

Hidden Hodge symmetries and Hodge correlators

A.B. Goncharov

To Don Zagier for his 60-th birthday

1 Introduction

There is a well known parallel between Hodge and étale theories, still incomplete and rather mysterious:

<i>l</i> -adic Étale Theory	Hodge Theory
Category of <i>l</i> -adic Galois modules	Abelian category $\mathcal{MH}_{\mathbb{R}}$ of real mixed Hodge structures
Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$	Hodge Galois group $G_{\text{Hod}} :=$ Galois group of the category $\mathcal{MH}_{\mathbb{R}}$
$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H_{\text{et}}^*(\overline{X}, \mathbb{Q}_l)$, where X is a variety over \mathbb{Q}	$H^*(X(\mathbb{C}), \mathbb{R})$ has a functorial real mixed Hodge structure
étale site	??
$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the étale site, and thus on categories of étale sheaves on X , e.g. on the category of <i>l</i> -adic perverse sheaves	?? ?? ??
$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -equivariant perverse sheaves	Saito's Hodge sheaves

The current absence of the “Hodge site” was emphasized by A.A. Beilinson [B].

The Hodge Galois group. A weight n pure real Hodge structure is a real vector space H together with a decreasing filtration $F^\bullet H_{\mathbb{C}}$ on its complexification satisfying

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} F^p H_{\mathbb{C}} \cap \overline{F^q} H_{\mathbb{C}}.$$

A real Hodge structure is a direct sum of pure ones. The category $\mathcal{PH}_{\mathbb{R}}$ real Hodge structures is equivalent to the category of representations of the real algebraic group $\mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$. The group of complex points of $\mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$ is $\mathbb{C}^* \times \mathbb{C}^*$; the complex conjugation interchanges the factors.

A real mixed Hodge structure is given by a real vector space H equipped with the weight filtration $W_\bullet H$ and the Hodge filtration $F^\bullet H_{\mathbb{C}}$ of its complexification, such that the Hodge filtration induces on $\text{gr}_n^W H$ a weight n real Hodge structure. The category

$\mathcal{MH}_{\mathbb{R}}$ of real mixed Hodge structures is an abelian rigid tensor category. There is a fiber functor to the category of real vector spaces

$$\omega_{\text{Hod}} : \mathcal{MH}_{\mathbb{R}} \longrightarrow \text{Vect}_{\mathbb{R}}, \quad H \longrightarrow \bigoplus_n \text{gr}_n^W H.$$

The Hodge Galois group is a real algebraic group given by automorphisms of the fiber functor:

$$G_{\text{H}} := \text{Aut}^{\otimes} \omega_{\text{Hod}}.$$

The fiber functor provides a canonical equivalence of categories

$$\omega_{\text{Hod}} : \mathcal{MH}_{\mathbb{R}} \xrightarrow{\sim} G_{\text{Hod}} - \text{modules}.$$

The Hodge Galois group is a semidirect product of the unipotent radical U_{Hod} and $\mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$:

$$0 \longrightarrow U_{\text{Hod}} \longrightarrow G_{\text{Hod}} \longrightarrow \mathbb{C}_{\mathbb{C}/\mathbb{R}}^* \longrightarrow 0, \quad \mathbb{C}_{\mathbb{C}/\mathbb{R}}^* \hookrightarrow G_{\text{Hod}}. \quad (1)$$

The projection $G_{\text{Hod}} \rightarrow \mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$ is provided by the inclusion of the category of real Hodge structures to the category of mixed real Hodge structures. The splitting $s : \mathbb{C}_m \rightarrow G_{\text{Hod}}$ is provided by the functor ω_{Hod} .

