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Explicit Construction of Characteristic Classes

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To my teacher Israel Moiseevich Gelfand on the occasion of his 80th birthday

ABSTRACT. Let E be a vector bundle over an algebraic manifold X. An explicit local construction is given of characteristic classes $c_n(E)$ with values in *bi-Grassmannian* cohomology that are defined in §1. In the special case $n = \dim E$ it reduces to the construction in [BMS] of $c_n(E)$ with values in the Grassmannian cohomology.

Our construction implies immediately an explicit construction of Chern classes with values in $H^n(X, \underline{\underline{K}}_n^M)$, where $\underline{\underline{K}}_n^M$ is the sheaf of Milnor's K-groups.

A construction of classes $c_n(E)$ with values in motivic cohomology is given for $n \leqslant 3$. For $n = 2$ it could be considered as a motivic analog of the local combinatorial formula of Gabrielov, Gelfand, and Losik for the first Pontryagin class (see [GGL]). The reason for the restriction $n \leqslant 3$ is the present lack of a good theory of n-logarithms for $n \geqslant 4$. Explicit constructions of the universal Chern classes $c_n \in H^n(BGL_{m^\bullet}, \underline{\underline{K}}_n^M)$ and, for $n \leqslant 3$, of classes $c_n \in H^{2n}_{\mathscr{M}}(BGL_{m^\bullet}, \mathbb{Z}(n))$ ($H^\bullet_{\mathscr{M}}$ is the motivic cohomology) are given.

§1. Introduction

1.1. Chern classes with values in $H^n(X, \underline{\underline{K}}_n^M)$. Let L be a line bundle over X. There is the following classical construction of $c_1(L) \in H^1(X, \mathscr{O}^*)$. Choose a Zariski covering $\{U_i\}$ of X such that $L|_{U_i}$ is trivial. Choose nonzero sections $s_i \in \Gamma(U_i, L)$. Then $s_i/s_j \in \Gamma(U_i \cap U_j, \mathscr{O}^*)$ satisfies the cocycle relation and hence defines a cohomology class $c_1(L) \in H^1(X, \mathscr{O}^*)$.

Let us define the presheaf of Milnor's K-groups on X as follows: its section over an open set U is the quotient group of $\underbrace{\mathscr{O}^*(U) \otimes \cdots \otimes \mathscr{O}^*(U)}_{n \text{ times}}$ by the subgroup generated by elements

$$g_1 \otimes \cdots \otimes g_k \otimes f \otimes (1-f) \otimes g_{k+3} \otimes \cdots \otimes g_n, \qquad g_i, f, 1-f \in \mathscr{O}^*(U).$$

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Let us denote by $\underline{\underline{K}}_n^M$ the sheaf associated with this presheaf. We will denote by $\{f_1, \ldots, f_n\}$ the image of $f_1 \otimes \cdots \otimes f_n \in \mathscr{O}^*(U)^{\otimes n}$ in $\underline{\underline{K}}_n^M(U)$.

In § 3 for any vector bundle E over X an explicit construction of the Chern classes $c_n(E) \in H^n(X, \underline{\underline{K}}_n^M)$ will be given.

The construction of $c_n(E^n)$ for an n-dimensional vector bundle E^n follows from the results of [S1] and [BMS, Chapter 1]. More precisely, let U_i be a Zariski covering such that $E^n|_{U_i}$ is trivial. Choose a section $s_i \in \Gamma(U_i, E^n)$ such that $s_{i_1}(x), \ldots, s_{i_{n+1}}(x)$ are in general position on $U_{i_1 \ldots i_{n+1}} := U_{i_1} \cap \cdots \cap U_{i_{n+1}}$. Then $s_{i_{n+1}} = \sum_{k=1}^n a_{i_k}(x) \cdot s_{i_k}(x)$ and

$$\{a_{i_1}(x), \ldots, a_{i_n}(x)\} \in K_n^M(U_{i_1 \ldots i_{n+1}})$$

is a cocycle in the Čech complex.

I will generalize this construction to vector bundles of arbitrary dimension and show that, being applied to $c_1(E)$, it gives the above cocycle for $c_1(\det E)$.

1.2. An application. There is a canonical map of sheaves

$$\underline{\underline{K}}_n^M \to \Omega^n_{\log} \hookrightarrow \Omega^n_{\mathrm{cl}} \hookrightarrow \Omega^n,$$
$$\{f_1, \ldots, f_n\} \mapsto d\log f_1 \wedge \cdots \wedge d\log f_n.$$

Here $\Omega^n_{\log}$ (respectively Ω^n_{cl}) is the sheaf of n-forms with logarithmic singularities at infinity (respectively closed n-forms). Therefore we get a construction of characteristic classes with values in $H^n(X, \Omega^n_{\log})$ and $H^n(X, \Omega^n_{\mathrm{cl}})$. Note that Atiyah's construction provides us with characteristic classes in $H^n(X, \Omega^n)$ ([A], see also [Har]).

1.3. The Grassmannian bicomplex and bi-Grassmannian cohomology (see [G1, G2], compare with [GGL, BMS, S3]). Let Y be a set and $\widetilde{C}_n(Y)$ be the free abelian group generated by elements $(y_0, \ldots, y_n)$ of $Y^{n+1} := \underbrace{Y \times \cdots \times Y}_{n+1}$. There is a complex $(\widetilde{C}_*(Y), d)$, where

$$d(y_0, \ldots, y_n) := \sum_{i=0}^{n} (-1)^i (y_0, \ldots, \widehat{y}_i, \ldots, y_n). \tag{1.1}$$

This is just the simplicial complex of the simplex whose vertices are labeled by elements of Y. Suppose that a group G acts on Y. Let us call elements of the quotient set $G\backslash Y^{n+1}$ *configurations* of elements of Y. Denote by $C_n(Y)$ the free abelian group generated by configurations of $(n+1)$ elements of Y. There is a complex $(C_*(Y), d)$, where d is defined by the same formula (1.1) and $C_*(Y) = \widetilde{C}_*(Y)_G$. We will also apply this construction to subsets of $G \backslash Y^{n+1}$ of "configurations in general position".

Now let us denote by $C_n(m)$ the free abelian group generated by configurations of $n+1$ vectors in general position in an m-dimensional vector space V^m over F (i.e., any m vectors of the configuration are linearly independent). It does not depend on the choice of vector space V_m. In this case there is another map:

$$d' : C_n(m) \to C_{n-1}(m-1)$$

$$d' : (v_0, \dots, v_n) \mapsto \sum_{i=0}^{n} (-1)^i (v_i \mid v_0, \dots, \widehat{v}_i, \dots, v_n).$$

Here $(v_i \mid v_0, \dots, \widehat{v}_i, \dots, v_n)$ is a configuration of vectors in $V_m/\langle v_i \rangle$ obtained by projection of vectors $v_j \in V^m$, $j \neq i$. Then there is the following bicomplex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & & C_{n+4}(n+2) & \xrightarrow{d} & C_{n+3}(n+2) & \xrightarrow{d} & C_{n+2}(n+2) \\
 & & \downarrow{d'} & & \downarrow{d'} & & \downarrow{d'} \\
\cdots \longrightarrow & & C_{n+3}(n+1) & \xrightarrow{d} & C_{n+2}(n+1) & \xrightarrow{d} & C_{n+1}(n+1) \\
 & & \downarrow{d'} & & \downarrow{d'} & & \downarrow{d'} \\
\cdots \longrightarrow & & C_{n+2}(n) & \xrightarrow{d} & C_{n+1}(n) & \xrightarrow{d} & C_n(n)
\end{array} \qquad (1.2)$$

We will call it the *Grassmannian bicomplex* (over $X = \operatorname{Spec} F$).

There is a subcomplex $(C_*(n), d)$

$$\to C_{n+2}(n) \xrightarrow{d} C_{n+1}(n) \xrightarrow{d} C_n(n), \qquad (1.3)$$

of the bicomplex (1.2). This is the *Grassmannian complex* introduced in [S1, BMS], see also [Q2].

Let us denote by $(BC_*(n), \partial)$ the total complex associated with the bicomplex (1.2); in particular, $BC_n(n) := C_n(n)$. We assume that $BC_k(n)$ placed in degree k and ∂ has degree -1.

Note that

$$BC_*(n+1) = BC_*(n)/C_*(n)$$

and there is a sequence of surjective homomorphisms of complexes

$$BC_*(1) \twoheadrightarrow BC_*(2) \twoheadrightarrow BC_*(3) \twoheadrightarrow \cdots.$$

The complex $BC_*(n)$ is "homological", i.e., its differential has degree -1. Let us make from it a "cohomological" complex $BC^*(n)$ with differential ∂ of degree $+1$. By definition $BC^i(n) := BC_{2n-i}(n)$,

$$\cdots \xrightarrow{\partial} BC^0(n) \xrightarrow{\partial} BC^1(n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} BC^n(n).$$

Here $BC^i(n)$ is placed at degree i. There is also a "cohomological" version $C^*(n)$ of the Grassmannian complex $C_*(n)$.

Now let us give a more geometrical interpretation of the Grassmannian bicomplex that also explains the name.

Let $(e_1, \dots, e_{p+q+1})$ be a basis in a vector space V. Let us denote by $\widehat{\mathbf{G}}^p_q$ the open subset of the Grassmannian of q-dimensional subspaces of P^{p+q} consisting of subspaces that are transverse to coordinate hyperplanes. In [M] R. MacPherson constructed an isomorphism

$$m\colon \widehat{\mathbf{G}}^p_q \xrightarrow{\sim} \left\{ \begin{matrix} \text{configurations of } p+q+1 \text{ vectors in general} \\ \text{position in a } p\text{-dimensional vector space} \end{matrix} \right\}. \tag{1.4}$$

Namely, $m(\xi)$ is a configuration formed by the images of e_i in V/ξ.

Remark. The set $\widehat{\mathbf{G}}^p_q$ is defined in terms of projective geometry. However, the isomorphism (1.4) depends on the choice of the vectors e_i. This additional data can be also visualized inside P^{p+q}: one must add a generic hyperplane (affinization).

Let

$$\mathbb{Z}\colon \mathrm{Var} \to \mathrm{Ab} \tag{1.5}$$

be a functor from the category of algebraic varieties over F to the category of abelian groups that sends a variety X to the free abelian group generated by F-points of X. Applying it to (1.4) we get an isomorphism

$$\mathbb{Z}[\widehat{\mathbf{G}}^p_q] \xrightarrow{\sim} C_{p+q}(p). \tag{1.6}$$

For each integer i such that $0 \leqslant i \leqslant p+q$, there are intersection maps a_i and projection maps b_i:

$$\begin{array}{ccc} \widehat{\mathbf{G}}^p_q & \xrightarrow{a_i} & \widehat{\mathbf{G}}^p_{q-1} \\ \downarrow{\scriptstyle b_i} & & \\ \widehat{\mathbf{G}}^{p-1}_q & & \end{array}$$

Here the subspace $a_i(\xi)$ is the intersection of ξ with the ith coordinate hyperplane and the subspace $b_i(\xi)$ is the image of ξ under the projection with the center at the ith vertex of the simplex. We get a bi-Grassmannian $\widehat{\mathbf{G}}(n)$:

$$\widehat{\mathbf{G}}(n)\colon \quad \begin{array}{ccccccc} & & & & & & \downarrow\vdots\downarrow \\ & & & & & \overset{\cdots}{\rightrightarrows} & \widehat{\mathbf{G}}^{n+2}_0 \\ & & & & \downarrow\vdots\downarrow & & {\scriptstyle b_0}\downarrow\vdots\downarrow{\scriptstyle b_{n+1}} \\ & & & \overset{\cdots}{\rightrightarrows} & \widehat{\mathbf{G}}^{n+1}_1 & \underset{a_{n+1}}{\overset{a_0}{\rightrightarrows}} & \widehat{\mathbf{G}}^{n+1}_0 \\ & \downarrow\vdots\downarrow & & & {\scriptstyle b_0}\downarrow\vdots\downarrow{\scriptstyle b_{n+1}} & & {\scriptstyle b_0}\downarrow\vdots\downarrow{\scriptstyle b_n} \\ \overset{\cdots}{\rightrightarrows} & \widehat{\mathbf{G}}^n_2 & \underset{a_{n+1}}{\overset{a_0}{\rightrightarrows}} & & \widehat{\mathbf{G}}^n_1 & \underset{a_n}{\overset{a_0}{\rightrightarrows}} & \widehat{\mathbf{G}}^n_0 \end{array} \tag{1.7}$$

Applying functor (1.5) to it, considering differentials $d = \sum(-1)^i a_i$ and $d' = \sum(-1)^i b_i$ and using isomorphism (1.6) we get the Grassmannian bicomplex.

Now let us sheafify these constructions.

A bicomplex $\underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}(n)]$ of sheaves on X, called the Grassmannian bicomplex, is constructed as follows. For a point $x \in X$, the stalk of $\underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}(n)]$ at x is the formal linear combinations of germs at x of maps from X to $\widehat{\mathbf{G}}^p_q$. The corresponding bicomplex looks as follows:

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \downarrow & & \downarrow \\ \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}(n)]: \cdots & \longrightarrow & \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}^{n+1}_1] & \xrightarrow{d} & \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}^{n+1}_0] \\ & & \downarrow d' & & \downarrow d' \\ \cdots & \longrightarrow & \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}^{n}_1] & \xrightarrow{d} & \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}^{n}_0] \end{array} \tag{1.8}$$

Here $\underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}^n_0]$ is placed at degree $(n, 0)$ and d (respectively d') has degree $(1, 0)$ (respectively $(0, 1)$). The hypercohomology of the total complex associated with this bicomplex of sheaves is the *bi-Grassmannian cohomology* of X. We will denote it by $H^*(X, \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}(n)])$ or $H^*(X, \underline{\underline{BC}}^*(n))$. Note that the Grassmannian cohomology of [BMS] maps canonically to the bi-Grassmannian one, but there is no inverse map.

In § 2 we construct characteristic classes $c_n(E) \in H^{2n}(X, \underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}(n)])$. There is a homomorphism of complexes of sheaves

$$\underline{\underline{\mathbb{Z}}}[\widehat{\mathbf{G}}(n)] \to \underline{\underline{K}}^M_n[-n] \tag{1.9}$$

(see § 3) that provides a construction of characteristic classes

$$c_n(E) \in H^n(X, \underline{\underline{K}}^M_n).$$

1.4. Polylogarithms (compare with [GGL, BMS, HM]). Now let $F = \mathbb{C}$. Note that $\widehat{\mathbf{G}}^n_0$ is almost canonically isomorphic to $(\mathbb{C}^*)^n$ (see Remark in 1.3 above). Indeed, according to (1.4) a point $\xi \in \widehat{\mathbf{G}}^n_0$ defines an (ordered) configuration of $n+1$ vectors in general position in $\mathbb{C}^n$: $m(\xi) = (v_0, \dots, v_n)$. So $v_0 = \sum_{i=1}^n z_i v_i$ and the map $\xi \mapsto (z_1, \dots, z_n)$ provides an isomorphism $\widehat{\mathbf{G}}^n_0 \xrightarrow{\sim} (\mathbb{C}^*)^n$. Therefore there is a canonical multivalued holomorphic $n-1$ form

$$w^n_0 := \frac{1}{n}\sum_{i=1}^n (-1)^i \log z_i \, d\log z_1 \wedge \cdots \wedge \widehat{d\log z_i} \wedge \cdots \wedge d\log z_n \tag{1.10}$$

on $\widehat{\mathbf{G}}^n_0$.

