# POLYLOGARITHMS IN ARITHMETIC AND GEOMETRY

A.B.GONCHAROV

The classical polylogarithms were invented in correspondence of Leibniz with Joh.Bernoulli in 1696 ( [Lei]). They are defined by the series

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad |z| < 1$$

and continued analytically to a covering of $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$:

$$Li_n(z) := \int_0^z Li_{n-1}(t)\frac{dt}{t}, \qquad Li_1(z) = -\log(1-z)$$

**1. The dilogarithm**. It was studied by Spence, Abel, Kummer, Lobachevsky, ..., Rogers,Ramanujan, ([L]). The main discovery was that the dilogarithm satisfies many functional equations. For example Rogers' version of the dilogarithm $L_2(x) := Li_2(x) + \frac{1}{2}\log(x)\log(1-x) - \frac{\pi^2}{6}$ for $1 > x > y > 0$ satisfies the relation

$$L_2(x) - L_2(y) + L_2(y/x) - L_2(\frac{1-x^{-1}}{1-y^{-1}}) + L_2(\frac{1-x}{1-y}) = 0 \tag{1}$$

After a century of neglect the dilogarithm appeared twenty years ago in works of Gabrielov-Gelfand-Losik [GGL] on a combinatorial formula for the first Pontryagin class, Bloch on K-theory and regulators [Bl1] and Wigner on Lie groups.

The dilogarithm has a single-valued cousin : the Bloch - Wigner function

$$\mathcal{L}_2(z) := \text{Im}Li_2(z) + \arg(1-z)\log|z|.$$

Let $r(x_1, ..., x_4)$ be the cross-ratio of 4 distinct points on $\mathbb{C}P^1$. Then

$$\sum_{i=0}^{4}(-1)^i \mathcal{L}_2(r(z_0, ..., \hat{z}_i, ..., z_4)) = 0 \qquad z_i \in \mathbb{C}P^1 \tag{2}$$

If $(z_1, ..., z_5) = (\infty, 0, 1, x, y)$ the arguments here are the same as in (1).

Choose $x \in \mathbb{C}P^1$. Then (2) just means that the function $c_3(g_0, ..., g_3) := \mathcal{L}_2(r(g_0x, ..., g_3x))$, where $g_i \in GL_2(\mathbb{C})$ and $g_ix \neq g_jx$, is a *measurable* 3-cocycle on $GL_2(\mathbb{C})$. (Wigner).

The function $\log|x|$ is characterized by its functional equation $\log|xy| = \log|x| + \log|y|$. The 5-term equation plays a similar role for the dilogarithm: any measurable function $f(z), z \in \mathbb{C}$ satisfying (2) is proportional to $\mathcal{L}_2(z)$ ([Bl1]). Moreover, any functional equation for $\mathcal{L}_2(z)$ is a formal consequence of (2).

---

The author gratefully acknowledges travel grant from ICM-94.

For a set $X$ denote by $\mathbb{Z}[X]$ the free abelian group generated by symbols $\{x\}$, $x \in X$. Let $F$ be a field. Consider the homomorphism ([Bl1])

$$\tilde{\delta}_2 : \mathbb{Z}[F^*\backslash 1] \longrightarrow \Lambda^2 F^*, \quad \{x\} \longmapsto (1-x)\wedge x$$

By Matsumoto theorem Coker$\tilde{\delta}_2 = K_2(F)$.

Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^*\backslash 1]$ generated by the elements $\sum(-1)^i\{r(z_0,...,\hat{z}_i,...,z_4)\}$ where $z_i \neq z_j \in P^1_F$. One can check that $\tilde{\delta}_2(R_2(F)) = 0$. So setting $B_2(F) := \mathbb{Z}[F^*\backslash 1]/R_2(F)$ we get the Bloch complex ( [DS], [S1])

$$B_2(F) \stackrel{\delta_2}{\longrightarrow} \Lambda^2 F^*, \quad \{x\} \mapsto (1-x)\wedge x \tag{3}$$

For an abelian group $A$ put $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. Suslin proved that Ker$\delta_2 \otimes \mathbb{Q} = K_3^{ind}(F)_{\mathbb{Q}}$ ([S1]). Here $K_3^{ind}(F)$ is the cokernel of the multiplication $K_1(F)^{\otimes 3} \rightarrow K_3(F)$.