The complexified Lie algebra of U_{Hod} has *canonical* generators $G_{p,q}$, $p, q \geq 1$, satisfying the only relation $\overline{G}_{p,q} = -G_{q,p}$, defined in [G1]. For the subcategory of Hodge-Tate structures they were defined in [L]. Unlike similar but different Deligne's generators [D], they behave nicely in families. So to define an action of the group G_{Hod} one needs to have an action of the subgroup $\mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$ and, in addition to this, an action of a single operator

$$G := \sum_{p,q \geq 1} G_{p,q}.$$

Denote by \mathbb{C}^* the real algebraic group with the group of complex points \mathbb{C}^* . The extension induced from (1) by the diagonal embedding $\mathbb{C}^* \subset \mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$ is the *twistor Galois group*. It is a semidirect product of the groups U_{Hod} and \mathbb{C}^* .

$$0 \longrightarrow U_{\text{Hod}} \longrightarrow G_{\text{T}} \xrightarrow{\leftarrow} \mathbb{C}^* \longrightarrow 0. \quad (2)$$

It is not difficult to prove

Lemma 1.1 *The category of representations of G_{T} is equivalent to the category of mixed twistor structures defined by Simpson [Si2].*

We suggest the following fills the ??-marks in the dictionary related the Hodge and Galois. Below X is a smooth projective complex algebraic variety.

Conjecture 1.2 *There exists a functorial homotopy action of the twistor Galois group G_{T} by A_{∞} -equivalences of an A_{∞} -enhancement of the derived category of perverse sheaves on X such that the category of equivariant objects is equivalent to Saito's category real mixed Hodge sheaves.¹*

¹We want to have a natural construction of the action first, and get Saito's category real mixed Hodge sheaves as a consequence, not the other way around.

Denote by $D_{\text{sm}}^b(X)$ the category of smooth complexes of sheaves on X , i.e. complexes of sheaves on X whose cohomology are local systems.

Theorem 1.3 *There exists a functorial for pull-backs homotopy action of the twistor Galois group G_T by A_∞ -equivalences of an A_∞ -enhancement of the category $D_{\text{sm}}^b(X)$.*

The action of the subgroup \mathbb{C}^* is not algebraic. It arises from Simpson’s action of \mathbb{C}^* on semisimple local systems [Si1]. The action of the Lie algebra of the unipotent radical U_{Hod} is determined by a collection of numbers, which we call the *Hodge correlators for semisimple local systems*. Our construction uses the theory of harmonic bundles [Si1]. The Hodge correlators can be interpreted as correlators for a certain Feynman integral. This Feynman integral is probably responsible for the “Hodge site”.

For the trivial local system the construction was carried out in [G2]. A more general construction for curves, involving the constant sheaves and delta-functions, was carried out in [G1].

In the case when X is the universal modular curve, the Hodge correlators contain the special values $L(f, n)$ of weight $k \geq 2$ modular forms for $GL_2(\mathbb{Q})$ outside of the critical strip – it turns out that the simplest Hodge correlators in this case coincide with the Rankin-Selberg integrals for $L(f, n)$. There is an analog of Hodge correlators for the modular hyperbolic threefolds with the same properties. Its algebraic-geometric meaning is obscure.

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2 Hodge correlators for local systems

2.1 An action of G_T on the “minimal model” of $\mathcal{D}_{\text{sm}}(X)$.

Tensor products of irreducible local systems are semisimple local systems. The *category of harmonic bundles* Har_X is the graded category whose objects are semi-simple local systems on X and their shifts, and morphisms are given by graded vector spaces

$$\text{Hom}_{\text{Har}_X}^\bullet(V_1, V_2) := H^\bullet(X, V_1^\vee \otimes V_2). \quad (3)$$

Here is our main result.