Consider the multivalued Deligne complex $\widetilde{\mathbb{Q}}(n)_Y$ on a variety Y ($\mathbb{Q}$ is placed at degree 0, d has degree $+1$):

$$\mathbb{Q} \xrightarrow{(2\pi i)^n} \widetilde{\Omega}^0(Y) \xrightarrow{d} \widetilde{\Omega}^1(Y) \xrightarrow{d} \cdots \xrightarrow{d} \widetilde{\Omega}^{p-1}(Y) \to 0.$$

Here $\widetilde{\Omega}^i$ denotes multivalued holomorphic differential forms, i.e., holomorphic differential forms defined on the universal covering space $\widetilde{Y}$ of Y. We want to consider a triple complex $\mathbb{D}$, which is the multivalued complex $\widetilde{\mathbb{Q}}(n)_{\widehat{\mathbf{G}}(n)}$ in the vertical direction and is a double complex constructed from the bi-Grassmannian $\widehat{\mathbf{G}}(n)$ in the horizontal directions. All differentials have degree $+1$.

A $2n$-cocycle in the complex $\mathbb{D}$ is just a collection of $(2n-1-p-q)$-forms $\{\omega_q^p\}$ such that

$$d\omega_q^p = \sum(-1)^i a_i^* \omega_{q-1}^p + \sum(-1)^i b_i^* \omega_q^{p-1}. \tag{1.11}$$

CONJECTURE 1.1. *There exists a $2n$-cocycle $\mathbb{L}_n$ in the triple complex $\mathbb{D}$ such that its ω_0^n-component is given by formula* (1.10).

The collection of forms $\{\omega_q^n\}$ is, of course, the Grassmannian n-logarithm conjectured in [BMS, HM]. However, for an explicit construction of the Chern classes in Deligne cohomology we have to construct the entire bi-Grassmannian n-logarithm and it is *not* sufficient to construct only its Grassmannian part. The main construction of this paper (see § 2) yields a construction of

$$c_n(E) \in H_{\mathcal{D}}^{2n}(X, \mathbb{Q}(n)),$$

using the bi-Grassmannian polylogarithm $\mathbb{L}_n$. The coincidence of this class with the one constructed by A. A. Beilinson [B2] is guaranteed by formula (1.10) (see Theorem 5.11). The problem of construction of a collection of forms $\{\omega_q^p\}$ satisfying (1.11) goes back to [GGL], see also [You], where the real-valued forms on the corresponding manifolds over $\mathbb{R}$ were considered (forms $S^{p,q}$).

The most interesting component of $\mathbb{L}_n$ is a multivalued function $P_n := \omega_{n-1}^n$ on $\widehat{\mathbf{G}}_{n-1}^n$. The cocycle condition means that it should satisfy two "$(2n+1)$-term" functional equations

$$\sum_{i=0}^{2n}(-1)^i a_i^* P_n = (2\pi i)^n q_1, \tag{1.12a}$$

$$\sum_{i=0}^{2n}(-1)^i b_i^* P_n = (2\pi i)^n q_2, \tag{1.12b}$$

where $q_1, q_2 \in \mathbb{Q}$. Note that a_i^*, b_i^* make sense after lifting maps a_i, b_i to the simply connected covering spaces.

For a much more precise "motivic" version of Conjecture 1.1 see Conjecture 6.1. It is formulated for any field F and implies Conjecture 1.1, when $F = \mathbb{C}$.

Instead of the Deligne complex $\underline{\underline{\widetilde{\mathbb{Q}}}}(n)_{\mathscr{D}}$ one could consider the real Deligne complex $\underline{\underline{\mathbb{R}}}(n)_{\mathscr{D}}$ that is the total complex of the bicomplex

$$\underline{\underline{\mathbb{R}}}(n)_{\mathscr{D}}: \quad \begin{array}{ccccccccccc} S^0_X & \xrightarrow{d} & S^1_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & S^n_X & \xrightarrow{d} & S^{n+1}_X & \xrightarrow{d} & \cdots \\ & & & & & & \uparrow{\scriptstyle \pi_n} & & \uparrow{\scriptstyle \pi_n} & & \\ & & & & & & \Omega^n_X & \xrightarrow{\partial} & \Omega^{n+1}_X & \xrightarrow{\partial} & \cdots \end{array} \tag{1.13}$$

where $(S^{\bullet}_X, d)$ is the de Rham complex of the real-valued forms, $(\Omega^{\bullet}, \partial)$ is the de Rham complex of the holomorphic forms with logarithmic singularities at infinity $\pi_n = \mathrm{Re}$ for odd n and $\pi_n = \mathrm{Im}$ for even n, and S^0_X is placed at degree 1.

One can consider the triple complex $\mathbb{D}$, which is the complex $\underline{\underline{\mathbb{R}}}(n)_{\mathscr{D}}$ in the vertical direction and is the double complex constructed from the bi-Grassmannian $\underline{\underline{\widehat{G}}}(n)$ in the horizontal directions. Actually it is more natural to consider the complex that computes hypercohomology of the bi-Grassmannian $\widehat{G}(n)$ with coefficients in $\underline{\underline{\mathbb{R}}}(n)_{\mathscr{D}}$ (for this we should replace the complex $(\Omega^{\geqslant n}_X, \partial)$ in (1.13) by its Dolbeaux resolution $(\mathscr{D}^{\geqslant n, q}_X)$ for example), but it is not important for our purposes.

Conjecture 1.1$'$. *There exists a* $2n$*-cocycle* $\mathbb{L}'_n$ *in the triple complex* $\mathbb{D}'$ *such that its component over* $\widehat{G}^n_0$ *is given by the following formulas:*

$$\begin{aligned} \omega^{n'}_0 &= \pi_n \operatorname{Alt}\Big(\sum_{k=0}^{[(n-1)/2]} \frac{i^{n-2k-1}}{(2k+1)!(n-2k-1)!} \log|z_1|\, d\log|z_2| \wedge \cdots \\ &\qquad \wedge d\log|z_{2k+1}| \wedge d\arg z_{2k+2} \wedge \cdots \wedge d\arg z_n \Big) \in S^{n-1}_{\widehat{G}^n_0}, \\ \omega^{n''}_0 &= d\log z_1 \wedge \cdots \wedge d\log z_n \in \Omega^n_X \end{aligned} \tag{1.14}$$

$(d\omega^{n'}_0 = \pi_n(\omega^{n''}_0))$. *The corresponding component* P'_n *of* $\mathbb{L}'_n$ *on* $\widehat{G}^n_{n-1}$ *should satisfy the "clean"* $(2n+1)$*-term equation*

$$\sum_{i=0}^{2n} (-1)^i a^*_i P'_n = 0, \tag{1.14a}$$

$$\sum_{i=0}^{2n} (-1)^i b^*_i P'_n = 0. \tag{1.14b}$$

Remark. $(\omega^{n'}_0, \omega^{n''}_0)$ is just the product in the real Deligne cohomology of 1-cocycles $(\log|z_i|, d\log z_i) \in H^1(\widehat{G}^n_0, \mathbb{R}(1)_{\mathscr{D}})$.

In [GM] I. M. Gelfand and R. MacPherson constructed for even n a function $\widetilde{P}_n$ on the real "middle Grassmannian" $\widehat{G}^n_{n-1}(\mathbb{R})$, that satisfies the conditions (1.12).

On the other hand, there are the *classical* polylogarithms $\mathrm{Li}_n(z)$ that are functions in *one* complex variable z. They were defined by L. Euler as functions on the unit disc $|z| \leqslant 1$ given by absolutely convergent series

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

and can be continued analytically to a multivalued function on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ using the formulas

$$\mathrm{Li}_1(z) = -\log(1-z),$$
$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t)\frac{dt}{t}.$$

It turns out that $\mathrm{Li}_n(z)$ has a remarkable single-valued version ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, ... are Bernoulli numbers)

$$\mathscr{L}_n(z) = \begin{matrix}\mathrm{Re}(n\colon \text{odd})\\ \mathrm{Im}(n\colon \text{even})\end{matrix}\left(\sum_{k=0}^{n} \frac{B_k \cdot 2^k}{k!} \log^k|z| \cdot \mathrm{Li}_{n-k}(z)\right), \qquad n \geqslant 2,$$
$$\mathscr{L}_1(z) = \log|z|.$$

For example

$$\mathscr{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\cdot\log|z|$$

is the Bloch-Wigner function, and

$$\mathscr{L}_3(z) = \mathrm{Re}(\mathrm{Li}_3(z) - \log|z|\cdot \mathrm{Li}_2(z) + 1/3\log^2|z|\cdot \mathrm{Li}_1|z|)$$

was used in [G1]. The functions $\mathscr{L}_n(z)$ for arbitrary n were written by D. Zagier [Z]. Explicit formulas expressing the bi-Grassmannian polylogarithms $\mathbb{L}_n$, $\mathbb{L}'_n$ in terms of the classical polylogarithms for $n \leqslant 3$ were given in [G1] (see also [G2] and §§5, 6 of this paper). For example $\mathbb{L}'_3$, that is a function on the 9-dimensional manifolds $\widehat{G}^3_2$, is expressed in terms of $\mathscr{L}_3(z)$. However for $n \geqslant 4$ the "natural" cocycle $\mathbb{L}_n$ can*not* be expressed by the classical polylogarithms (the reason was explained in §1 of [G1]).

An interesting geometrical construction of the Grassmannian 2- and 3-logarithms was suggested by M. Hanamura and R. MacPherson [HaM]. The existence of the Grassmannian n-logarithms for $n \leqslant 3$ was proved in [HM].

It is interesting that in formulas for $\mathbb{L}'_n$, $n \leqslant 3$, from §9 of [G1] all forms ω_q^{n+i} vanish for $i > 0$. This means that the bi-Grassmannian n-logarithms for $n \leqslant 3$ reduces essentially to its Grassmannian part $\{\omega_q^n\}$. This is a nontrivial fact about the Grassmannian n-logarithms, $n \leqslant 3$. But this is *not true* for $n \geqslant 4$. For example, already forms ω_1^{n+1} cannot be chosen equal to zero for $n \geqslant 4$. This is another important difference between cases $n \leqslant 3$

and $n > 4$. It shows why we must enlarge the Grassmannian polylogarithms to the bi-Grassmannian ones.

1.5. The universal Chern classes $c_n \in H^n(BGL_{m\bullet}, \underline{\underline{K}}_n^M)$. Recall that the classifying space for a group G can be represented by the simplicial scheme

$$BG_\bullet : * \Leftarrow G \Lleftarrow G^2 \cdots .$$

In § 4 an explicit construction will be given for the universal Chern classes $c_n \in H^n(BGL_m(F)_\bullet, \underline{\underline{K}}_n^M)$, $m \geqslant n$. It is a refinement of the construction from § 2 and, of course, implies it immediately. More precisely, a Zariski covering $\{U_i\}_{i\in I}$ defines a simplicial scheme $U_\bullet$:

$$\coprod_{i\in I} U_i \Leftarrow \coprod_{i_0<i_1\in I} U_{i_0 i_1} \Lleftarrow \coprod_{i_0<i_1<i_2\in I} U_{i_0 i_1 i_2} \cdots .$$

A G-bundle E over X given by its transition functions $g_{ij} \in \Gamma(U_{ij}, G)$ defines a canonical map of simplicial schemes $u: U_\bullet \to BG_\bullet$. Our G-bundle is the inverse image of the canonical G-bundle $EG_\bullet \xrightarrow{G} BG_\bullet$ over $BG_\bullet$ and $c_n(E) = u^* c_n$.

As a byproduct, an explicit algebraic construction of cohomology classes generating the ring $H^*(GL_m)$ is obtained. The existence of such a description of the usual topological cohomology of GL_m was conjectured by A. A. Beilinson [B3].

1.6. The universal motivic Chern classes. In § 4 an explicit construction of such Chern classes

$$c_n \in H^{2n}_{\mathscr{M}}(BGL_{m}\bullet, \mathbb{Z}(n)), \qquad n \leqslant 3,$$

will be given. It implies, in particular, an explicit construction of the Chern classes $c_n(E)$ with values in Deligne cohomology $H^{2n}_{\mathscr{D}}(X, \mathbb{Z}(n))$ in terms of classical n-logarithms ($n \leqslant 3$). A cocycle representing the usual topological characteristic class $c_n(E) \in H^{2n}(X, \mathbb{Z})$ in the Čech complex was constructed by J.-L. Brylinski and D. MacLaughlin [BM2].

A local combinatorial formula for all Pontryagin classes was suggested by I. M. Gelfand and R. MacPherson [GM2].

Let $H^*_{\mathrm{cts}}(G, R)$ be the ring of continuous cohomology of a Lie group G. It is known that

$$H^*_{\mathrm{cts}}(GL_m(\mathbb{C}), R) = \Lambda^*_R(b_1^{(m)}, b_3^{(m)}, \dots, b_{2m-1}^{(m)}),$$
$$b_{2k-1}^{(m)} \in H^{2k-1}_{\mathrm{cts}}(GL_m(\mathbb{C}), \mathbb{R}).$$

As a byproduct of the construction of the universal Chern classes $c_n \in H^{2n}_{\mathscr{D}}(BGL_m(\mathbb{C}), \mathbb{R}(n))$, we get an explicit formula for (measurable) cocycles representing classes $b_{2n-1}^{(m)}$ for $n \leqslant 3$ and arbitrary $m \geqslant 2n-1$ using the classical n-logarithm. The formula for b_1 is well known: $b_1(g) := \log|\det g|$,

$g \in GL_m(\mathbb{C})$, is a 1-cocycle. The formula for $b_3^{(2)}$ was given by D. Wigner in the middle of the seventies, and the formula for $b_5^{(3)}$ by the author ([G1], see also [G2]). A formula for $b_3^{(m)}$ was written also by Kioshi Igusa (unpublished).

Note that there is a canonical map

$$H_n(X, \underline{\underline{K}}_n^M) \to H^n(X, \underline{\underline{K}}_n),$$

and it was shown by Soulé [So] and by Nesterenko and Suslin [NS] that this map is an isomorphism modulo torsion. This together with characteristic classes $c_n(E) \in H^n(X, \underline{\underline{K}}_n)$ of Gillet [Gil] proves the existence of $c_n(E) \in H^n(X, \underline{\underline{K}}_n^M)$ but does not give any precise construction.

This work was initiated by A. A. Beilinson who explained to me that there is no explicit construction of the Chern classes with values in $H^n(X, \underline{\underline{K}}_n^M)$ as well as in $H^n(X, \Omega_{\log}^n)$ or $H^n(X, \Omega_{\log}^n)$ and emphasized importance of such a construction.

I hope it is clear from the introduction how much I benefited from the paper [GGL] of A. M. Gabrielov, I. M. Gelfand, and M. V. Losik.

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§2. Affine flags and Chern classes in bi-Grassmannian cohomology

2.1. Affine flags. Let V be a vector space over a field F. By definition a *p-flag* in V is a sequence of subspaces

$$0 \subset L^1 \subset L^2 \subset \cdots \subset L^p, \qquad \dim L^i = i.$$

An *affine p-flag* $L^\bullet$ is a p-flag together with choice of vectors $l^i \in L^i/L^{i-1}$, $i = 1, \ldots, p$ ($L^0 = 0$). We will denote affine p-flags as $(l^1, \ldots, l^p)$. Subspaces L^i can be recovered as the ones generated by $l^1, \ldots, l^i$: $L^i = \langle l^1, \ldots, l^i \rangle$.