If $F = \mathbb{C}$ any real-valued function $f(z)$ defines a homomorphism $\tilde{f} : \mathbb{Z}[\mathbb{C}] \longrightarrow \mathbb{R}, \quad \{z\} \longmapsto f(z)$. Thanks to (2) we have a homomorphism $\tilde{\mathcal{L}}_2 : B_2(\mathbb{C}) \rightarrow \mathbb{R}$. Combined with the above homomorphism $K_3(\mathbb{C}) \rightarrow$ Ker$\delta_2$ it gives an explicit formula for the Borel regulator $K_3(\mathbb{C}) \rightarrow \mathbb{R}$ and hence ([Bo2]) a formula for $\zeta_F(2)$ for any number field $F$ (see s.5 below).

Let $\mathcal{H}^3$ be the 3-dimensional hyperbolic space. Denote by $I(z_0,...,z_3)$ the ideal geodesic symplex with vertices at points $z_0,...,z_3$ of the absolute $\partial\mathcal{H}^3 = \mathbb{C}P^1$. Then $volI(z_0,...,z_3) = 3/2\mathcal{L}_2(r(z_0,...,z_3))$ (Lobachevsky).

Any complete hyperbolic 3-fold of finite volume $V^3$ can be represented as a formal sum of ideal geodesic simplices. So $volV^3 = 3/2\sum\mathcal{L}_2(x_i)$. It turns out the condition "$V^3$ is a manifold" implies $\delta_2\sum\{x_i\} = 0$. (Thurston, [DS], [NZ]).

At first glance many features of this picture seem special for the dilogarithm. For example the classical n-logarithms are functions of just 1 variable, but for $n > 2$ $GL_n$ does not act on $P^1$ , $\partial\mathcal{H}^n$ is no longer a complex manifold and so on. In this lecture I will explain how most of these facts about the dilogarithm are generalized to the trilogarithm and outline what should happen in general.

**2. The trilogarithm and $\zeta_F(3)$ ([G2]).** A single-valued version of $Li_3(z)$ is

$$\mathcal{L}_3(z) := Re\Big(Li_3(z) - Li_2(z)\cdot\log|z| + \frac{1}{3}Li_1(z)\cdot\log^2|z|\Big)$$

Denote by $\{x\}_2$ the image of $\{x\}$ in $B_2(F)$. Set

$$\mathbb{Z}[F^*] \stackrel{\delta_3}{\longrightarrow} B_2(F)\otimes F^*, \qquad \delta_3 : \{x\} \mapsto \{x\}_2 \otimes x, \quad \{1\} \mapsto 0 \tag{4}$$

Let $F$ be a number field with $r_1$ real and $r_2$ complex places, $\{\sigma_j\}$ be the set of all possible embeddings $F \hookrightarrow \mathbb{C}$ numbered so that $\sigma_{r_1+k} = \overline{\sigma_{r_1+r_2+k}}$ and $d_F$ be the discriminant of $F$. For $x \in \mathbb{Z}[F^*]$ one get numbers $\tilde{\mathcal{L}}_3(\sigma_j(x))$ defined via the composition $\mathbb{Z}[F^*] \stackrel{\sigma_j}{\hookrightarrow} \mathbb{Z}[\mathbb{C}^*] \stackrel{\tilde{\mathcal{L}}_3}{\longrightarrow} \mathbb{R}$.

**Theorem 0.1.** *For any number field $F$ there exist elements $y_1, \ldots y_{r_1+r_2} \in$ Ker$\delta_3 \otimes \mathbb{Q} \subset \mathbb{Q}[F^*]$ such that*

$$\zeta_F(3) = \pi^{3r_2} d_F^{-\frac{1}{2}} \det|\tilde{\mathcal{L}}_3(\sigma_j(y_i))| \,, \quad (1 \le i,j \le r_1+r_2)\,. \tag{5}$$

It was conjectured by Zagier, who gave many numerical examples ([Z1]). Here is one them:

$$\zeta_{\mathbb{Q}(\sqrt{5})}(3) = \frac{24}{25\sqrt{5}} \cdot \mathcal{L}_3(1) \cdot \Big(\mathcal{L}_3(\frac{1+\sqrt{5}}{2}) - \mathcal{L}_3(\frac{1-\sqrt{5}}{2})\Big)$$

If $\alpha := \frac{1+\sqrt{5}}{2}$ and $\bar{\alpha} := \frac{1-\sqrt{5}}{2}$ then $\alpha \cdot \bar{\alpha} = -1, \alpha + \bar{\alpha} = 1$, so $\{\alpha\}_2 \otimes \alpha - \{\bar{\alpha}\}_2 \otimes \bar{\alpha} = (\{\alpha\}_2 + \{1-\alpha\}_2) \otimes \alpha = 0$ modulo torsion because $6 \cdot (\{x\}_2 + \{1-x\}_2) \in R_2(F)$.

