in the form $P \cdot \exp(\mathfrak{N}_-)$, we identify a neighborhood of the point $P \cdot e$ in X with $\mathfrak{N}_-$ and let us consider an N_--invariant trivialization of E_λ over $\mathfrak{N}_-$. For $t \in \mathscr{G}$ the operator $\mathscr{L}_t$ in Sec. 3 is the coordinate expression of an appropriate action of $\mathscr{G}$. If λ is the same as in Theorem 1 (Sec. 3), then the Fourier transform (see Sec. 2) $F(\mathscr{L}_t)$ of the operator $\mathscr{L}_t$ lies in $\mathscr{D}(K^*)$. The reason for this remarkable phenomenon is Lemma 1.

Let us observe that $F(\mathscr{L}_n)$ for $n \in \mathfrak{N}$ has order 2 and cannot be expressed in terms of operators of order 1 from $\mathscr{D}(K^*)$.

1b. The coadjoint representation of G has exactly one nonzero orbit O_G of smallest dimension. It passes through the vector of highest weight in $\mathscr{G}^*$. The constructed representation corresponds to O_G in the sense that $2 \cdot \dim K^* = \dim O_G$. But O_G does not have polarization for $\mathscr{G} \neq \mathrm{sl}(n, C)$ and, therefore, the usual methods for construction of a representation with respect to an orbit do not work. The problem of construction of "minimal representations" has been considered by various authors (see [2, 3]), but the construction has almost always been obtained by the restriction of the Weil representation of sp (2n) to $\mathscr{G} \subset \mathrm{sp}\,(2n)$. Our construction is more universal, and a new construction of the Weil representation has been obtained for $\mathscr{G} = \mathrm{sp}\,(2n, C)$. It is easily carried over to a wide class of simple Lie algebras, e.g., to the class of the Lie algebras that can be split over a field k of characteristic zero and which have a parabolic subalgebra with Abelian radical.

In the following discussions we will consider the analytical side of the matter: the construction of a unitary representation of G, and so on.

I. N. Bernshtein, I. M. Gel'fand, and S. I. Gel'fand have observed that $so\,(8, C)$ is embedded in the algebra of differential operators on the basic affine space A for SL (3, C) (not published). We have

$$A \simeq \{(z, w) \in C^6 \mid z_1 w_1 + z_2 w_2 + z_3 w_3 = 0;\ z \neq 0,\ w \neq 0\},$$

after which we can easily write out the embedding of $so\,(2n + 2, C)$ in the differential operators on $\{(z, w) \in C^{2n} \mid z_1 w_1 + \ldots + z_n w_n = 0\}$. (Although the connection with a compact Hermitian symmetric space was not revealed.) I am thankful to I. N. Bernshtein for communicating this to me.

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Using this opportunity, I thank I. N. Bernshtein for a series of consultations about various problems of representation theory and S. G. Gindikin for assistance.

2. Preliminary Results. Let V be a finite-dimensional linear space over $\mathbf{C}$ and $\langle\ ,\ \rangle$ be the pairing of V and V*. For $v \in V$ we define a derivation of the ring S(V*) by setting $\partial_v(v^*) = \langle v, v^*\rangle$ for $v^* \in V^*$. Let ψ_V denote the linear function on V* defined by the vector v and let $\mathscr{D}(V)$ denote the algebra of regular differential operators on V.

Let us define an isomorphism of algebras $F: \mathscr{D}(V) \to \mathscr{D}(V^*)$ by specifying it on the generators by the equations $F(\partial_v) = i\psi_v$ and $F(\psi_{v^*}) = i\partial_{v^*}$. This definition is correct, since $F([\partial_v, \psi_{v^*}]) = \langle v, v^*\rangle$ and $[F(\partial_v), F(\psi_{v^*})] = [i\psi_v, i\partial_{v^*}] = \langle v, v^*\rangle$.

Let I denote the ideal in $S(\mathfrak{N}^*)$ that defines the variety $K^* \cup 0$; and let $I^{(k)}$ be the space of polynomials of degree of homogeneity k in I.

Proposition 2. a) $I^{(2)}$ is an irreducible M-module that occurs in $\mathfrak{N}^* \otimes \mathfrak{N}^*$ with multiplicity one.

b) $I^{(2)} \cdot S(\mathfrak{N}^*) = I$.

We omit the proof for want of space.

For $t \in \mathscr{G}$ let L_t be the vector field on $\mathfrak{N}_-$ that originates by the action of G on X ($\mathfrak{N}_-$ is embedded in X as in Sec. 1). Let us consider $S(\mathfrak{N}_-^*) \otimes \mathfrak{N}_-$ as vector fields on $\mathfrak{N}_-$. Henceforth $n \in \mathfrak{N}$, and $y_1, y_2 \in \mathfrak{N}_-$.