We say that an $(n+1)$-tuple of affine flags

$$L_0^\bullet = (l_0^1, \ldots, l_0^p), \ldots, L_n^\bullet = (l_n^1, \ldots, l_n^p) \tag{2.1}$$

is generic (or in the general position)

$$\dim(L_0^{i_0} + \cdots + L_n^{i_n}) = i_0 + \cdots + i_n \quad \text{whenever} \quad i_0 + \cdots + i_n \leqslant \dim V. \tag{2.2}$$

Let $A^p(m)$ be the manifold of all affine p-flags in an m-dimensional vector space V_m. It is a $GL(V_m)$-set, so as usual (see 1.3 of the Introduction) one

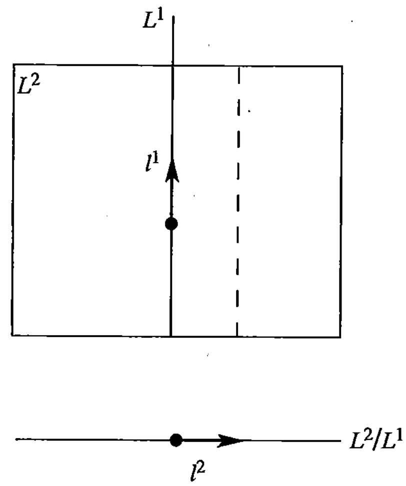


FIGURE 1. An affine 2-flag

can consider free abelian groups $C_n(A^p(m))$ of configurations of $(n+1)$-tuples of affine p-flags in general position in V_m. Further, there is a complex of affine p-flags $C_*(A^p(m))$:

$$\cdots \xrightarrow{d} C_{n+1}(A^p(m)) \xrightarrow{d} C_n(A^p(m)) \xrightarrow{d} C_{n-1}(A^p(m)) \xrightarrow{d} \cdots , \\ d\colon (L_0^\bullet, \dots, L_n^\bullet) \mapsto \sum_{i=0}^{n} (-1)^i (L_0^\bullet, \dots, \widehat{L_i^\bullet}, \dots, L_n^\bullet). \tag{2.3}$$

In particular, $C_*(A^1(m)) \equiv C_*(m)$. Let us define a map of complexes

$$T\colon C_*(A^{p+1}(n+p)) \to BC_*(n) \tag{2.4}$$

as follows: for

$$a_k^{p+1} = (v_0^1, \dots, v_0^{p+1}; \dots; v_k^1, \dots, v_k^{p+1}) \in C_k(A^{p+1}(n+p)) \qquad (k \geqslant n),$$

set

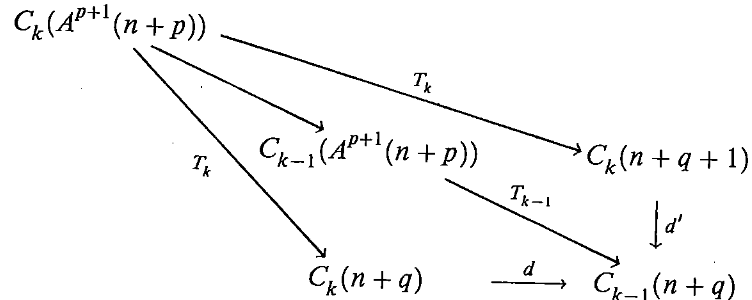
$$T(a_k^{p+1}) := \bigoplus_{q=0}^{k-n} \sum_{\substack{i_0+\cdots+i_k=p-q \\ i_k \geqslant 0}} (L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} \mid v_0^{i_0+1}, \dots, v_k^{i_k+1}) \\ \in \bigoplus_{q=0}^{k-n} C_k(n+q) =: BC_k(n). \tag{2.5}$$

Here $(L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} \mid v_0^{i_0+1}, \dots, v_k^{i_k+1})$ is the configuration of vectors in the space $V_I := V_m / \bigoplus_{\alpha=0}^{k} L_\alpha^{i_\alpha}$ equal to projections of vectors $v_0^{i_0+1}, \dots, v_k^{i_k+1}$. (Since $v_\alpha^{i_\alpha+1}$ is a vector in $V_m/L_\alpha^{i_\alpha}$, we can project it to V_I.)

KEY LEMMA 2.1. *T is a morphism of complexes.*

PROOF. Let $T_k(n+q)\colon C_k(A^{p+1}(n+p)) \to C_k(n+q)$ be the $C_k(n+q)$-component of the map T. We have to prove that (see (2.6))

$$d \circ T_k(n+q) = T_{k-1}(n+q) - d' \circ T_k(n+q+1),$$



(2.6)

For a given partition $i_0 + \cdots + i_k = p - q$ let us consider the expression

$$\begin{aligned} &d(L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} \mid v_0^{i_0+1}, \dots, v_k^{i_k+1}) \\ &= \sum_{j=0}^{k} (-1)^j (L_0^{i_0} \oplus \cdots \oplus L_k^{i_k} \mid v_0^{i_0+1}, \dots, \widehat{v_j^{i_j+1}}, \dots, v_k^{i_k+1}). \end{aligned} \tag{2.7}$$

For $i_j = 1$ the corresponding term in (2.6) will appear in the formula for $T_{k-1}(n+q)(a_k^{p+1})$. For $i_j > 1$ such a term appears in the formula for

$$d'(L_0^{i_0} \oplus \cdots \oplus L_j^{i_j-1} \oplus L_k^{i_k} \mid v_0^{i_0+1}, \dots, v_j^{i_j}, \dots, v_k^{i_k+1}).$$

2.2. A construction of Chern classes in bi-Grassmannian cohomology. Let us denote by $\mathscr{A}_E^p(X)$ the bundle of affine p-flags in fibers of a vector bundle E over X. Choose a Zariski covering $\{U_i\}$ of X such that $E|_{U_i}$ is trivial. Choose sections

$$L_i^\bullet(x) \in \Gamma(U_i, \mathscr{A}_E^p(x))$$

such that for any $i_0 < \cdots < i_k$ affine p-flags $L_{i_0}^\bullet(x), \dots, L_{i_k}^\bullet(x)$ are in general position for every $x \in U_{i_0 \dots i_k}$.

Let $(\mathscr{F}(n)_X^*, \partial)$ be a complex of sheaves in the Zariski topology on X obtained by sheafification from a complex $(\mathscr{F}(n)^*, \partial)$, where ∂ has degree $+1$. Denote by $\mathscr{F}(n)_*$ the corresponding homological complex:

$$\cdots \xrightarrow{\partial} \mathscr{F}(n)_i \xrightarrow{\partial} \mathscr{F}(n)_{i-1} \xrightarrow{\partial} \mathscr{F}(n)_{i-2} \xrightarrow{\partial} \cdots,$$

where, by definition, $\mathscr{F}(n)_i := \mathscr{F}(n)^{2n-i}$.

Suppose that we are given a morphism of complexes

$$T_n\colon C_*(A^{p+1}(n+p)) \to \mathscr{F}(n)_*. \tag{2.8}$$

Then

$$T_n(L_{i_0}(x), \dots, L_{i_k}(x)) \in \mathscr{F}(n)^{2n-k}|_{U_{i_0 \dots i_k}} \tag{2.9}$$

is a $2n$-cocycle in the Čech complex for the covering $\{U_i\}$ with values in the complex of sheaves $\mathscr{F}(n)^*_X$. (In fact, the chain (2.9) is a cocycle if and only if the map (2.8) is a morphism of complexes.)

A different choice of sections $L_i^\bullet(x)$ gives a cocycle that is canonically cohomologous to the previous one. Therefore the cohomology class

$$c_n(E) \in H^{2n}(\check{C}(U_\bullet, \mathscr{F}(n)^*))$$

is well defined.

Applying this construction to the bi-Grassmanian cohomology and using the Key Lemma 2.1 we immediately obtain the following theorem.

THEOREM 2.2. *$T(L_{i_0}^\bullet(x), \dots, L_{i_k}^\bullet(x)) \in \underline{\underline{BC}}^{2n-k}(n)\,|_{U_{i_0\dots i_k}}$ is a cocycle in the Čech complex for the covering $\{U_i\}$ with values in the bi-Grassmannian complex.*

The homomorphism (2.8) for other cohomology theories $\mathscr{F}(n)^*$, including the motivic one (hence Deligne, l-adic, K^M, etc. cohomology), should be obtained as a composition

$$C_*(A^{p+1}(n+p)) \xrightarrow{T} BC_*(n) \longrightarrow \mathscr{F}(n)_*.$$

This will be proved for K_n^M-cohomology in § 3 and for motivic cohomology of weight $\leqslant 3$ in § 5.

However, I want to emphasize that the bi-Grassmannian cohomology is certainly different (larger) than the motivic one.

§3. Chern classes with values in $H^n(X, \underline{\underline{K}}_n^M)$

3.1. In § 2 we constructed Chern classes with values in $H^{2n}(\underline{\underline{BC}}^*(n))$. To obtain Chern classes with values in $H^n(X, \underline{\underline{K}}_n^M)$ it is sufficient to define a morphism of complexes

$$BC_*(n) \to K_n^M(F)[-n], \tag{3.1}$$

i.e., a homomorphism $\overline{f}_n(n)\colon C_n(n) \to K_n^M(F)$ such that $f_n(n) \circ d = f_n(n) \circ d' = 0$:

$$\begin{array}{ccccc}
 & & & & \downarrow{\scriptstyle d'} \\
 & & C_{n+2}(n+1) & \xrightarrow{d} & C_{n+1}(n+1) \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
 & \xrightarrow{d} & C_{n+1}(n) & \xrightarrow{d} & C_n(n) \\
 & & & & \downarrow{\scriptstyle f_n(n)} \\
 & & & & K_n^M(F)
\end{array}$$

Now let us define a homomorphism

$$f_n(n)\colon C_n(n) \to \Lambda^n F^*,$$

as follows (compare with § 3.2 in [G2]). Choose a volume form $w \in \det(V_n)^* \equiv \Lambda^n(V_n)^*$ (where $\dim V^n = n$). Set

$$\Delta(v_1, \dots, v_n) := \langle w, v_1 \wedge \cdots \wedge v_n \rangle \in F^*, \qquad v_i \in V_n,$$
$$f_n(n)(v_0, \dots, v_n) := \mathrm{Alt} \bigwedge_{1 \leqslant i \leqslant n} \Delta(v_0, \dots, \widehat{v}_i, \dots, v_n) \in \Lambda^n F^*. \tag{3.2}$$

Here $\mathrm{Alt}\, g(v_0, \dots, v_n) := \sum_{\sigma \in S_{n+1}} (-1)^{|\sigma|} g(v_{\sigma(0)}, \dots, v_{\sigma(n)})$. For example, up to a 2-torsion we have

$$f_2(2)(v_0, v_1, v_2) := 2(\Delta(v_0, v_2) \wedge \Delta(v_0, v_1)$$
$$- \Delta(v_1, v_2) \wedge \Delta(v_0, v_1) + \Delta(v_0, v_2) \wedge \Delta(v_1, v_2)).$$

LEMMA 3.1. *$f_n(n)(v_0, \dots, v_n)$ does not depend on w.*

PROOF. Let $f_n'(n)$ be a homomorphism defined using another volume form $w' = \lambda w$. Then

$$(f_n(n) - f_n'(n))(v_0, \dots, v_n) = \lambda \wedge \sum \Lambda_{i,j},$$

where $\Lambda_{i,j} \in \Lambda^{n-1} F^*$ depends on $v_0, \dots, \widehat{v}_i, \dots, \widehat{v}_j, \dots, v_n$. So $\Lambda_{i,j}$ is symmetric on v_i, v_j. But the left-hand side is antisymmetric by definition. Therefore $\Lambda_{i,j} = 0$.

LEMMA 3.2. *The composition*

$$C_{n+1}(n+1) \xrightarrow{d'} C_n(n) \xrightarrow{f_n(n)} \Lambda^n F^* \tag{3.3}$$

is equal to zero modulo 2-torsion.

PROOF (Compare with the proof of Lemma 3.4 in [G1]).

$$f_n(n) \circ d'(v_0, \dots, v_{n+1}) = \mathrm{Alt} \bigwedge_{j=2}^{n+1} \Delta(v_0, v_1, \dots, \widehat{v}_j, \dots, v_{n+1}) = 0,$$

because $\Delta(v_0, v_1, \dots, \widehat{v}_j, \dots, v_{n+1})$ is invariant under the transposition of v_0 and v_1 modulo 2-torsion.

PROPOSITION 3.3. *The composition*

$$C_{n+1}(n) \xrightarrow{d} C_n(n) \xrightarrow{\overline{f_n(n)}} K_n^M(F) \tag{3.4}$$

is equal to zero.

PROOF (Compare with the proof of Proposition 2.4 in [S1]). There exists a duality $*\colon C_{m+n-1}(m) \to C_{m+n-1}(n)$, $*^2 = \mathrm{id}$, that satisfies the following properties (see § 3.8 in [G2]).

1. $*$ commutes with the action of the permutation group S_{m+n}.

2. If $*(l_1, \dots, l_{m+n}) = (l'_1, \dots, l'_{m+n})$ then
$$*(l_1, \dots, \widehat{l_i}, \dots, l_{m+n}) = (l'_i \mid l'_1, \dots, \widehat{l_i}, \dots, \widehat{l}_{m+n}).$$
3. Choose volume forms in V_m and V_n and consider the partition
$$\{1, \dots, m+n\} = \{i_1 < \dots < i_m\} \cup \{j_1 < \dots < j_n\}.$$
Then $\dfrac{\Delta(l_{i_1}, \dots, l_{i_m})}{\Delta(l_{j_1}, \dots, l_{j_n})}$ does not depend on a partition.

This duality can be defined as follows. A configuration of $(m+n)$ vectors in an m-dimensional coordinate vector space can be represented as columns of the $m \times (m+n)$ matrix (I_m, A). The dual configuration is represented by the $n \times (m+n)$ matrix $(-A^t, I_n)$. Using the duality we can reformulate Proposition 3.3 as follows: the composition
$$C_{n+1}(2) \xrightarrow{d'} C_n(1) \xrightarrow{\widetilde{f}_n(n)} K_n^M(F)$$
is equal to 0. Here
$$\widetilde{f}_n(n)(v_0, \dots, v_n) := \operatorname{Alt}\Delta(v_0) \wedge \Delta(v_1) \wedge \dots \wedge \Delta(v_{n-1}) \in \Lambda^n F^*.$$
Consider the following diagram:
$$\begin{array}{ccc} C_{n+1}(2) & \xrightarrow{d'} & C_n(1) \\ \downarrow{\scriptstyle \widetilde{f}_{n+1}(n)} & & \downarrow{\scriptstyle \widetilde{f}_n(n)} \\ \mathbb{Z}[P_F^1 \setminus \{0, 1, \infty\}] \otimes \Lambda^{n-2} F^* & \xrightarrow{\delta} & \Lambda F^* \end{array}$$
Here $\mathbb{Z}[P_F^1 \setminus \{0, 1, \infty\}]$ is a free abelian group generated by symbols $\{x\}$, $x \in P_F^1 \setminus \{0, 1, \infty\}$, $\delta: \{x\} \otimes y_1 \wedge \dots \wedge y_{n-2} \mapsto (1-x) \wedge x \wedge y_1 \wedge \dots \wedge y_{n-2}$. Note that by definition $\operatorname{Coker}\delta = K_n^M(F)$. The homomorphism $\widetilde{f}_{n+1}(n)$ is defined as follows:
$$\widetilde{f}_{n+1}(n)(v_0, \dots, v_{n+1}) := n!\,[v_0, \dots, v_{n+1}],$$
where $[v_0, \dots, v_{n+1}]$ is defined by induction:
$$[v_0, v_1, v_2, v_3] := \{r(v_0, v_1, v_2, v_3)\} \in \mathbb{Z}[P_F^1 \setminus \{0, 1, \infty\}],$$
$$[v_0, \dots, v_{n+1}] := \gamma_n^{-1} \cdot \operatorname{Alt}\Big(\varepsilon_1 \cdot \binom{n+1}{1}[v_1, \dots, v_{n+1}] \otimes \Delta(v_0, v_1)$$
$$+ \sum_{k=2}^{n-2} \varepsilon_k \binom{n+1}{k}[v_0, v_{k+1}, \dots, v_{n+1}] \otimes \Delta(v_0, v_1) \wedge \dots \wedge \Delta(v_0, v_k)\Big).$$
Here $\varepsilon_i = \pm 1$. More precisely, $\gamma_n = 2^{n+1} - (2 + \binom{n+1}{n-1} + \binom{n+1}{n} + \binom{n+1}{n+1})$, $\varepsilon_1 = +1$, and $\varepsilon_i = (-1)^i$, $i > 1$, for even n and $\gamma_n = 2^{n+1} - (\binom{n+1}{n-1} + \binom{n+1}{n} + \binom{n+1}{n+1})$,

$\varepsilon_1 = -1$, $\varepsilon_i = +1$, $i > 1$, for odd n. To prove the last formula one can write

$$[v_0, \dots, v_{n+1}] := \mathrm{Alt}\Big(\alpha_1 \cdot [v_1, \dots, v_{n+1}] \otimes \Delta(v_0, v_1) + \sum_{k=2}^{n-3} \alpha_k [v_0, v_k, \dots, v_{n+1}] \otimes \Delta(v_0, v_1) \wedge \cdots \wedge \Delta(v_0, v_k)\Big)$$

with some unknown α_i. Then the condition

$$\delta[v_0, \dots, v_{n+1}] = (1/n!)\, \mathrm{Alt}\, \Delta(v_0, v_1) \wedge \cdots \wedge \Delta(v_0, v_n)$$

gives exactly $n-3$ simple linear equations on α_i.