Let $\Delta : G \to G \times G$ be the diagonal map. An element $x \in H_n(G)$ is called primitive if $\Delta_*(x) = x \otimes 1 + 1 \otimes x$. For any field $F$ one can define $K_n(F)_{\mathbb{Q}}$ as the subspace of primitive elements in $H_n(GL(F), \mathbb{Q})$.

Let $H_c^*(G, \mathbb{R})$ be continous cohomology of a Lie group $G$. It is known that $H_c^*(GL(\mathbb{C}), \mathbb{R}) = \Lambda_{\mathbb{R}}^*(c_1, c_3, ...)$ where $c_{2n-1} \in H_c^{2n-1}(GL(\mathbb{C}), \mathbb{R})$ are certain classes.. For example $c_1(g_1, g_2) = \log|detg_1^{-1}g_2|$. Considered as a functional on homology $c_{2n-1}$ induces a map $r_{\mathbb{C}}(n) : K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \to \mathbb{R}$. It is called the Borel regulator [Bo]. Let $F$ be a number field. Then the image of the composition

$$r(n) : K_{2n-1}(F) \longrightarrow \oplus_{Hom(F,\mathbb{C})} K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \overset{r_{\mathbb{C}}(n)\otimes\mathbb{R}(n-1)}{\longrightarrow} \mathbb{Z}^{Hom(F,\mathbb{C})} \otimes \mathbb{R}(n-1)$$

is invariant under the complex conjugation. So we get a regulator map $r(n) : K_{2n-1}(F) \longrightarrow \mathbb{R}(n-1)^{d_n}$. Here $d_n = r_1 + r_2$ for odd $n$ and $r_2$ for even. We will use notation $a \sim b$ if $a/b \in \mathbb{Q}^*$. Borel proved that $r(n)(K_{2n-1}(F))$ is a lattice with covolume $\sim d_F^{1/2} \zeta_F(n)(\pi i)^{-nd_{n-1}}$.

The proof of our theorem is based on an explicit description of the regulator $K_5(\mathbb{C}) \to \mathbb{R}$ by means of the trilogarithm $\mathcal{L}_3$. The key step is a formula for a measurable 5-cocycle of $GL(\mathbb{C})$ representing the class $c_5$. For $GL_3(\mathbb{C})$ it looks as follows.

Let $V^3$ be a 3-dimensional vector space over $F$. Choose a volume form $\omega \in \wedge^3(V^3)^*$. For 6 vectors $l_1, \dots, l_6$ in generic position in $V^3$ set $\Delta(l_i, l_j, l_k) := \langle \omega, l_i \wedge l_j \wedge l_k \rangle \in F^*$. Let $\mathrm{Alt}_6 f(l_1, \dots, l_6) := \sum_{\sigma \in S_6} (-1)^{|\sigma|} f(l_{\sigma(1)}, \dots, f_{\sigma(6)})$. Set

$$r_3(l_1, \dots, l_6) := \mathrm{Alt}_6 \left\{ \frac{\Delta(l_1,l_2,l_4)\Delta(l_2,l_3,l_5)\Delta(l_3,l_1,l_6)}{\Delta(l_1,l_2,l_5)\Delta(l_2,l_3,l_6)\Delta(l_3,l_1,l_4)} \right\} \in \mathbb{Z}[F^*] \qquad (6)$$

$r_3(l_1, \dots, l_6)$ clearly does not depend on the lengths of vectors $l_i$ and so is a generalized cross-ratio of 6 points on the projective plane.