Theorem 2.1 *There is a homotopy action of the twistor Galois group G_T by A_∞ -equivalences of the graded category Har_X , such that the action of the subgroup \mathbb{C}^* is given by Simpson’s action of \mathbb{C}^* on semi-simple local systems.*

This immediately implies Theorem 1.3. Indeed, given a small A_∞ -category \mathcal{A} , there is a functorial construction of the triangulated envelope $\text{Tr}(\mathcal{A})$ of \mathcal{A} , the smallest triangulated

category containing \mathcal{A} . Since $\mathcal{D}_{\text{sm}}^b(X)$ is generated as a triangulated category by semi-simple local systems, the category $\text{Tr}(\text{Har}_X)$ is equivalent to $\mathcal{D}_{\text{sm}}^b(X)$ as a triangulated category, and thus is an A_∞ -enhancement of the latter. On the other hand, the action of the group G_T from Theorem 2.1 extends by functoriality to the action on $\text{Tr}(\text{Har}_X)$.

Below we recall what are A_∞ -equivalences of DG categories and then define the corresponding data in our case.

2.2 A_∞ -equivalences of DG categories

The Hochschild cohomology of a small dg-category \mathcal{A} . Let \mathcal{A} be a small dg category. Consider a bicomplex whose n -th column is

$$\prod_{[X_i]} \text{Hom}\left(\mathcal{A}(X_0, X_1)[1] \otimes \mathcal{A}(X_1, X_2)[1] \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n)[1], \mathcal{A}(X_0, X_n)[1]\right), \quad (4)$$

where the product is over isomorphism classes $[X_i]$ of objects of the category \mathcal{A} . The vertical differential d_1 in the bicomplex is given by the differential on the tensor product of complexes. The horizontal one d_2 is the degree 1 map provided by the composition

$$\mathcal{A}(X_i, X_{i+1}) \otimes \mathcal{A}(X_{i+1}, X_{i+2}) \longrightarrow \mathcal{A}(X_i, X_{i+2}).$$

Let $\text{HC}^*(\mathcal{A})$ be the total complex of this bicomplex. Its cohomology are the Hochschild cohomology $\text{HH}^*(\mathcal{A})$ of \mathcal{A} . Let $\text{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A})$ be the space of A_∞ -functors from \mathcal{A} to itself. Lemma 2.2 can serve as a definition of A_∞ -functors considered modulo homotopy equivalence.

Lemma 2.2 *One has*

$$H^0 \text{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A}) = \text{HH}^0(\mathcal{A}). \quad (5)$$

Indeed, a cocycle in $\text{HC}^0(\mathcal{A})$ is the same thing as an A_∞ -functor. Coboundaries corresponds to the homotopic to zero functors.

The cyclic homology of a small rigid dg-category \mathcal{A} . Let $(\alpha_0 \otimes \dots \otimes \alpha_m)_\mathcal{C}$ be the projection of $\alpha_0 \otimes \dots \otimes \alpha_m$ to the coinvariants of the cyclic shift. So, if $\bar{\alpha} := \text{deg} \alpha$,

$$(\alpha_0 \otimes \dots \otimes \alpha_m)_\mathcal{C} = (-1)^{\bar{\alpha}_m(\bar{\alpha}_0 + \dots + \bar{\alpha}_{m-1})} (\alpha_1 \otimes \dots \otimes \alpha_m \otimes \alpha_0)_\mathcal{C}.$$

We assign to \mathcal{A} a bicomplex whose n -th column is

$$\prod_{[X_i]} \left(\mathcal{A}(X_0, X_1)[1] \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n)[1] \otimes \mathcal{A}(X_n, X_0)[1] \right)_\mathcal{C}.$$

The differentials are induced by the differentials and the composition maps on Hom's. The cyclic homology complex $\text{CC}_*(\mathcal{A})$ of \mathcal{A} is the total complex of this bicomplex. Its homology are the cyclic homology of \mathcal{A} .