3.2. We get the following construction of the Chern classes $c_n(E) \in H^n(X, \underline{\underline{K}}_n^M)$. Choose a Zariski covering $\{U_i\}$ of X such that $E|_{U_i}$ is trivial. Choose sections $L_i^\bullet(x) \in \Gamma(U_i, \mathscr{A}_E^p(x))$ such that for any $i_0 < \cdots < i_n$ affine flags $L_{i_0}^\bullet(x), \dots, L_{i_n}^\bullet(x)$ are in general position for every $x \in U_{i_0 \dots i_n}$.

THEOREM 3.4. *The chain*

$$\overline{f}_n(n)(T(L_{i_0}^\bullet(x), \dots, L_{i_n}^\bullet)) \in \underline{\underline{K}}_n^M(\mathscr{O}^*(U_{i_0 \dots i_n})) \tag{3.5}$$

is a cocycle in the Čech complex for the covering $\{U_i\}$.

PROOF. Follows from Lemmas 3.2, 3.3 and Theorem 2.2.

By definition, $c_n(E)$ is the cohomology class of the cocycle from Theorem 3.4. It does not depend on the choice of sections $L_i^\bullet(x)$.

EXAMPLE 3.5. Recall that $c_1(E) = c_1(\det E)$. So $c_1(E)$ can be computed as follows: choose $m = \dim E$ linearly independent sections $l_i^\alpha(X)$ ($1 \leqslant \alpha \leqslant m$) of $E|_{U_i}$. Then $(l_i^\alpha(x)) = g_{ij}(x) \cdot (l_i^\beta(x))$, where $g_{ij}(x) \in GL_n(F)$ is the transition matrix and $\det g_{ij}(x)$ is a 1-cocycle representing $c_1(E)$.

Now let $(l_i^1, \dots, l_i^m)$ be the affine flag corresponding to the m-tuple of vectors $(l_i^1; \dots; l_i^m)$. Let us prove that the cocycle (3.5) computed for these flags cooincides with $\det g_{ij}$.

PROPOSITION 3.6. *We have* $f_1(1)(c((l_i^1, \dots, l_i^m), (l_j^1, \dots, l_j^m))) = \det g_{ij}$.

PROOF. Let us say that a frame $(f^1; \dots; f^m)$ is associated with an affine m-flag $(l^1, \dots, l^m)$ if

$$\langle f^1, \dots, f^k \rangle = \langle l^1, \dots, l^k \rangle \equiv L^k,$$

and the images of f^{k+1} and l^{k+1} in L^{k+1}/L^k coincide.

The set of all frames associated with a given m-flag is a principal homogeneous space over the group of unipotent matrices.

LEMMA-CONSTRUCTION 3.7. *For two affine m-flags*

$$L_1^\bullet = (v_1, \dots, v_m) \quad \text{and} \quad L_2^\bullet = (w_1, \dots, w_m),$$

in general position in V^m *there exist exactly two frames associated with both of them.*

PROOF. We have the following isomorphisms of one-dimensional vector spaces:

$$s_1 \colon L_1^k / L_1^{k-1} \xrightarrow{\sim} L_1^k \cap L_2^{m-k+1},$$
$$s_2 \colon L_2^{m-k+1} / L_2^{m-k} \xrightarrow{\sim} L_1^k \cap L_2^{m-k+1}.$$

Put $f_1^k := s_1(v_k)$, $f_2^{m-k+1} := s_2(w_{m-k+1})$. Then the frames $(f_1^1; \dots; f_1^m)$ and $(f_2^1; \dots; f_2^m)$ are associated with both $L_1^\bullet$ and $L_2^\bullet$.

Let $f_1^k = \lambda_k \cdot f_2^k$, $\lambda_k \in F^*$, and

$$(v_1, \dots, v_m) = g \cdot (w_1, \dots, w_m), \qquad g \in GL_m(F).$$

Then $\det g = \prod_{k=1}^m \lambda_k$ since $g = n_+ \cdot \lambda \cdot n_-$:

$$(w_i) \xrightarrow{n_-} (f_2^k) \xrightarrow{\lambda = (\lambda_k)} (f_1^k) \xrightarrow{n_+} (v_j),$$

where n_- (n_+) is a lower (upper) unipotent matrix and λ is the diagonal matrix with entries λ_k (the Gauss decomposition).

On the other hand, the left-hand side of the formula in Proposition 3.6 is equal to

$$f_1(1)\Big(\sum_{k=1}^m (L_1^k \oplus L_2^{m-k} \mid l_1^k, l_2^{m-k+1}) = f_1(1)(f_1^k, f_2^k)\Big) = \prod_{k=1}^m \lambda_k.$$

§4. The universal Chern class $c_n \in H^n(BGL(m)_\bullet, \underline{\underline{K}}_n^M)$

4.1. The Gersten resolution in Milnor's K-theory ([Ka]). Let F be a field with a discrete valuation v and the residue class field $\overline{F}_v (= \overline{F})$. The group of units U has a natural homomorphism $U \to \overline{F}^*$, $u \mapsto \overline{u}$. An element $\pi \in F^*$ is prime if $\operatorname{ord}_v(\pi) = 1$. There is a canonical homomorphism (see [M1]):

$$\partial \colon K_{n+1}^M(F) \to K_n^M(\overline{K}_v) \qquad (n \geqslant 0),$$

uniquely determined by the properties ($u_i \in U$)

1. $\partial(\{\pi, u_1, \dots, u_n\}) = \{\overline{u}_1, \dots, \overline{u}_n\}$;
2. $\partial(\{\pi, u_1, \dots, u_{n+1}\}) = 0$.

Let X be an excellent scheme [EGA, 3, IV, §7], $X_{(i)}$ the set of all codimension i points x, $F(x)$ the field of functions corresponding to a point $x \in X_{(i)}$.

There is a sequence of groups $\mathscr{K}(n)_\bullet$; here $K_n^M(x) := K_n^M(F(x))$:

$$K_n^M(F(X)) \xrightarrow{\partial} \bigoplus_{x\in X_{(1)}} K_{n-1}^M(x) \xrightarrow{\partial} \bigoplus_{x\in X_{(2)}} K_{n-2}^M(x) \to \cdots \to \bigoplus_{x\in X_{(n)}} \mathbb{Z}. \qquad (4.1)$$

We follow [Ka] in the definition of ∂. Let us define for $y \in X_{(i)}$ and $x \in X_{(i+1)}$ a homomorphism $\partial_x^y\colon K_{*+1}^M(y) \to K_*^M(x)$ as follows. Let Y be the normalization of the reduced scheme $\{\overline{y}\}$. Set $\partial_x^y := \sum_{x'} N_{F(x')/F(x)} \circ \partial_{x'}$, where x' ranges over all points of Y lying over x, $\partial_{x'}\colon K_{*+1}^M(y) \to K_*(x)$ is the tame symbol associated with the discrete valuation ring $\mathscr{O}_{Y,x'}$, and $N_{F(x')/F(x)}$ is the norm map $K_*^M(x') \to K_*^M(x)$ (see [BT, Chapter I, § 5] and [Ka, § 1.7]). The coboundary ∂ is by definition the sum of these homomorphism ∂_x^y.

PROPOSITION 4.1. $\partial^2 = 0$.

PROOF. See the proof of Proposition 1 in [Ka].

THEOREM 4.2. *The complex $\mathscr{K}(n)_\bullet$ is exact.*

PROOF. See [So, Chapter 6] or [NS].

4.2. Explicit formula for a class $c \in H^n(BGL(m)_\bullet, \underline{\underline{K}}_n^M)$. Set $G^n := \underbrace{G\times\cdots\times G}_{n \text{ times}}$. Recall that

$$BG_\bullet := \mathrm{pt} \underset{s_1}{\overset{s_0}{\leftleftarrows}} G \underset{s_2}{\overset{s_0}{\Lleftarrow}} G^2 \ldots$$

is the simplicial scheme representing the classifying space for a group G. We will compute $H^n(BG_\bullet, \underline{\underline{K}}_n^M)$ using the Gersten resolution (4.1). Therefore, the cochain we must construct lives in the following bicomplex ($G := GL(m)$):

$$\begin{array}{ccccc}
\vdots & & & & \\
\uparrow\partial & & & & \\
\bigoplus_{x\in G_{(2)}^{n-2}} K_{n-2}^M(F(x)) & \xrightarrow{s^*} & \bigoplus_{x\in G_{(2)}^{n-1}} K_{n-2}^M(F(x)) & & \\
& & \uparrow\partial & & \\
& & \bigoplus_{x\in G_{(1)}^{n-1}} K_{n-1}^M(F(x)) & \xrightarrow{s^*} & \bigoplus_{x\in G_{(1)}^{n}} K_{n-1}^M(F(x)) \\
& & & & \uparrow\partial \\
& & & & K_n^M(F(G^n))
\end{array} \qquad (4.2)$$

For each partition $j_0+\cdots+j_r=m-n$ we define a codimension $(n-r)$ irreducible subvariety $D(j_0, \ldots, j_r) \in G^r_{(n-r)}$ and an element $\omega(j_0, \ldots, j_r) \in K^M_r(D(j_0, \ldots, j_r))$ such that a collection of all these elements forms a cocycle in (4.2).

Recall that $A^{m-n+1}(m)$ is the manifold of affine $(m-n+1)$-flags in V_m. For a partition $j_0+\cdots+j_r=m-n$ define a codimension $n-r$ manifold

$$\widetilde{D}_{j_0,\ldots,j_r} \subset \underbrace{A^{m-n+1}(m)\times\cdots\times A^{m-n+1}(m)}_{r+1 \text{ times}}$$

as follows: $(L^\bullet_0, \ldots, L^\bullet_r) \in \widetilde{D}_{j_0,\ldots,j_r}$ if and only if

$$\dim\Big(\mathop{+}_{p=0}^{r} L_p^{j_p+1}\Big) = r+\sum_{p=0} j_p = \dim\Big(\bigoplus_{p=0}^{r} L_p^{j_p+1}\Big)-1.$$

Note that for generic $(L^\bullet_0, \ldots, L^\bullet_r) \in \widetilde{D}_{j_0,\ldots,j_r}$ the sum $+_{p=0}^{r} L_p^{j_p}$ is direct and the configurations of $r+1$ vectors

$$\Big(\bigoplus_{p=0}^{r} L_p^{j_p} \mid l_0^{j_0+1}, \ldots, l_r^{j_r+1}\Big) \tag{4.3}$$

in $V_m/\bigoplus_{p=0}^{r} L_p^{j_p}$ generates a subspace of dimension r. Recall that there is a homomorphism (see (3.2))

$$\overline{f}_r(r)\colon C_r(r) \to \Lambda^r F^* \to K^M_r(F).$$

Applying it to the configuration of $r+1$ vectors (4.3) we obtain an element

$$\widetilde{\omega}_{j_0,\ldots,j_r} \in K^M_r(F(\widetilde{D}_{j_0,\ldots,j_r})). \tag{4.4}$$

Now choose $a \in A^{m-n+1}(m)$. Set

$$D_{j_0,\ldots,j_r;a} := \{(g_1, \ldots, g_r) \in G^r \mid (a, g_1a, \ldots, g_ra) \in \widetilde{D}_{j_0,\ldots,j_r}\}.$$

Then $D_{j_0,\ldots,j_r;a} \in G^r_{(n-r)}$ and $\widetilde{\omega}_{j_0,\ldots,j_r}$ induces an element

$$\omega_{j_0,\ldots,j_r;a} \in K^M_r(F(D_{j_0,\ldots,j_r;a})). \tag{4.5}$$

Set

$$\widetilde{\omega}_r := \sum_{j_0+\cdots+j_r=m-n} \widetilde{\omega}_{j_0,\ldots,j_r} \in \bigoplus_{j_0+\cdots+j_r=m-n} K^M_r(F(\widetilde{D}_{j_0,\ldots,j_r})),$$

$$\omega_r := \sum_{j_0+\cdots+j_r=m-n} \omega_{j_0,\ldots,j_r} \in \bigoplus_{j_0+\cdots+j_r=m-n} K^M_r(F(D_{j_0,\ldots,j_r})).$$

THEOREM 4.3. *Collection of elements ω_r defines a cocycle in the bicomplex* (4.2).

PROOF. Choose a partition $i_0+\cdots+i_r=m-n+r$. Let $\widetilde{\mathscr{E}}_{i_0,\ldots,i_r}$ be a subvariety in the manifold of $(r+1)$-tuples of affine $(m-n+1)$-flags in V^m

defined as follows:

$$\widetilde{\mathscr{E}}_{i_0,\dots,i_r} := \left\{(L_0^\bullet, \dots, L_r^\bullet) \mid \dim\left(\overset{r}{\underset{p=0}{+}} L_p^{i_p}\right) = \left(\sum_{p=0}^r i_p\right) - 1\right\}.$$

This is an irreducible subvariety of codimension $n - r + 1$.

Proposition 4.4. *The component of $\partial\widetilde{\omega}_r$ on $\widetilde{\mathscr{E}}_{i_0,\dots,i_r}$ can be nonzero only if $i_k = 0$ for some k but $i_p > 0$ for $p \neq k$. In this case it is equal to*

$$\overline{f}_{r-1}(r-1)\left(\bigoplus_{p\neq k} L_p^{i_p-1} \mid l_0^{i_0}, \dots, \widehat{l_k^{i_k}}, \dots, l_r^{i_r}\right). \tag{4.6}$$

Proof. Let $j_0 + \cdots + j_r = m - n$ and

$$(l_0^1, \dots, l_0^{m-n+1}; \dots; l_r^1, \dots, l_r^{m-n+1}) \equiv (L_0^\bullet, \dots, L_r^\bullet) \in \widetilde{D}_{j_0,\dots,j_r}.$$

Choose a volume form in subspace $\langle l_0^1, \dots, l_0^{j_0+1}, \dots, l_r^1, \dots, l_r^{j_r+1}\rangle$ of codimension n. Then we can compute the determinant $\Delta(v_1, \dots, v_{m-n+r})$ for any $m - n + r$ vectors in this subspace, Set

$$\Delta(j_{k+1}) := \Delta(l_0^1, \dots, l_0^{j_0+1}, \dots, \widehat{l_k^{j_k+1}}, \dots, l_r^1, \dots, l_r^{j_r+1}).$$

By definition,

$$\widetilde{\omega}_{j_0,\dots,j_r} = \sum_{k=0}^r (-1)^k \{\Delta(j_0+1), \dots, \widehat{\Delta}(j_k+1), \dots, \Delta(j_r+1)\}. \tag{4.7}$$

The coboundary $\partial\widetilde{\omega}_{j_0,\dots,j_r}$ can be nonzero on divisors $\Delta(j_k+1) = 0$ in $\widetilde{D}_{j_0,\dots,j_r}$ only. The component of $\partial\widetilde{\omega}_{j_0,\dots,j_r}$ on the divisor $\Delta(j_k+1) = 0$ is equal to

$$\overline{f}_{r-1}(r-1)\left(\bigoplus_{p=0}^r L_p^{j_p} \oplus l_k^{j_{k+1}} \mid l_0^{j_0+1}, \dots, \widehat{l_k^{j_{k+1}}}, \dots, l_r^{j_r+1}\right). \tag{4.8}$$

This formula implies immediately that the component of $\partial\widetilde{\omega}_r$ on $\widetilde{\mathscr{E}}_{i_0,\dots,i_r}$ is zero if $i_{k_1} = i_{k_2} = 0$ for some $k_1 \neq k_2$.