Proposition 3. a) $L_n \in S^2(\mathfrak{N}_-^*) \otimes \mathfrak{N}_-$.

b) $L_n(y_1, y_2) = {}^1/_2\, [[n, y_1], y_2]$.

▲a) $[L_{y_1}, L_n] = L_{[y_1, n]} \in \mathfrak{N}_-^* \otimes \mathfrak{N}_-$, since $[y_1, n] \in \mathfrak{M} \subset \mathfrak{N}_-^* \otimes \mathfrak{N}_-$.

b) Since $\mathfrak{N}_-$ is Abelian, by virtue of the Jacobi identity the right-hand side of b) defines an element of $S^2(\mathfrak{N}_-^*) \otimes \mathfrak{N}_-$. It remains to verify that we get the same thing on commuting both the sides first with L_{y_1} and then with L_{y_2}. ■.

3. Basic Construction. Let $\mathscr{D}_k^l$ denote the space of the differential operators from $\mathscr{D}(\mathfrak{N}_-) \equiv \mathscr{D}$ of order at most k and homogeneity l. For example, $L_{y_1} = \partial_{y_1} \in \mathscr{D}_1^{-1}$. Let us agree to write an operator $\mathfrak{D} \in \mathscr{D}$ in the form

$$\mathfrak{D} = \sum \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n} a(y).$$

It is clear from the homogeneity arguments that

$$[L_n, f] = \mathfrak{D}_2^{-1} + \mathfrak{D}_1^{-1} \qquad (\mathfrak{D}_k^l \in \mathscr{D}_k^l) \tag{1}$$

for $f \in F(I^{(2)})$.

Proposition 4. $\mathfrak{D}_2^{-1} \in F(I^{(2)}) \cdot \mathscr{D}$.

▲ Let $\sigma_2(\mathfrak{D})$ be the 2-symbol of the operator $\mathfrak{D}$. This is a function on $T^*\mathfrak{N}_-$. Let $\tilde{L}_n$ denote the vector field on $T^*\mathfrak{N}_-$ that corresponds to the field L_n on $\mathfrak{N}_-$. Let K*(y) be the shift of $K^* \subset T_0^*\mathfrak{N}_- \simeq \mathfrak{N}$ at a point $y \in \mathfrak{N}_-$. It can be verified that K*(y) is identified with K_y^* under the embedding of $\mathfrak{N}_-$ in X (as in Sec. 1a). Therefore, it follows from Lemma 1 that $\sigma_2([L_n, f]) = \tilde{L}_n(\sigma(f))$ vanishes on each cone K*(y). ■.

Let us set $\mathscr{D}(K^*) = \mathrm{Norm}\,(I\mathscr{D})/I\mathscr{D}$, where $\mathrm{Norm}\,(I \cdot \mathscr{D}) = \{\mathfrak{D} \in \mathscr{D} \mid \mathfrak{D}I \subset I\mathscr{D}\}$. Let $n \in \mathfrak{N}$, $y \in \mathfrak{N}_-$, and $\psi_n \in \mathfrak{N}_-^*$ (see Sec. 2). We set

$$\mathscr{L}_n = L_n + \lambda \cdot \psi_n, \qquad \mathscr{L}_{[n, y]} = L_{[n, y]} - \lambda \langle n, y\rangle, \qquad \mathscr{L}_y = L_y.$$

For $t \in \mathscr{G}$ we have the isomorphism of Lie algebras $t \mapsto \mathscr{L}_t$ (cf. Sec. 1a).

THEOREM 1. There exists a $\lambda \in \mathbf{C}$, such that $F(\mathscr{L}_y) \in \mathscr{D}(K^*)$.

▲ Let us associate the operator $\mathfrak{D}^{-1}$ with L_n and $f \in F(I^{(2)})$ [see (1)]. We obtain a morphism of M-modules $A: \mathfrak{N} \otimes F(I^{(2)}) \to \mathfrak{N}_-$. Let us consider the morphism $\bar{A}: n \otimes f \mapsto [\psi_n, f]$ of the same M-modules. It follows from the statement a) of Proposition 2 that $\dim \mathrm{Hom}_M(\mathfrak{N} \otimes F(I^{(2)}), \mathfrak{N}_-) = 1$. Since $\bar{A} \neq 0$, it follows that $A = \lambda\bar{A}\ \exists\lambda \in \mathbf{C}$. This is the desired λ, since $[\mathscr{L}_n, F(I^{(2)})] \subset F(I^{(2)}) \cdot \mathscr{D}$ and $F(I) = F(I^{(2)}) \cdot F(S(\mathfrak{N}_-^*))$ for it. ■.