Assume that there are functorial pairings

$$\mathcal{A}(X, Y)[1] \otimes \mathcal{A}(Y, X)[1] \longrightarrow \mathcal{H}^*.$$

Then there is a morphism of complexes

$$\mathrm{HC}^*(\mathcal{A})^* \longrightarrow \mathrm{CC}_*(\mathcal{A}) \otimes \mathcal{H}. \quad (6)$$

For the category of harmonic bundles Har_X there is such a pairing with

$$\mathcal{H} := H_{2n}(X)[-2].$$

It provides a map

$$\varphi : \mathrm{Hom}\left(H_0(\mathrm{CC}_*(\mathrm{Har}_X) \otimes \mathcal{H}, \mathbb{C})\right) \longrightarrow \mathrm{HH}^0(\mathrm{Har}_X) \stackrel{(5)}{=} H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X). \quad (7)$$

2.3 The Hodge correlators

Theorem 2.3 *a) There is a linear map, the Hodge correlator map*

$$\mathrm{Cor}_{\mathrm{Har}_X} : H_0(\mathrm{CC}_*(\mathrm{Har}_X) \otimes \mathcal{H}) \longrightarrow \mathbb{C}. \quad (8)$$

Combining it with (7), we get a cohomology class

$$\mathbf{H}_{\mathrm{Har}_X} := \varphi(\mathrm{Cor}_{\mathrm{Har}_X}) \in H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X). \quad (9)$$

b) There is a homotopy action of the twistor Galois group G_{T} by A_∞ -autoequivalences of the category Har_X such that

- *Its restriction to the subgroup \mathbb{C}^* is the Simpson action [Si1] on the category Har_X .*
- *Its restriction to the Lie algebra $\mathrm{LieU}_{\mathrm{Hod}}$ is given by a Lie algebra map*

$$\mathbb{H}_{\mathrm{Har}_X} : \mathrm{L}_{\mathrm{Hod}} \longrightarrow H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X), \quad (10)$$

uniquely determined by the condition that $\mathbb{H}_{\mathrm{Har}_X}(G) = \mathbf{H}_{\mathrm{Har}_X}$.

c) The action of the group G_{T} is functorial with respect to the pull backs.

2.4 Construction.

To define the Hodge correlator map (8), we define a collection of degree zero maps

$$\mathrm{Cor}_{\mathrm{Hod}_X} : \left(H^\bullet(X, V_0^* \otimes V_1)[1] \otimes \dots \otimes H^\bullet(X, V_m^* \otimes V_0)[1]\right)_{\mathbb{C}} \otimes \mathcal{H} \longrightarrow \mathbb{C}. \quad (11)$$

The definition depends on some choices, like harmonic representatives of cohomology classes. We prove that it is well defined on HC^0 , i.e. its restriction to cycles is independent of the choices, and coboundaries are mapped to zero.

We picture an element in the source of the map (11) by a polygon P , see Fig 1, whose vertices are the objects V_i , and the oriented sides $V_i V_{i+1}$ are graded vector space $\mathrm{Ext}^*(V_i, V_{i+1})(1)$.

Figure 1: A decorated plane trivalent tree; V_i are harmonic bundles.

Green currents for harmonic bundles. Let V be a harmonic bundle on X . Then there is a Dolbeault bicomplex $(\mathcal{A}^\bullet(X, V); D', D'')$ where the differentials D', D'' are provided by the complex structure on X and the harmonic metric on V . It satisfies the D', D'' -lemma.

Choose a splitting of the corresponding de Rham complex $\mathcal{A}^\bullet(X, V)$ into an arbitrary subspace $\mathcal{H}ar^\bullet(X, V)$ isomorphically projecting onto the cohomology $H^\bullet(X, V)$ ("harmonic forms") and its orthogonal complement. If $V = \mathbb{C}_X$, we choose $a \in X$ and take the δ -function δ_a at the point $a \in X$ as a representative of the fundamental class.