It follows from (4.8) that if $i_p > 0$ for all p, the component of $\partial\widetilde{\omega}_r$ on $\widetilde{\mathscr{E}}_{i_0,\dots,i_r}$ is

$$\overline{f}_{r-1}(r-1)\left(\sum_{k=0}^r (-1)^k \left(\bigoplus_{p=0}^r L_p^{i_p-1} + l_k^{i_k} \mid l_0^{i_0}, \dots, \widehat{l_k^{i_k}}, \dots, l_r^{i_r}\right)\right). \tag{4.9}$$

Note that $(\bigoplus_{p=0}^r L^{i_p-1} \mid l_0^{i_0}, \dots, l_r^{i_r})$ is a configuration of $r+1$ vectors in an r-dimensional space. Therefore (4.9) is equal to

$$\overline{f}_{r-1}(r-1) \circ d'\left(\bigoplus_{p=0}^r L_p^{i_p-1} \mid l_0^{i_0}, \dots, l_r^{i_r}\right).$$

But this vanishes according to Lemma 3.2.

Now suppose that $i_k = 0$, $i_p \neq 0$ for $p \neq k$. Then (4.8) implies that the component of $\partial(\widetilde{\omega}_r)$ on $\widetilde{\mathscr{E}}_{i_0, \dots, i_r}$ is exactly (4.6). Proposition 4.4 is proved.

Theorem 4.3 follows immediately from Proposition 4.4.

4.3. Relation to the classical construction of Chern cycles. Assume that a vector bundle E over X has sufficiently many sections. Consider first of all the case when $\dim E = n$ and we are interested in $c_n(E) \in CH^n(X)$. Choose a section $s_0(x) \in \Gamma(X, E)$ that is transversal to the zero section of E. Then the subvariety

$$D_0 := \{x \in X \mid s_0(x) = 0\}$$

has codimension n and represents the class $c_n(E) \in CH^n(X)$. Now let $s_1(x)$ be another generic section of E (i.e., it is transversal to the zero section of E too). Then

$$D_1 := \{x \in X \mid s_1(x) = 0\}$$

should represent the same class in $CH^n(X)$. To see this let us consider a codimension $(n-1)$ subvariety

$$D_{01} := \{x \in X \mid \lambda_0 s_0(x) + \lambda_1 s_1(x) = 0 \text{ for some } \lambda_0, \lambda_1 \in \mathbb{C}\}.$$

There is a canonical rational function

$$\lambda_{01} := \frac{\lambda_0}{\lambda_1} \in F(D_{01}) \quad \text{and} \quad \operatorname{Div}(\lambda_{01}) = D_0 - D_1.$$

So D_0 and D_1 are canonically rationally equivalent cycles. Now let $s_2(x)$ be a third generic section of E. Put

$$D_{012} = \{x \in X \mid \dim\langle s_0(x), s_1(x), s_2(x)\rangle = 2\}.$$

Then $\operatorname{codim} D_{012} = n - 2$ and there is a canonical element

$$\lambda_{012} := f_2(2)(s_0, s_1, s_2) \in K_2(F(D_{012})),$$
$$\partial(\lambda_{012}) = \lambda_{01} - \lambda_{02} + \lambda_{12},$$

where $\partial \colon K_2(F(Y)) \to \coprod_{y \in Y_{(1)}} F(y)^*$ is the tame symbol. Continuing this process we get, for a generic $(r+1)$-tuple of sections $s_0(x), \dots, s_r(x)$ of E, a codimension $(n-r)$ subvariety

$$D_{01\cdots r} := \{x \in X \mid \dim\langle s_0(x), \dots, s_r(x)\rangle = r\},$$

and a canonical element

$$\lambda_{01\cdots r} := \overline{f}_r(z)(s_0, \dots, s_r) \in K_r^M(F(D_{01\cdots r})),$$

satisfying the relation

$$\partial(\lambda_{01\cdots r}) = \sum_{i=0}^{r} (-1)^i \lambda_{01\cdots\widehat{i}\cdots r}$$

(∂ is the differential in complex (4.1)).

Now let E be a vector bundle of dimension $m > n$ and $p = m - n + 1$. Let

$$L_0^\bullet(x) = (l_0^1(x), \dots, l_0^p(x))$$

be a *generic* section of the bundle of affine p-flags of X. Put

$$D_0 := \{x \in X \mid l_0^1(x) \wedge \dots \wedge l_0^p(x) = 0 \text{ but } l_0^1(x) \wedge \dots \wedge l_0^{p-1}(x) \neq 0\}.$$

It is well known (see, for example, § 3 of Chapter III in [GH]) that the image of the cycle D_0 in the Chow group $CH^n(X)$ is just $c_n(E)$. Let $L_0^\bullet, \dots, L_r^\bullet$ be $r+1$ generic sections of the bundle of affine p-flags. For any partition $j_0 + \dots + j_r = p - 1$, $j_k \geqslant 0$, put

$$D(j_0, \dots, j_r) := \Big\{x \in X \mid \text{an } (r+1)\text{-tuple of vectors } (L_0^{j_0} + \dots + L_r^{j_r} \mid l_0^{j_0+1}, \dots, l_r^{j_r+1}) \text{ generates an } r\text{-dimensional vector space and } \dim \mathop{+}_{p=0}^{r} L_k^{j_k} = \sum_{k=0}^{r} j_k\Big\}. \tag{4.10}$$

Then $D(j_0, \dots, j_r)$ is a codimension $n-r$ cycle in X. There is a canonical element

$$\overline{f}_r(r)((L_0^{j_0} \oplus \dots \oplus L_r^{j_r} \mid l_0^{j_0+1}, \dots, l_r^{j_r+1})) \in K_r^M(F(D(j_0, \dots, j_r))). \tag{4.11}$$

Let us define an element

$$\lambda_{01\dots r} \in \coprod_{j_0+\dots+j_r=p-1} K_r^M(F(D(j_0, \dots, j_r))) \subset \coprod_{x \in X_{(n-r)}} K_r^M(F(x)),$$

as the sum of elements (4.11):

$$\lambda_{01\dots r} := \sum_{j_0+\dots+j_r=p-1} \overline{f}_r(r)\Big(\bigoplus_{k=0}^{r} L_k^{j_k} \mid l_0^{j_0+1}, \dots, l^{j_r+1}\Big).$$

THEOREM 4.5. $\partial(\lambda_{01\dots r}) = \sum_{i=0}^{r}(-1)^i \lambda_{01\dots\widehat{i}\dots r}$.

PROOF. Follows immediately from the proof of Proposition 4.4.

4.4. An algebraic construction of the ring $H^*(GL_m(\mathbb{C}))$. I will construct a nonzero class in $W_0H^{2n-1}(GL_m(\mathbb{C}), \mathbb{Q}(n))$. This vector space is one-dimensional for $m \geqslant n$. Let us define for any $0 \leqslant j \leqslant m-n$ a subvariety $\widetilde{D}_j \subset A^{m-n+1}(m) \times A^{m-n+1}(m)$ as follows:

$$\widetilde{D}_j := \{(L_1^\bullet, L_2^\bullet) \text{ such that } (L_1^j + L_2^{m-n-j} \mid l_1^{j+1}, l_2^{m-n-j+1}) \text{ is a pair of collinear nonzero vectors}\}. \tag{4.12}$$

The ratio $\dfrac{\text{first vector}}{\text{second vector}}$ in (4.12) gives canonical invertible function $\widetilde{f}_j$ on $\widetilde{D}_j$. Now we choose an affine $(m-n+1)$-flag $L^\bullet$ in V_m. Set

$$GL(V_m) \supset D_j := \{g \in GL(V_m) \mid (gL^\bullet, L^\bullet) \in \widetilde{D}_j\}.$$

There is a canonical function $f_j \in \mathcal{O}(D_j)^*$.

THEOREM 4.6. *The current* $\sum_j d\log f_j$ *represents a nonzero class in the space*

$$W_0H^{2n-1}(GL_m(\mathbb{C}), \mathbb{Q}(n)).$$

PROOF. Let us prove that $\sum_j \operatorname{div} f_j = 0$, where $\operatorname{div} f_j$ is the divisor of f_j on $\overline{D}_j$ considered as a codimension n cycle on $GL(V_m)$. Note that $\operatorname{div}\widetilde{f}_j = Z_j^+ - Z_j^-$, where

$$Z_j^+ = \{(L_1^\bullet, L_2^\bullet) \mid \langle L_1^{j+1}, L_2^{m-n-j}\rangle = \langle L_1^j, L_2^{m-n-j}\rangle \text{ and } L_1^j \cap L_2^{m-n-j} = 0\},$$
$$Z_j^- = \{(L_1^\bullet, L_2^\bullet) \mid \langle L_1^{j}, L_2^{m-n-j+1}\rangle = \langle L_1^j, L_2^{m-n-j}\rangle \text{ and } L_1^j \cap L_2^{m-n-j} = 0\}.$$

Therefore it is easy to see that $\sum_j \operatorname{div}\widetilde{f}_j = 0$ and hence $\sum_j \operatorname{div} f_j = 0$. So the current $\sum_j d\log f_j$ represents a class in $W_0H^{2n-1}(GL_m(\mathbb{C}), \mathbb{Q}(n))$. It remains to prove that it is nontrivial.

Let $\mathrm{Gr}(N-m, N)$ be the Grassmannian of codimension m subspaces in V_N. There is a canonical m-dimensional bundle E over it: the fiber over plane h is V_N/h. Let us choose an affine $(m-n+1)$-flag $L^1 \subset \cdots \subset L^{m-n+1}$ in V_N. It determines a Chern cycle $c_m(E; L^\bullet) \subset \mathrm{Gr}(N-m, N)$. Let $\pi: \widetilde{E} \to \mathrm{Gr}(N-m, N)$ be the bundle of frames $(e_1, \dots, e_n)$ in fibers of E. This is a principal GL_m-bundle. Let us construct a cycle $B_m \subset \widetilde{E}$ together with a rational function $g_m \in F(B_m)$ such that

$$\operatorname{div} g_m = \pi^{-1}(c_m(E; L^\bullet)), \tag{4.13}$$

and for generic $h \in \mathrm{Gr}(N-m, N)$ the intersection

$$(B_m, g_m) \cap \pi^{-1}(h) \text{ coincides with } \sum_j (D_j, f_j), \tag{4.14}$$

constructed using the projection of the flag $L^\bullet$ onto V_N/h. (More precisely, a frame $(e_1, \dots, e_m)$ defines an affine $(m-n+1)$-flag $(e_1; \dots; e_{m-n+1})$ and this flag together with the projection of $L^\bullet$ should satisfy 4.12.) Conditions (4.13) and (4.14) just mean that the cohomology class of the current $\sum_j d\log f_j$ is the transgression of the mth Chern class of the universal bundle. Moreover, they give a precise description of the cycle B_m: it is closure of the union of cycles $\sum D_j \subset \bigcup_h \pi^{-1}(h)$ constructed using the projection of $L^\bullet$; here h runs through an open part in $\mathrm{Gr}(N-m, N)$. It is easy to see that for the natural invertible function g_m on B_m (4.13) holds.

§5. Explicit formulas for the universal motivic Chern classes $c_n \in H^{2n}(BGL_{m\bullet}, \mathbb{Q}(n))$ for $n \leqslant 3$

First of all let us recall what the motivic complexes are. So for convenience of the reader I will reproduce in 5.1–5.3 basic definitions and results from [G1, G2].

5.1. Motivic complexes. Let F be an arbitrary field. Denote by $\mathbb{Z}[P^1_F]$ a free abelian group generated by symbols $\{x\}$, where x runs through all F-points of P^1. Let us define subgroups $R_n(F) \subset \mathbb{Z}[P^1_F]$ $(n \leqslant 3)$ as follows:

- $R_1(F)$ is a subgroup generated by $\{xy\} - \{x\} - \{y\}$, where x, y run through all elements of F^*.
- $R_2(F)$ is a subgroup generated by $\sum_{i=0}^4 (-1)^i \{r(x_0, \dots, \widehat{x}_i, \dots, x_4)\}$, where $(x_0, \dots, x_4)$ run through all configurations of five distinct points of P^1_F and $r(x_1, \dots, x_4) := \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$ is the cross-ratio.
- $R_3(F)$ is a subgroup generated by $\sum_{i=0}^6 (-1)^i \{r_3(l_0, \dots, \widehat{l}_i, \dots, l_0)\}$, where $(l_0, \dots, l_6)$ run through all configurations of seven points in P^2_F in general position and $r_3(l_1, \dots, l_6) \in \mathbb{Z}[P^1_F]$ is the *generalized cross-ratio*:

$$r_3(l_1, \dots, l_6) := \mathrm{Alt}\left\{\frac{\Delta(\tilde{l}_1\tilde{l}_2\tilde{l}_4)\cdot\Delta(\tilde{l}_2\tilde{l}_3\tilde{l}_5)\cdot\Delta(\tilde{l}_3\tilde{l}_1\tilde{l}_6)}{\Delta(\tilde{l}_1\tilde{l}_2\tilde{l}_5)\cdot\Delta(\tilde{l}_2\tilde{l}_3\tilde{l}_6)\cdot\Delta(\tilde{l}_3\tilde{l}_1\tilde{l}_4)}\right\}, \tag{5.1}$$

where $\mathrm{Alt}\, f(l_1, \dots, l_6) := \sum_{\sigma \in S_6} (-1)^{|\sigma|} f(l_{\sigma(1)}, \dots, l_{\sigma(6)})$.

Here $\tilde{l}_i$ are vectors in $V^3 \setminus 0$ that project to the points $l_i \in P(V_3)$. The right-hand side of (5.1) does not depend on the volume form in V_3 and on lengths of vectors l_i. So the cross-ratio of six points in P^2_F is well defined. Put

$$B_n(F) := \frac{\mathbb{Z}[P^1_F]}{R_n(F),\ \{0\},\ \{\infty\}}.$$

There is a canonical isomorphism $B_1(F) \xrightarrow{\sim} F^*$ given by the map $\{x\} \mapsto x$; $\{0\}$, $\{\infty\} \mapsto 1$. Let us consider the following complexes $B_F(n)$:

$$\begin{aligned} &B_F(1)\colon F^*,\\ &B_F(2)\colon B_2(F) \xrightarrow{\delta_2} \Lambda^2 F^*,\\ &B_F(3)\colon B_3(F) \xrightarrow{\delta_3} B_2(F)\otimes F^* \xrightarrow{\delta_3} \Lambda^3 F^*. \end{aligned} \tag{5.2}$$

Here

$$\begin{aligned} &\delta_2(\{x\}) = (1-x)\wedge x,\\ &\delta_3(\{x\}) = \{x\}\otimes x; \qquad \delta_3(\{x\}\otimes y) = (1-x)\wedge x\wedge y, \end{aligned} \tag{5.3}$$

and by definition $\delta_n(\{0\}) = \delta_n(\{\infty\}) = 0$, $n = 2, 3$. Note that $\delta_3 \circ \delta_3(\{x\}) = (1-x)\wedge x \wedge x = 0$ modulo 2-torsion, so $B_F(3)$ is a complex.