Example. $\mathscr{G} \simeq so(2n+2, \mathbf{C})$; X is a quadric in $\mathbf{C}P^{2n+1}$, $K^* \simeq \{(z_1, \ldots, z_n; w_1, \ldots, w_n) \mid z_1w_1 + \ldots + z_nw_n = 0\}$. Let us set $\Delta = \partial_{z_1}\partial_{w_1} + \cdots + \partial_{z_n}\partial_{w_n}$, $J_z = z_1\partial_{z_1} + \ldots + z_n\partial_{z_n}$ and $J_w = w_1\partial_{w_1} + \ldots + w_n\partial_{w_n}$. The operators $W_k = w_k\Delta - (J_z + J_w - (n-1))\partial_{z_k}$, $Z_k = z_k\Delta - (J_z + J_w - (n-1))\,\partial_{w_k}$, $[W_k, z_j]$, $[W_k, w_j]$, and $[Z_k, z_j]$ and the operators of multiplication by a linear function give a representation of $so\,(2n+2, \mathbb{C})$ in terms of regular function on K^*.

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PARTIALLY ORDERED SETS OF FINITE GROWTH

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Let $\mathfrak{N} = \{a_1, \ldots, a_n\}$ be a finite partially ordered set and k be a commutative ring. We say that a representation of the set $\mathfrak{N}$ over the ring k is given if a submodule V_i of a certain finitely generated k-module V is associated with each element $a_i \in \mathfrak{N}$ such that if $a_i \le a_j$, then $V_i \subseteq V_j$.

The representations of the partially ordered set $\mathfrak{N}$ form an additive category, in which equivalent and indecomposable representations are defined naturally. If k is a field, then the $(n+1)$-dimensional integral vector $d = (d_0, d_1, \ldots, d_n)$, where d_0 is the dimension of the space V and d_i is the dimension of the factor space $V_i / \sum\limits_{a_j < a_i} V_j$, is called the dimension of the representation $S = (V, V_1, \ldots, V_n)$.

As in [1-3], we will study representations of partially ordered sets over fields. We need representations over rings only to recall the definition of partially ordered sets of tame type and to define partially ordered sets of finite growth.

As the algebras [4] and the quivers [5, 6], the partially ordered sets of infinite type (i.e., having infinitely may nonequivalent indecomposable representations) are divided into two disjoint classes: the tame ones (admitting a classification of representations) and the wild ones ("containing in them" a classical unsolved problem about a pair of linear operators).

We will say that a representation $S = (V, V_1, \ldots, V_n)$ of the partially ordered set $\mathfrak{N}$ over the field k is generated by the representation $\overline{S} = (\overline{V}, \overline{V}_1, \ldots, \overline{V}_n)$ of the same set over the ring k[X] of the polynomials in one variable if there exists a finite-dimensional k[X]-module B such that

$$V = \overline{V} \underset{k[X]}{\otimes} B, \qquad V_i = \operatorname{Im}\,(\overline{V}_i \underset{k[X]}{\otimes} B \xrightarrow{f_i \otimes 1} \overline{V} \underset{k[X]}{\otimes} B),$$

where $f_i : \overline{V}_i \to \overline{V}$ are the natural embeddings. The partially ordered set $\mathfrak{N}$ has tame type over the field k if all indecomposable representations of each dimension are generated by a finite number of representations of $\mathfrak{N}$ over k[X].*

It is proved in [3] that a partially ordered set has tame type (over an arbitrary field k) if and only if it does not contain any one of the following sets as a subset: (1, 1, 1, 1, 1), (1, 1, 1, 2), (2, 2, 3), (1, 3, 4), (1, 2, 6), and $\overline{N} = \{a_1 < a_2 > b_1 < b_2;\ c_1 < c_2 < c_3 < c_4 < c_5\}$, where $(l_1, \ldots, l_m)$ is the cardinal sum of m linearly ordered sets that consist of $l_1, \ldots, l_m$ elements, respectively.

Let $\mu(d)$ denote the least possible number of the representations of the set $\mathfrak{N}$ over k[X] that generate almost (i.e., all except a finite number of) indecomposable representations of $\mathfrak{N}$ over k of dimension d. A detailed study of the partially ordered sets of tame type shows that they are divided into two classes:

*It is clear that this definition is suitable only for the case of an infinite field k. In the finite case the field k is replaced by its separable closure $\bar{k}$.

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