Let δ_Δ be the Schwarz kernel of the identity map $V \rightarrow V$ given by the δ -function of the diagonal, and P_{Har} the Schwarz kernel of the projector onto the space $\mathcal{H}ar^\bullet(X, V)$, realized by an (n, n) -form on $X \times X$. Choose a basis $\{\alpha_i\}$ in $\mathcal{H}ar^\bullet(X, V)$. Denote by $\{\alpha_i^\vee\}$ the dual basis. Then we have

$$P_{\text{Har}} = \sum \alpha_i^\vee \otimes \alpha_i, \quad \int_X \alpha_i \wedge \alpha_j^\vee = \delta_{ij}.$$

Let $p_i : X \times X \rightarrow X$ be the projections onto the factors.

Definition 2.4 A Green current $G(V; x, y)$ is a $p_1^*V^* \otimes p_2^*V$ -valued current on $X \times X$,

$$G(V; x, y) \in \mathcal{D}^{2n-2}(X \times X, p_1^*V^* \otimes p_2^*V), \quad n = \dim_{\mathbb{C}} X,$$

which satisfies the differential equation

$$(2\pi i)^{-1} D'' D' G(V; x, y) = \delta_\Delta - P_{\text{Har}}. \quad (12)$$

The two currents on the right hand side of (12) represent the same cohomology class, so the equation has a solution by the $D'' D'$ -lemma.

Remark. The Green current depends on the choice of the "harmonic forms". So if $V = \mathbb{C}$, it depends on the choice of the base point a . Solutions of equation (12) are well defined modulo $\text{Im} D'' + \text{Im} D' + \mathcal{H}ar^\bullet(X, V)$.

Construction of the Hodge correlators. *Trees.* Take a plane trivalent tree T dual to a triangulation of the polygon P , see Fig 1. The complement to T in the polygon P is a union of connected domains parametrized by the vertices of P , and thus decorated by the harmonic bundles V_i . Each edge E of the tree T is shared by two domains. The corresponding harmonic bundles are denoted V_{E-} and V_{E+} . If E is an external edge, we assume that V_{E-} is before V_{E+} for the clockwise orientation.

Given an internal vertex v of the tree T , there are three domains sharing the vertex. We denote the corresponding harmonic bundles by V_i, V_j, V_k , where the cyclic order of the bundles agrees with the clockwise orientation. There is a natural trace map

$$\mathrm{Tr}_v : V_i^* \otimes V_j \otimes V_j^* \otimes V_k \otimes V_k^* \otimes V_i \longrightarrow \mathbb{C}. \quad (13)$$

It is invariant under the cyclic shift.

Decorations. For every edge E of T , choose a graded splitting of the de Rham complex

$$\mathcal{A}^\bullet(X, V_{E-}^* \otimes V_{E+}) = \mathcal{H}ar^\bullet(X, V_{E-}^* \otimes V_{E+}) \bigoplus \mathcal{H}ar^\bullet(X, V_{E-}^* \otimes V_{E+})^\perp.$$

Then a decomposable class in $\left(\otimes_{i=0}^m H^*(X, V_i^\bullet \otimes V_{i+1})[1]\right)_\mathcal{C}$ has a harmonic representative

$$W = \left(\alpha_{0,1} \otimes \alpha_{1,2} \otimes \dots \otimes \alpha_{m,0}\right)_\mathcal{C}.$$

We are going to assign to W a top degree current $\kappa(W)$ on

$$X^{\{\text{internal vertices of } T\}}. \quad (14)$$

Each external edge E of the tree T is decorated by an element

$$\alpha_E \in \mathcal{H}ar^\bullet(X, V_{E-}^* \otimes V_{E+}).$$

Put the current α_E to the copy of X assigned to the internal vertex of the edge E , and pull it back to (14) using the projection p_{α_E} of the latter to the X . Abusing notation, we denote the pull back by α_E . It is a form on (14) with values in the bundle $p_{\alpha_E}^*(V_{E-}^* \otimes V_{E+})$

Green currents. We assign to each internal edge E of the tree T a Green current

$$G(V_{E-}^* \otimes V_{E+}; x_-, x_+). \quad (15)$$

The order of (x_-, x_+) agrees with the one of (V_{E-}^*, V_{E+}) as on Fig 2: the cyclic order of $(V_{E-}^*, x_-, V_{E+}, x_+)$ agrees with the clockwise orientation. The Green current (15) is symmetric:

$$G(V_{E-}^* \otimes V_{E+}; x_-, x_+) = G(V_{E+}^* \otimes V_{E-}; x_+, x_-). \quad (16)$$

So it does not depend on the choice of orientation of the edge E .