THEOREM 5.1. $\delta_n(R_n(F)) = 0$.

PROOF. See § 3 in [G2]; see also §5.3 below.

In complexes (5.2) the groups $B_n(F)$ are placed at degree 1 and δ_n has degree $+1$.

The complex $B_F(2)$ is the well-known Bloch-Suslin complex.

5.2. The motivic complexes $\Gamma(X; n)$ for a regular scheme X ($n \leqslant 3$). Let F be a field with a discrete variation v and the residue class field $\overline{F}_v$. Let us construct a canonical homomorphism of complexes

$$\partial_v : B_F(n) \to B_{\overline{F}_v}(n-1)[-1].$$

There is a homomorphism $\theta: \Lambda^n F^* \to \Lambda^{n-1}\overline{F}_v^*$ uniquely defined by the following properties ($u_i \in U$, $u \mapsto \overline{u}$ is the natural homomorphism $U \to \overline{F}_v^*$, and π is a prime, $\operatorname{ord}_v \pi = 1$):

1. $\theta(\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \overline{u}_1 \wedge \cdots \wedge \overline{u}_{n-1}$;
2. $\theta(u_1 \wedge \cdots \wedge u_n) = 0$.

It clearly does not depend on the choice of π.

Let us define a homomorphism $s_v : \mathbb{Z}[P_F^1] \to \mathbb{Z}[P_{\overline{F}_v}^1]$ as follows:

$$s_v\{x\} = \begin{cases} \{\overline{x}\} & \text{if } x \text{ is a unit,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

PROPOSITION 5.2. *Homomorphism* (5.4) *induces a homomorphism*

$$s_v : B(F) \to B_n(\overline{F}_v), \qquad n = 2, 3.$$

PROOF. Straightforward but tedious computations using formula (3.17) from [G2] for generators of the subgroup $R_3(F)$.

To avoid these computations one can consider subgroups $\mathscr{R}_n(F) \subset \mathbb{Z}[P_F^1]$ defined in § 1.4 of [G2]. Then, essentially by definition, $s_v(\mathscr{R}(F)) = \mathscr{R}_n(\overline{F}_v)$ and $\delta(R_n(F)) = 0$. So we get the groups $\mathscr{B}_n(F) := \mathbb{Z}[P_F^1]/\mathscr{R}_n(F)$ together with homomorphisms $s_v : \mathscr{B}_n(F) \to \mathscr{B}_n(\overline{F}_v)$.

Set

$$\partial_v := s_v \otimes \theta : B_k(F) \otimes \Lambda^{n-k} F^* \to B_k(\overline{F}_v) \otimes \Lambda^{n-k-1}\overline{F}_v^*. \tag{5.5}$$

LEMMA 5.3. *The homomorphism ∂_v commutes with the coboundary δ and hence defines a homomorphism of complexes* (5.3).

PROOF. Straightforward computation. See also § 1.14 in [G1], where the corresponding fact is proved for groups $\mathscr{B}_n(F)$.

Now let X be an arbitrary regular scheme, $X_{(i)}$ the set of all codimension i points of X, $F(x)$ the field of functions corresponding to a point $x \in X_{(i)}$. We define the motivic complexes $\Gamma(X, n)$ as the total complexes associated with the following bicomplexes:

$$\Gamma(X, 1): \quad F(X)^* \xrightarrow{\partial_1} \coprod_{x \in X_{(1)}} \mathbb{Z}$$

$$\begin{array}{ccccc} \Gamma(X, 2): \quad \Lambda^2 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x \in X_{(1)}} F(x)^* & \xrightarrow{\partial_2} & \coprod_{x \in X_{(2)}} \mathbb{Z} \\ \uparrow \delta & & & & \\ B_2(F(X)) & & & & \end{array}$$

$$\begin{array}{ccccccc}
\Gamma(X,3): & \Lambda^3 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x\in X_{(1)}} \Lambda^2 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x\in X_{(2)}} F(x)^* \xrightarrow{\partial_2} \coprod_{x\in X_{(3)}} \mathbb{Z} \\
& \uparrow\delta & & \uparrow\delta & & \\
& B_2(F(X))\otimes F(X)^* & \xrightarrow{\partial_1} & B_2(F(X)) & & \\
& \uparrow\delta & & & & \\
& B_3(F(X)) & & & &
\end{array}$$

where $B_n(F(X))$ is placed at degree 1 and coboundaries have degree $+1$.

The coboundaries ∂_i are defined as follows. $\partial_1 := \coprod_{x\in X_{(i)}} \partial_{v_x}$. The others are a little bit more complicated. Let $x \in X_{(k)}$ and $v_1(y), \dots, v_m(y)$ be all discrete valuations of the field $F(x)$ over a point $y \in X_{(k+1)}$, $y \in \overline{x}$. Then $\overline{F(x)}_i := \overline{F(x)}_{v_i(y)} \supset F(y)$. (If $\overline{x}$ is nonsingular at the point y, then $\overline{F(x)}_i = F(y)$ and $m = 1$.) Let us define a homomorphism $\partial_2 : \Lambda^2 F(x) \to F(y)^*$ as the composition

$$\Lambda^2 F(x) \xrightarrow{\oplus\partial_{v_i(y)}} \bigoplus_{i=1}^{m} \overline{F(x)}_i^* \xrightarrow{\oplus N_{\overline{F(x)}_i/F(y)}} F(y)^*, \tag{5.6}$$

and $F(x)^* \xrightarrow{\oplus\partial_{v_i}} \oplus_{i=1}^m \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}$.

5.3. Motivic Chern classes $c_n \in H^{2n}_{\mathcal{M}}(BGL_m(F)_\bullet, \mathbb{Z}(n))$, $n \leqslant 3$. Recall that

$$BG_\bullet := \mathrm{pt} \overset{s_0}{\underset{s_1}{\Longleftarrow}} G \Lleftarrow G^2 \Lleftarrow G^3 \cdots.$$

We must construct a $2n$-cocycle c_n in the bicomplex

$$\Gamma(G;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^n;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^{2n-1};n), \tag{5.7}$$

where $s^* = \sum(-1)^i s_i$. Its components in

$$\Gamma(G;n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^n;n),$$

should be in the following part of the bicomplex:

$$\begin{array}{cccc}
\bigoplus_{x\in G_{(n)}} \mathbb{Z} & & & \\
\uparrow \partial & & & \\
\bigoplus_{x\in G_{(n-1)}} F(x)^* \xrightarrow{s^*} & \bigoplus_{x\in G^2_{(n-1)}} F(x)^* & & \\
& \uparrow \partial & & \\
& \bigoplus_{x\in G^2_{(n-2)}} \Lambda^2 F(x)^* \xrightarrow{s^*} \cdots & & \\
& & \cdots \xrightarrow{s^*} & \bigoplus_{x\in G^n_{(1)}} \Lambda^{n-1} F(x)^* \\
& & & \uparrow \partial \\
& & & \Lambda^n F(G^n)^*.
\end{array} \tag{5.8}$$

In fact the components of c_n in (5.8) were already constructed in § 4. Recall this construction. Let a be an affine $(m-n+1)$-flag in an m-dimensional vector space V^m. For each partition $j_0 + \cdots + j_r = m-n$, irreducible subvarieties

$$D_{j_0,\dots,j_r;a} \in G^r_{(n-r)},$$

together with elements

$$\widetilde{\omega}_{j_0,\dots,j_r;a} \in \Lambda^r F(D_{j_0,\dots,j_r;a})^*, \tag{5.9}$$

were constructed. More precisely, if

$$(L_0^\bullet, \dots, L_r^\bullet) := (a, g_1 a, \dots, g_r a),$$

where $(g_1, \dots, g_r) \in D_{j_0,\dots,j_r;a} \subset G^r$, then

$$\Big(\bigoplus_{p=0}^{r} L_p^{j_p} \mid l_0^{j_0+1}, \dots, l_r^{j_r+1}\Big)$$

is a configuration of $r+1$ vectors in an r-dimensional vector space. Applying to it the homomorphism $f_r(r)\colon C_r(r) \to \Lambda^r F^*$, we get the element (5.9). The collection of elements

$$\begin{aligned}\widetilde{\omega}_r := \sum_{j_0+\cdots+j_r=m-n} \widetilde{\omega}_{j_0,\dots,j_r;a} \in & \bigoplus_{j_0+\cdots+j_2=m-n} \Lambda^r F(D_{j_0,\dots,j_r;a})^* \\ & \in \bigoplus_{x\in G^r_{(n-r)}} \Lambda^r F(x)^* \end{aligned} \tag{5.10}$$

forms a cocycle in the bicomplex (5.8). (The proof of this fact is similar to the proof of Theorem 4.3.) The components of c_n in the bicomplex

$$\Gamma(G^n; n) \xrightarrow{s^*} \Gamma(G^{n+1}; n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \Gamma(G^{2n-1}; n) \tag{5.11}$$

are constructed as follows. There is a homomorphism of complexes (see (2.4), (2.5))

$$T\colon C_*(A^{m-n+1}(m)) \to BC_*(n),$$

where $BC_*(n)$ is the total complex for the Grassmannian bicomplex (1.2).

We will construct morphisms of complexes

$$f(n)\colon BC^*(n) \to B_F(n) \qquad (n \leqslant 3) \tag{5.12}$$

such that for $r \geqslant n+1$ the ∂-coboundaries of elements

$$f(n) \circ T(a, g_1 a, \dots, g_r a) \tag{5.13}$$

are equal to zero. The collection of elements (5.10) and (5.13) form a cocycle c_n in the bicomplex (5.7).

Let us describe the construction of the homomorphism (5.12).

a) $n = 1$. $f_1(1)\colon C_1(1) \to F^*$ is the only homomorphism we need. It is easy to check that both $f_1(1) \circ d'\colon C_2(2) \to F^*$ and $f_1(1) \circ d\colon C_2(1) \to F^*$ are zero homomorphisms, so that we get a homomorphism $f(1)\colon BC^*(1) \to F^*[-1]$.

b) $n = 2$. We must construct a homomorphism from the total complex associated with the bicomplex

$$\begin{array}{ccccc}
 & & \downarrow & & \downarrow \\
\longrightarrow & & C_4(3) & \xrightarrow{d} & C_3(3) \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
\xrightarrow{d} & & C_3(2) & \xrightarrow{d} & C_2(2)
\end{array}$$

to the complex

$$0 \to B_2(F) \to \Lambda^2 F^*.$$

A homomorphism $f_2(2)\colon C_2(2) \to \Lambda^2 F^*$ was defined by formula (3.2). Lemma 3.2 shows that one can take a map from $C_3(3)$ to $B_2(F)$ equal to zero. Let us define a homomorphism

$$f_3(2)\colon C_3(2) \to B_2(F)$$

setting

$$(l_0, \dots, l_3) \mapsto \{r(\bar{l}_0, \dots, \bar{l}_3)\}_2,$$

where $(\bar{l}_0, \dots, \bar{l}_3)$ is a configuration of four points in P^1_F corresponding to the one $(l_0, \dots, l_3)$ of four vectors in V^2. Then $f_3(2) \circ d\colon C_4(2) \to B_2(F)$ is zero by definition of the group $B_2(F)$.

LEMMA 5.4. $f_3(2) \circ d' = 0$.

PROOF. We have to prove that for $(l_0, \dots, l_4) \in C_4(3)$,

$$\sum_{i=0}^{4} (-1)^i \{r(\overline{l}_i \mid \overline{l}_0, \dots, \widehat{\overline{l}_i}, \dots, \overline{l}_4)\}_2 = 0 \quad \text{in } B_2(F). \tag{5.14}$$

There is a conic (a curve of order 2) passing through five points $\overline{l}_0, \dots, \overline{l}_4$ in P_F^2. Let us consider it as a projective line. Then (5.14) is just the 5-term relation for five points $\overline{l}_i$ on *this* projective line.

So we have defined a homomorphism $f(2)\colon BC^*(2) \to B_F(2)$. It is nonzero only on the Grassmannian subcomplex $C^*(2) \subset BC^*(2)$.

c) $n = 3$. We have to define a homomorphism from the total complex associated with the bicomplex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{d} & C_6(4) & \xrightarrow{d} & C_5(4) & \xrightarrow{d} & C_4(4) \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
\cdots & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) & \xrightarrow{d} & C_3(3)
\end{array}$$

to the complex

$$B_3(F) \xrightarrow{\delta_3} B_2(F) \otimes F^* \xrightarrow{\delta_3} \Lambda^3 F^*.$$

A homomorphism $f_3(3)\colon C_3(3) \to \Lambda^3 F^*$ was defined by formula (3.2). Set

$$\begin{aligned} &f_4(3)\colon C_4(3) \to B_2(F) \otimes F^*, \\ &f_4(3)\colon (l_0, \dots, l_4) \mapsto (1/2)\,\mathrm{Alt}(\{r(\overline{l}_0 \mid \overline{l}_1, \dots, \overline{l}_u)\}_2 \otimes \Delta(l_0, l_1, l_2)). \end{aligned} \tag{5.15}$$

PROPOSITION 5.5. $f_4(3)$ *does not depend on the choice of the volume form* $\omega_3 \in \Lambda^3(V^3)^*$ *in the definition of* $\Delta(l_i, l_j, l_k)$.

PROOF. The difference between the right-hand sides of (5.15) computed using $\lambda \cdot \omega_3$ and ω_3 is proportional to (left-hand side of (5.14)) $\otimes\lambda$. So it is zero by Lemma 5.4.

PROPOSITION 5.6. *We have* $f_3(3) \circ d = \delta \circ f_4(3)$.

PROOF. Direct computation using the formula

$$r(\overline{l}_1, \dots, \overline{l}_4) = \frac{\Delta(l_1, l_3) \cdot \Delta(l_2, l_4)}{\Delta(l_1, l_4) \cdot \Delta(l_2, l_3)}.$$

Now set

$$\begin{aligned} &f_5(3)\colon C_5(3) \to B_3(F) \\ &f_5(3)\colon (l_0, \dots, l_5) \mapsto \mathrm{Alt}\left\{\frac{\Delta(l_0, l_1, l_3) \cdot \Delta(l_1, l_2, l_4) \cdot \Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4) \cdot \Delta(l_1, l_2, l_5) \cdot \Delta(l_2, l_0, l_3)}\right\}_3. \end{aligned} \tag{5.16}$$

THEOREM 5.7. *We have* $f_4(3) \circ d = \delta \circ f_5(3)$.

PROOF. See the proof of Theorem 3.10 in [G2].

PROPOSITION 5.8. $f_k(3) \circ d' = 0$ *for* $k = 3, 4, 5$.

PROOF. For $k = 3$ this is Lemma 3.2. For $k = 4, 5$ see Theorem 3.12 in [G2].

PROPOSITION 5.9. $f_5(3) \circ d = 0$ *in* $B_3(F)$.

PROOF. Follows immediately from the definition of the group $B_3(F)$.

So one can define a homomorphism $f(3)\colon BC^*(3) \to B_F(3)$ using homomorphisms $f_k(3)$ on the subcomplex $C^*(3) \subset BC^*(3)$ and zero homomorphisms otherwise.

Now consider an element

$$f_4(3) \circ T(a, g_1 a, \dots, g_4 a) \in B_2(F(G^4)) \otimes F(G^4)^*.$$

Then

$$\partial_1 \circ f_4(3) \circ T(a, g_1 a, \dots, g_4 a) \in \bigoplus_{x \in G^4_{(1)}} B_2(F(x)). \tag{5.17}$$

LEMMA 5.10. *The left-hand side of* (5.17) *is equal to zero.*

PROOF. It follows from the definition (5.5) of ∂_v and the following remark: the term $\Delta(l_0, l_1, l_2)$ appears in formula (5.15) with the coefficient

$$\{r(\bar{l}_3 \mid \bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_4)\}_2 - \{r(\bar{l}_4 \mid \bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_3)\}_2$$

which is obviously zero if $\Delta(l_0, l_1, l_2) = 0$.

So we have proved that the collection of elements (5.10) and (5.13) form a cocycle in the bicomplex (5.7). The cohomology class of this cocycle does not depend on the choice of an affine $(m - n + 1)$-flag a. (Different flags yield canonically cohomologous cocycles.)

5.4. Chern classes in Deligne cohomology. Let us assume that there exists a $2n$-cocycle $\mathbb{L}'_n$ from Conjecture 1.1′. (A precise construction of this cocycle for $n \leqslant 3$ can be found in § 9 of [G1], see also [G2] and 6.3 below.) The main construction of § 2 gives an explicit construction of Chern classes in bi-Grassmannian cohomology and hence, applying $\mathbb{L}'_n$, in real Deligne cohomology. We will see in the next section that these Chern classes coincide with the classical ones (see Theorem 5.11).

5.5. The universal Chern classes in Deligne cohomology. Assuming the existence of $\mathbb{L}'_n$ we construct

$$c_n \in H^{2n}_{D}(BGL_m(\mathbb{C}), \mathbb{R}(n)).$$

The Dolbeaux resolution of the complex associated with the bicomplex (1.13) provides us with a complex computing real Deligne cohomology of an algebraic manifold over $\mathbb{C}$. We will denote this complex by $\mathbb{R}(X, n)$. We have to construct a $2n$-cocycle in the bicomplex

$$\mathbb{R}(G, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \mathbb{R}(G^n, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \mathbb{R}(G^{2n-1}, n) \qquad (5.18)$$

(compare with (4.7)). First of all let us construct its components in

$$\mathbb{R}(G, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \mathbb{R}(G^n, n). \qquad (5.19)$$

For a subvariety $Y \hookrightarrow X$ of codimension d, there is a canonical morphism of complexes $i_*: \mathbb{R}(Y, n) \to \mathbb{R}(X, n+d)[2d]$. In 5.3 we have constructed a chain (5.10) in the bicomplex (5.8) corresponding to an affine $(m-n+1)$-flag a in V_m. Each component of this chain lies in $\Lambda^r F(x)^*$, where x is a codimension $n-r$ point in G^r. There is a canonical map

$$\Lambda^r \mathbb{C}(x)^* \to \mathbb{R}(\operatorname{Spec} \mathbb{C}(x), r),$$

$$f_1 \wedge \cdots \wedge f_r \mapsto \Big(\pi_r \operatorname{Alt}\Big(\sum_{k=0}^{[(n-1)/2]} \frac{1}{(2k+1)!\,(n-2k-1)!} \log|f_1|\, d\log|f_1| \wedge \cdots \wedge d\log|f_{2k+1}| \wedge d\, i \arg f_{2k+2} \wedge \cdots \wedge d\, i \arg f_r, \; d\log f_1 \wedge \cdots \wedge d\log f_r\Big)\Big)$$

commuting with residue homomorphisms. So we get a chain in (5.19).

The components of c_n in the bicomplex

$$\mathbb{R}(G^n, n) \xrightarrow{s^*} \cdots \xrightarrow{s^*} \mathbb{R}(G^{2n-1}, n)$$

are constructed as a composition of the homomorphism of complexes

$$T: C_*(A^{m-n+1}(m)) \to BC_*(n),$$

with the $2n$-cocycle $\mathbb{L}'_n$ that lives on $BC_*(n)$ (or, better, on the bi-Grassmannian $\widehat{\mathrm{G}}(n)$). More precisely, to construct the $\mathbb{R}(G^k, n)$-component of c_n we must restrict homomorphism T to elements $(a, g_1 a, \ldots, g_k a)$, where a is a given affine $(m-n+1)$-flag in V_m.

THEOREM 5.11. a) *The constructed chain* c_n *is a cocycle in* (5.18).

b) *The class of* c_n *equals the usual Chern class in* $H_D^{2n}(BGL_m(\mathbb{C}), \mathbb{R}(n))$.

PROOF. Part a) follows from the definition and previous results.

b) (Compare with the proof of Theorem 5.10 in [G2].) Let $\pi: EG_\bullet \to BG_\bullet$ be the universal G-bundle. Then $EG_{(p)} = BG_{(p+1)}$ and so any i-cochain $c_{(\bullet)}$ for $BG_\bullet$ defines an $(i-1)$-cochain $\widetilde{c}_{(\bullet)}$ for $EG_\bullet$: $\widetilde{c}_{(p)} := c_{(p+1)}$. Moreover, if $c_{(0)} = 0$ and $c_{(\bullet)}$ is a cocycle then $d\widetilde{c}_{(\bullet)} = c_{(\bullet)}$. Therefore $c_{(1)} = \widetilde{c}|_G$ is the transgression of the cocycle $c_{(\bullet)}$.

Applying this result to the cocycle c_n, we get a cocycle c_n' in $H_D^{2n-1}(GL_m(\mathbb{C}), \mathbb{R}(n))$ The usual exact sequence for Deligne cohomology gives us

$$\cdots \longrightarrow H_D^{2n-1}(GL_m(\mathbb{C}), \mathbb{R}(n)) \xrightarrow{\alpha} \\ \longrightarrow H^{2n-1}(GL_m(\mathbb{C}), \mathbb{R}(n)) \cap H^{2n-1}(GL_m(\mathbb{C}), \Omega^{\geqslant n}).$$

It follows from definitions that $\alpha(c_n')$ coincides with the class constructed in 4.4. It is nontrivial according to Theorem 4.6.

This proves that the cocycle c_n coincides with the usual Chern class in $H_D^{2n}(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n))$ modulo decomposable elements. (Decomposable elements go to zero under transgression.)

To prove that c_n coincides with the Chern class exactly, consider the standard long exact sequence for Deligne cohomology:

$$\cdots \to H^{2n-1}(BGL_m(\mathbb{C})_\bullet, \mathbb{C}) \to H_D^{2n}(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n)) \to \\ \xrightarrow{\beta_1+\beta_2} H^{2n}(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n)) \oplus H^{2n}(BGL_m(\mathbb{C})_\bullet, \Omega^{\geqslant n}) \to \cdots.$$

It is known that $H^{2n-1}(BGL_m(\mathbb{C})_\bullet, \mathbb{C}) = 0$. So we get an inclusion

$$H_D^{2n}(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n)) \overset{\beta_1+\beta_2}{\hookrightarrow} H^{2n}(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n)) \oplus H^{2n}(BGL_m(\mathbb{C})_\bullet, \Omega^{\geqslant n}).$$

(In fact both β_1 and β_2 are isomorphisms because $H^{2n}(BGL_m(\mathbb{C}), \mathbb{R}(n))$ is a pure Hodge structure of type $(0, 0)$.) To prove b) it suffices to prove the following

LEMMA 5.11$'$. *The class $\beta_2(c_n)$ coincides with the usual nth Chern class in $H^{2n}(BGL_m(\mathbb{C})_\bullet, \Omega^{\geqslant n})$.*

This lemma will be proved in 5.8 below.

5.6. Explicit formulas for measurable cocycles of $GL(\mathbb{C})$. We assume that there exists a function P_n' on $\widehat{G}_{n-1}^n$ satisfying $(2n-1)$-term relations (1.14). Recall that such a function can be considered as a function on configurations of $2n$ vectors in general position in $\mathbb{C}^n$ satisfying the equations

$$\sum_{i=0}^{2n}(-1)^i P_n'(l_0, \dots, \widehat{l_i}, \dots, l_{2n}) = 0, \tag{5.20a}$$

$$\sum_{i=0}^{2n}(-1)^i P_n'(l_i \mid l_0, \dots, \widehat{l_i}, \dots, l_{2n}) = 0. \tag{5.20b}$$

We assume also that P_n' is a component of a $2n$-cocycle $\mathbb{L}_n'$ from Conjecture 1.1$'$.

THEOREM 5.12. *Let a be an affine $(m-n+1)$-flag in V_m. Then $P_n(T(g_0a, \ldots, g_{2n-1}a))$ is a $2n$-cocycle of $GL_m(\mathbb{C})$. Its cohomology class coincides with the Borel class in $H^{2n-1}_{(m)}(GL_m(\mathbb{C}), \mathbb{R})$ $(m \geqslant n)$.*

Recall that here $T\colon C_*(A^{m-n+1}(n)) \to BC_*(n)$ is a homomorphism of complexes. The cocycle condition follows from this fact and $(2n+1)$-terms equations (5.20).

Let G^δ be a Lie group G made discrete. The morphism of groups $GL_m(\mathbb{C})^\delta \to GL_m(\mathbb{C})$ provides a morphism

$$e\colon BGL_m(\mathbb{C})^\delta_\bullet \to BGL_m(\mathbb{C})_\bullet.$$

Therefore,

$$\begin{aligned} e^*\colon H^{2n}_D(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n)) &\to H^{2n}_D(BGL_m(\mathbb{C})^\delta_\bullet, \mathbb{R}(n)) \\ &= H^{2n-1}(BGL_m(\mathbb{C})_\bullet, S^0) \equiv H^{2n-1}_{(m)}(GL_m(\mathbb{C}), \mathbb{R}(n-1)). \end{aligned}$$

Here S^0 is a sheaf of smooth functions. It is known that e^* maps the indecomposable class in $H^{2n}_D(BGL_m(\mathbb{C}), \mathbb{Z}(n))$ just to the Borel class in $H^{2n-1}_{(m)}(GL_m(\mathbb{C}), \mathbb{R}(n-1))$ (see [B2, DMZ]). The arguments in the proof of Theorem 5.11 show that the constructed class $c_n \in H^{2n}_D(BGL_m(\mathbb{C})_\bullet, \mathbb{R}(n))$ lies in

$$\operatorname{Im} H^{2n}_D(BGL_m(\mathbb{C})_\bullet, \mathbb{Z}(n)) \to H^{2n}_D(BGL_m(\mathbb{C})_\bullet, R(n)),$$

and in fact coincides with the image of the standard class in

$$H^{2n}_D(BGL_m(\mathbb{C})_\bullet, \mathbb{Z}(n)).$$

In our case $e^*(c_n)$ coincides with $P_n(T(g_0a, \ldots, g_{2n-1}a))$ just by definition. Theorem 5.12 is proved.

REMARK 5.13. Explicit formulas for functions P_n are known for $n \leqslant 3$:

$$\begin{aligned} P_2(l_1, \ldots, l_4) &:= \mathscr{L}_2(r(l_1, \ldots, l_4)), \\ P_3(l_1, \ldots, l_6) &:= \mathscr{L}_3(r_3(l_1, \ldots, l_6)). \end{aligned} \tag{5.21}$$

5.7. Chern classes $c^M_n(E) \in H^n(X, \underline{\underline{K}}^M_n)$ constructed in §§3–4 coincide with standard classes. Gersten resolution (4.1) (see § 6 in [So] or [NS]) provides us with an isomorphism

$$i\colon H^n(X, \underline{\underline{K}}^M_n) \otimes \mathbb{Q} \xrightarrow{\sim} CH^n(X) \otimes \mathbb{Q}.$$

Our next goal is to show that $i(c^M_n(E))$ coincides with the usual Chern class $c_n(E) \in CH^n(X)$.

Let us call a vector bundle E "nice" if there is a section $l = (l^1; \ldots; l^p)$ of the bundle of affine p-flags ($p = \dim E - n + 1$) such that the cycle

$$c_n(E; l) := \{x \in X \mid l^1(x) \wedge \cdots \wedge l^p(x) = 0 \text{ but } l^1(x) \wedge \cdots \wedge l^{p-1}(x) \neq 0\}$$

has codimension n. Then it represents the usual Chern class in $CH^n(X)$.

PROPOSITION 5.14. *For a "nice" vector bundle E one has*

$$i(c_n^M(E)) = c_n(E).$$

PROOF. Let $X = \bigcup U_i$ be a Zariski covering such that $E|_{U_i}$ is trivial and there are sections $s_i \in \Gamma(U_i, A^p(E))$ satisfying the condition that for any point x of $U_{i_0\cdots i_n}$ sections $s_{i_0}(x), \dots, s_{i_n}(x)$ are in general position. Then, according to Theorem 3.4,

$$c_n^M(E; s_i) := \overline{f}_n(n) \circ T(s_{i_0}, \dots, s_{i_n}) \in K_n^M(\mathscr{O}_{U_{i_0\cdots i_n}})$$

is a cocycle in the Čech complex $\check{C}(U_\bullet, \underline{\underline{K}}_n^M)$.

Let us consider the bicomplex computing the Čech hypercohomology for this covering with coefficients in the Gersten resolution for $\underline{\underline{K}}_n^M$.

For example in the case $n = 2$ it looks as follows:

$$\begin{array}{ccccc}
\bigoplus_i \coprod_{x\in U_{i(2)}} \mathbb{Z} & & \bigoplus_{i_0<i_1} \coprod_{x\in U_{i_0 i_1(2)}} \mathbb{Z} & & \\
\partial\uparrow & & \partial\uparrow & & \\
\bigoplus_i \coprod_{x\in U_{i(1)}} F(x)^* & \xrightarrow{\delta} & \bigoplus_{i_0<i_1} \coprod_{x\in U_{i_0 i_1(1)}} F(x)^* & \xrightarrow{\delta} & \cdots \\
\partial\uparrow & & \partial\uparrow & & \\
\bigoplus_i K_2(F(U_i)) & \xrightarrow{\delta} & \bigoplus_{i_0<i_1} K_2(F(U_{i_0 i_1})) & \xrightarrow{\delta} & \bigoplus_{i_0<i_1<i_2} K_2(F(U_{i_0 i_1 i_2})) \to \cdots
\end{array}$$

Here $X_{(n)}$ is the set of codimension n irreducible subvarieties in X, δ is Čech coboundary, ∂ and δ have degree $+1$, and the group $\bigoplus_i K_2^M(F(U_i))$ in the bottom left corner are placed in degree $(0, 0)$. Note that $c_n(E; l)$ can be considered as an element in $\coprod_{x\in U_{i(n)}} \mathbb{Z}$.

Let us construct a chain $b = \sum b_{k,n-1-k}$ of degree $n-1$ in the total complex associated with the bicomplex such that $(\partial + \delta)b = c_n(E; l) - c_n^M(E; s_i)$.

To explain the idea let me first construct b in the case $\dim E = n = 2$. Then

$$b_{1,0} := \sum_{i_0<i_1} \overline{f}_2(r)(l, s_{i_0}, s_{i_1}) \in \bigoplus_{i_0<i_1} K_2(F(U_{i_0 i_1})),$$

and

$$b_{1,0} := \sum_{i;x} \frac{l(x)}{s_i(x)} \in \bigoplus_i \coprod_{x\in U_{i(1)}} F(x)^*.$$

In the last formula summation is over all codimension 1 subvarieties of U_i such that l is collinear to s_i at points of this subvariety. In this situation there is a rational function $l/s_i \in F(x)^*$.

In the general case set

$$b_{n-1,0} = \sum_{i_0<\cdots<i_{n-1}} \overline{f}_n(n) \circ T(l, s_{i_0,\dots,s_{i_{n-1}}}) \in \bigoplus_{i_0<\cdots<i_{n-1}} K_n^M(F(U_{i_0\cdots i_{n-1}})),$$

and

$$b_{n-k,k-1} := \sum_{i_0<\cdots<i_{n-k-1}} \overline{f}_{n-k+1}(n-k+1) \circ \widetilde{T}(l(x), \widetilde{s}_{i_0}(x), \dots, \widetilde{s}_{i_{n-k}})$$
$$\in \bigoplus_{i_0<\cdots<i_{n-k}} \coprod_{x\in U_{i_0\cdots i_{n-k}}} K_{n-k+1}^M(F(x)).$$

To explain the meaning of the right-hand side of this formula we need several definitions. Let $(L_0^\bullet, \dots, L_r^\bullet)$ be a configuration of $r+1$ affine p-flags, $J = (j_0, \dots, j_r)$, and $j_0 + \cdots + j_r = p-1$. We say that this configuration is J-weakly degenerate if the sum $L_0^{j_0} + \cdots + L_r^{j_r}$ is direct, the configuration of $r+1$ vectors

$$(L_0^{j_0} + \cdots + L_r^{j_r} \mid l_0^{i_0+1}, \dots, l_r^{j_r+1}) \tag{5.22}$$

generates an r-dimensional vector space, and every r of these vectors generate the same vector space.