The map ξ . There is a degree zero map

$$\xi : \mathcal{A}^\bullet(X, V_0)[-1] \otimes \dots \otimes \mathcal{A}^\bullet(X, V_m)[-1] \longrightarrow \mathcal{A}^\bullet(X, V_0 \otimes \dots \otimes V_m)[-1]; \quad (17)$$

Figure 2: Decorations of the Green current assigned to an edge E .

$$\varphi_0 \otimes \dots \otimes \varphi_m \longmapsto \mathrm{Sym}_{\{0, \dots, m\}} \left(\varphi_0 \wedge D^c \varphi_1 \wedge \dots \wedge D^c \varphi_m \right). \quad (18)$$

The graded symmetrization in (18) is defined via isomorphisms $V_{\sigma(0)} \otimes \dots \otimes V_{\sigma(m)} \rightarrow V_0 \otimes \dots \otimes V_m$, where σ is a permutation of $\{0, \dots, m\}$. It is essential that $\deg D^c \varphi = \deg \varphi + 1$.

An outline of the construction. We apply the operator ξ to the product of the Green currents assigned to the internal edges of T . Then we multiply on (14) the obtained local system valued current with the one provided by the decoration W , with an appropriate sign. Applying the product of the trace maps (13) over the internal vertices of T , we get a top degree scalar current on (14). Integrating it we get a number assigned to T . Taking the sum over all plane trivalent trees T decorated by W , we get a complex number $\mathrm{Cor}_{\mathrm{Har}_X}(W \otimes \mathcal{H})$. Altogether, we get the map (8). One checks that its degree is zero. The signs in this definition are defined the same way as in [G2].

Theorem 2.5 *The maps (11) give rise to a well defined Hodge correlator map (8).*

Figure 3: The Hodge correlator for the Green function $G(V^* \otimes W; a, b)$, $a \neq b$.

3 Hodge correlators and Rankin-Selberg integrals

Let X be a smooth projective curve. An irreducible holonomic \mathcal{D} -module with regular singularities on X is either the δ -functions δ_a at a point a , or the intermediate extension of an irreducible local system on $X - S$, where S is a finite (perhaps empty) collection of

points. Given a cyclic collection of irreducible holonomic \mathcal{D} -modules on X one can define Hodge correlators for cyclic tensor products of the consecutive Ext^1 's. The simplest Hodge correlator of this kind is given by the Green function assigned to the diagram on Fig 3. It tells the Feynman rule needed to incorporate the \mathcal{D} -modules δ_a .

Figure 4: The Hodge correlator for the special value $L(f, k + m + 2)$ of a weight k modular form.

Now let $Y(N)$ be the level N modular curve, and $X(N)$ its smooth compactification. Let $p : \mathcal{E} \rightarrow Y(N)$ be the universal elliptic curve. Consider the local system $\mathcal{H} := R^1 p_* \mathbb{C}(1)$ on $Y(N)$. Then for each cusp $a \in X(N) - Y(N)$ there are extension classes

$$E(a, S^n \mathcal{H}) \in \text{Ext}^1(\delta_a, S^n \mathcal{H}), \quad E(S^n \mathcal{H}, a) \in \text{Ext}^1(S^n \mathcal{H}, \delta_a)$$

Let f be a weight $k + 2$ modular form. It provides a class $f \in \text{Ext}^1(\mathbb{C}, S^k \mathcal{H})$ Now the Hodge correlator

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