In the right-hand side of the formula for $b_{k,n-k-1}$ summation is over all codimension $k-1$ subvarieties of $U_{i_0\cdots i_{n-k}}$ such that the configuration of $(n-k+1)$-flags $(l, s_{i_0}, \dots, s_{i_{n-k}})$ at the generic point of x is J-weakly degenerate for some partition J. In this case $\widetilde{T}(l(x), s_{i_0}(x), \dots, s_{i_{n-k}}(x))$ is the sum over all such partitions J of configurations of vectors (5.22). Then it follows immediately from the proof of Proposition 4.4, that

$$(\partial + \delta)b = c_n(E; l) - c_n^M(E; s_i).$$

Proposition 5.14 is proved.

THEOREM 5.15. *For any vector bundle E over X the characteristic class*

$$c_n^M(E) \in H^n(X, \underline{\underline{K}}_n^M)$$

constructed in §§3–4 *coincides with the usual one, i.e.,*

$$i(c_n^M(E)) = c_n(E).$$

PROOF. We need the following result which is a particular case of Proposition 2.4 from a paper of V. V. Schechtman [Sch].

PROPOSITION 5.16. *Let*

$$\overline{c}_i = \overline{c}_i(E_m) \in H^i(BGL_{m^\bullet}, \underline{\underline{K}}_i), \qquad 1 \leqslant i \leqslant m,$$

be the Chern classes of the canonical m-dimensional vector bundle over $BGL_{m^\bullet}$. Then one has an isomorphism

$$\bigoplus_m H^m(BGL_{m^\bullet}, \underline{\underline{K}}_m) = \mathbb{Z}[\overline{c}_1, \dots, \overline{c}_m].$$

We have constructed classes

$$c_i^M \in H^i(BGL_{m\bullet}, \underline{\underline{K}}_i^M),$$

and proved (Proposition 5.14) that for "nice" vector bundles E over X one has

$$f_E^*(\overline{c}_i) = f_E^*(c_i^M).$$

Here $f_E^*\colon H^i(BGL_{m\bullet}, \underline{\underline{K}}_i) \to H^i(X, \underline{\underline{K}}_i)$ is the homomorphism induced by vector bundle E. This implies Theorem 5.15. Indeed, consider as an example of a "nice" vector bundle, for example, the canonical m-dimensional vector bundle over $\mathrm{Gr}(N-m, N)$, N is big. Then the corresponding homomorphism f_E^* is injective. Theorem 5.15 is proved.

5.8. Proof of Lemma 5.11′. Recall that there is canonical map of sheaves

$$d\log\colon \underline{\underline{K}}_n^M \to \Omega^n_{\log} \hookrightarrow \Omega^n_{cl},$$
$$\{f_1, \dots, f_n\} \mapsto d\log f_1 \wedge \cdots \wedge d\log f_n.$$

It follows from the definitions that

$$\beta_2(c_n(E)) = d\log(c_n^M(E)).$$

This together with Theorem 5.15 implies Lemma 5.11′.

§6. Conjecture

I will formulate a much stronger and more precise version of Conjectures 1.1 and 1.1′. It makes sense for any field F, implies these conjectures when $F = \mathbb{C}$, and explains how (and why) multivalued forms appear in Conjecture 1.1. For this I have to recall the definition of the Hopf algebra $A(F)_\bullet$ [BGSV, BMS] and its analytic counterpart, Aomoto polylogarithms [Ao].

6.1. The motivic Hopf algebra $A(F)_\bullet$. First of all let us reproduce from [BGSV] the definition of groups $A_n(F)$, $n = 0, 1, 2, \dots$. An n-simplex is a family of $n+1$ hyperplanes $L = (L_0, \dots, L_n)$ in P_F^n. An n-simplex is said to be nondegenerate if the hyperplanes are in general position. A face of an n-simplex is any nonempty intersection of hyperplanes from L. A pair of n-simplexes (L, M) is said to be admissible if L and M have no common faces.

Define the group $A_n(F)$ as the group with generators $(L; M)$, where $(L; M)$ runs over all admissible pairs of simplexes, and with the following relations:

(A1) If one of the simplexes L or M is degenerate, then $(L; M) = 0$.

(A2) *Skew symmetry.* For every permutation $\sigma\colon (0, 1, \dots, n) \longrightarrow (0, 1, 2, \dots, n)$ one has

$$(\sigma L; M) = (L; \sigma M) = \mathrm{sgn}(\sigma)(L; M),$$

where $\mathrm{sgn}\,\sigma$ is the parity of σ and $\sigma L = (L_{\sigma(0)}, \dots, L_{\sigma(n)})$.

(A3) *Additivity in* L. For every family of hyperplanes $(L_0, \dots, L_{n+1})$ and any n-simplex M such that all pairs $(\widehat{L^j}; M)$, where $\widehat{L^j} = (L_0, \dots, \widehat{L_j}, \dots, L_{n+1})$, are admissible

$$\sum_{j=0}^{j=n+1} (-1)^j (\widehat{L^j}; M) = 0.$$

Additivity in M. For every family $(M_0, \dots, M_{n+1})$ and any simplex L such that all $(L; \widehat{M^j})$ are admissible

$$\sum_{j=0}^{j=n+1} (-1)^j (L; \widehat{M^j}) = 0.$$

(A4) *Projective invariance.* For any $g \in PGL_{n+1}(F)$ one has $(gL; gM) = (L; M)$.

There is a canonical isomorphism

$$r\colon A_1(F) \longrightarrow F^*, \quad r\colon (L_0, L_1; M_0, M_1) \longrightarrow r(L_0, L_1, M_0, M_1).$$

By definition, $A_0 = \mathbb{Z}$. Set

$$A(F)_\bullet := \bigoplus_{n=0}^{n=\infty} A_n(F).$$

Then there is a multiplication $m\colon A_n \otimes A_m \longrightarrow A_{n+m}$ and a comultiplication $\Delta\colon A_n \longrightarrow \bigoplus_{k+l=n} A_k \otimes A_l$ that make $A(F)_\bullet$ a graded commutative Hopf algebra.

In the case $F = \mathbb{C}$ there is a canonical holomorphic differential form w_L with logarithmic singularities on the hyperplanes L_i. If $z_i = 0$ is the homogeneous equation of L_i, then $w_L = d\log(z_1/z_0) \wedge \cdots \wedge d\log(z_n/z_0)$. Let Δ_M be an n-cycle representing a generator of the group $H_n(P^n_{\mathbb{C}}, \bigcup M_j)$. Then

$$\alpha_n(L; M) := \int_{\Delta_M} w_L$$

is a multivalued analytic function called the Aomoto polylogarithm [Ao]. This integral depends on the choice of Δ_M but does not change under continuous deformation.

Let us denote by $\operatorname{Bar}\widetilde{A}(F)$ the complex that we would get applying the reduced bar construction to this Hopf algebra. Recall that

$$\operatorname{Bar}\widetilde{A}(F)\colon \widetilde{A}_\bullet \xrightarrow{\delta} \widetilde{A}_\bullet \otimes \widetilde{A}_\bullet \xrightarrow{\delta} \widetilde{A}_\bullet \otimes \widetilde{A}_\bullet \otimes \widetilde{A}_\bullet \xrightarrow{\delta} \cdots,$$

where

$$\delta(a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_n}) = \widetilde{\Delta} a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_n} - a_{i_1} \otimes \widetilde{\Delta} a_{i_2} \otimes \cdots \otimes a_{i_n} + \cdots.$$

Here $\widetilde{A} := \bigoplus_{i=1}^{\infty} A_i$, $\widetilde{\Delta}(a) := \Delta(a) - (a \otimes 1 + 1 \otimes a)$ is the reduced coproduct, A placed in degree 1 and the differential has degree $+1$. Let $\mathrm{Bar}_{(n)} \widetilde{A}(F)$ be the degree n part of this complex:

$$\mathrm{Bar}_{(n)} \widetilde{A}(F) \colon A_n \xrightarrow{\delta} \bigoplus_{i_1+i_2=n} A_{i_1} \otimes A_{i_2} \xrightarrow{\delta} \bigoplus_{i_1+i_2+i_3=n} A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \xrightarrow{\delta} \cdots .$$

6.2. Recall that there is a "cohomological" complex $BC^*(n)$ with a differential ∂ of degree $+1$:

$$\cdots \xrightarrow{\partial} BC^0(n) \xrightarrow{\partial} BC^1(n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} BC^n(n).$$

By definition, $BC^i(n) := BC_{2n-i}(n)$ and $BC^i(n)$ is placed at degree i.

CONJECTURE 6.1. *There is a canonical homomorphism of complexes*

$$h_n \colon BC^*(n) \longrightarrow \mathrm{Bar}_{(n)} \widetilde{A}(F)$$

such that the homomorphism

$$h_n(n) \colon BC^n(n) \longrightarrow \bigotimes^n A_1(F) = \bigotimes^n F^*$$

is given by the formula

$$h_n(n)(l_0, \ldots, l_n) := \mathrm{Alt} \bigotimes_{i=1}^{i=n} \Delta(l_0, \ldots, \widehat{l}_i, \ldots, l_n)$$

(*compare with formula* (3.2)).

6.3. Relation with Conjecture 1.1. Let $a \in A_n(\mathbb{C}(X))$ and $\alpha(a)$ be the corresponding Aomoto n-logarithm considered as a function on (an open part of) X.

PROPOSITION 6.2. *Let $\Delta^{n-1,1}(a) = \sum b_i \otimes c_i \in A_{n-1} \otimes A_1$ be the $(n-1, 1)$ component of the coproduct. Then*

$$d\alpha(a) = \sum \alpha(b_i) d \log \alpha(c_i).$$

PROOF. An exercise.

Now let us define a map of complexes

$$\mathcal{D} \colon \mathrm{Bar}_n \widetilde{A}(\mathbb{C}(X)) \longrightarrow \widetilde{\Omega}^\bullet(\mathrm{Spec}\, \mathbb{C}(X)),$$

as follows ($\widetilde{\Omega}^\bullet$ is the De Rham complex of multivalued forms). Let $a_1 \otimes \cdots \otimes a_n \in A_{i_1} \otimes \cdots \otimes A_{i_n}$. Then

$$\mathcal{D}(a_1 \otimes \cdots \otimes a_{i_n}) := \alpha(a_1) d \log \alpha(a_2) \wedge \cdots \wedge d \log \alpha(a_n)$$

if $i_2 = i_3 = \cdots = i_n = 1$ and zero otherwise.

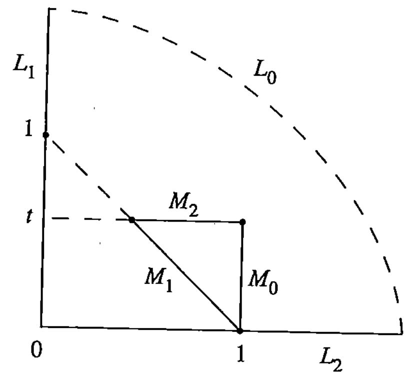


FIGURE 2. $L_2\{t\}$

LEMMA 6.3. *$\mathscr{D}$ is a homomorphism of complexes.*

PROOF. Follows immediately from Proposition 6.2.

The composition $\mathscr{D} \circ h_n$ gives us a multivalued n-logarithm L_n on the bi-Grassmannian satisfying all conditions of Conjecture 1.1.

Let me emphasize that the constructed n-logarithm L_n catches only a small part of the homomorphism h_n, because $\mathscr{D}$ is zero almost everywhere. Further, multivalued functions appear only at the last moment.

I know an explicit construction of homomorphism h_n for $n \leqslant 4$. For $n \leqslant 3$ it is the composition of homomorphism of complexes (5.12)

$$f(n)\colon BC_*(n) \longrightarrow B_F(n),$$

constructed in § 5.3 with canonical homomorphism of complexes

$$l(n)\colon B_F(n) \longrightarrow \mathrm{Bar}_{(n)}\widetilde{A}(F).$$

To give the definition of $l(n)$, let us first define a homomorphism

$$L_n\colon \mathbb{Z}[P_F^1] \longrightarrow A_n(F).$$

Let $x_1, \dots, x_n$ be coordinates in the affine space $P^n \backslash L_0$ such that L_i is the hyperplane $x_i = 0$. Consider the hyperplanes

$$0 = 1 - x_1; \quad 1 - x_1 = x_2; \quad x_2 = x_3; \dots; x_{n-1} = x_n; \quad x_n = t, \quad t \in F^*,$$

by $M_0, M_1, \dots, M_{n-1}, M_n(t)$. Set

$$(L, M(t)) = (L_0, \dots, L_n; M_0, \dots, M_{n-1}, M_n(t)),$$
$$L_n(\{t\}) := (L; M(t)), \qquad L_n(\{0\}) = L_n(\{\infty\}) = 0$$

(see Figure 2). Then $\alpha(L_n(\{t\})) = \mathrm{Li}_n(t)$.

I will also use a notation $z = L_1(\{1-z\})$. Now let us define a homomorphism of complexes $l(2)$ as follows:

$$\begin{array}{ccc} B_2(F) & \longrightarrow & \Lambda^2 F^* \\ \downarrow l_1(2) & & \downarrow l_2(2) \\ A_2 & \xrightarrow{\delta} & A_1 \otimes A_1 \end{array}$$

$$l_2(2): x \otimes y \mapsto (1/2)(x \otimes y - y \otimes x),$$
$$l_1(2): \{x\} \mapsto L_2(\{x\}) - (1/2)\,\mathrm{Li}_1(\{x\}) \cdot x.$$

THEOREM 6.4 [BGSV]. *$l(2)$ is a homomorphism of complexes.*

Let us define a homomorphism of complexes

$$\begin{array}{ccccc} B_3(F) & \longrightarrow & B_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* \\ \downarrow l_1(3) & & \downarrow l_2(3) & & \downarrow l_3(3) \\ A_3 & \xrightarrow{\delta} & A_2 \otimes A_1 \oplus A_1 \otimes A_2 & \xrightarrow{\delta} & \otimes^3 A_1 \end{array}$$

by the following formulas:

$$l_3(3): x_1 \otimes x_2 \otimes x_3 \mapsto (1/6)\,\mathrm{Alt}(x_1 \otimes x_2 \otimes x_3),$$
$$\begin{aligned} l_2(3): \{x\} \otimes y \mapsto\ & (1/2)(l_1(2)(\{x\}) \otimes y - y \otimes l_1(2)(\{x\})) \\ & - (1/12)(L_1(\{x\}) \cdot y \otimes x + x \otimes L_1(\{x\}) \cdot y) \\ & + (1/12)(L_1(\{x\}) \otimes x \cdot y + x \cdot y \otimes L_1(\{x\})), \end{aligned}$$
$$l_1(3): \{x\} \mapsto L_3(\{x\}) - (1/12)L_2(\{x\}) \cdot x + (1/2)L_1(x) \cdot x^2.$$

In these formulas the dot denotes multiplication in the Hopf algebra $A(F)_\bullet$.

THEOREM 6.5. *$l(3)$ is a homomorphism of complexes.*

There are similar formulas for homomorphism $l(n)$ for all n.

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