**Theorem 0.2.** *a) For any 7 points $(m_1, \dots, m_7)$ in generic position in $\mathbb{C}P^2$*

$$\sum_{i=1}^{7} (-1)^i \tilde{\mathcal{L}}_3(r_3(m_1, ..., \hat{m}_i, ..., m_7)) = 0$$

*b) Choose $x \in \mathbb{C}P^2$. Then the function $c_5(g_0, ..., g_5) := \tilde{\mathcal{L}}_3(r_3(g_0x, ..., g_5x))$ defined for $g_i \in GL_3(\mathbb{C})$ such that $(g_0x, ..., g_5x)$ is in general position, is a measurable 5-cocycle representing a nontrivial cohomology class of the group $GL_3(\mathbb{C})$.*

**3. Trilogarithm and algebraic K-theory**. Let $R_3(F)$ be the subgroup of $\mathbb{Z}[F^*]$ generated by $\{x\} + \{x^{-1}\}$ and $\sum_{i=1}^{7} (-1)^i r_3(m_1, ..., \hat{m}_i, ..., m_7)$ where $(m_1, ..., m_7)$ run through all generic configurations of 7 points in $P_F^2$. Then $\delta_3 R_3(F) =$

0. Let $B_3(F)$ be the quotient of $\mathbb{Z}[F^*]$ by $R_3(F)$. We get a complex $B_F(3)$

$$B_3(F) \xrightarrow{\delta_3} B_2(F) \otimes F^* \xrightarrow{\delta_2 \wedge id} \Lambda^3 F^*$$

placed in degrees [1,3]. ($\delta_3$,$\delta_2$ were defined in (3), (4)).

According to [S2] $H_n(GL_n(F), \mathbb{Q}) = H_n(GL(F), \mathbb{Q})$. Let

$$K_n^{(i)}(F)_{\mathbb{Q}} := K_n(F)_{\mathbb{Q}} \cap Im\Big(H_n(GL_{n-i}(F), \mathbb{Q}) \to H_n(GL_n(F), \mathbb{Q})\Big)$$

be the rank filtration. Denote by $K_n^{[i]}(F)_{\mathbb{Q}}$ its graded quotients.

**Theorem 0.3.** *There are canonical maps* $K_{6-i}^{[3-i]}(F)_{\mathbb{Q}} \longrightarrow H^i(B_F(3) \otimes \mathbb{Q})$

They should be isomorphisms. This is known for $i = 3$ ([S2]).

**4. Classical polylogarithms and motivic complexes**. The following single-valued version of $Li_n(z)$ was invented by Zagier [Z1], see also [BD].

$$\mathcal{L}_n(z) \quad := \quad \begin{matrix} \text{Re} & (n: \text{ odd}) \\ \text{Im} & (n: \text{ even}) \end{matrix} \left( \sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right) , \quad n \geq 2$$

It is continuos on $\mathbb{C}P^1$. Here $\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} \beta_k x^k$.

Let us define inductively for each $n \geq 1$ a subgroup $\mathcal{R}_n(F) \subset \mathbb{Z}[P^1_F]$, which for $F = \mathbb{C}$ will be the subgroup of *all* functional equations for $\mathcal{L}_n(z)$.

Put $\mathcal{B}_n(F) := \mathbb{Z}[P^1_F]/\mathcal{R}_n(F)$. Set $\mathcal{R}_1(F) := (\{x\}+\{y\}-\{xy\}; \{0\}; \{\infty\})$.Then $\mathcal{B}_1(F) = F^*$. Let $\{x\}_n$ be the image of $\{x\}$ in $\mathcal{B}_n(F)$. Consider homomorphisms

$$\mathbb{Z}[P^1_F] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : \quad n \geq 3 \\ \Lambda^2 F^* & : \quad n = 2 \end{cases} \tag{7}$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1} \otimes x & : \quad n \geq 3 \\ (1-x) \wedge x & : \quad n = 2 \end{cases} \qquad \delta_n : \{\infty\}, \{0\}, \{1\} \mapsto 0 \tag{8}$$

Set $\mathcal{A}_n(F) := \text{Ker } \delta_n$ . Any element $\alpha(t) = \Sigma n_i \{f_i(t)\} \in \mathbb{Z}[P^1_{F(t)}]$ has a specialization $\alpha(t_0) := \Sigma n_i \{f_i(t_0)\} \in \mathbb{Z}[P^1_F]$ at each point $t_0 \in P^1_F$.

**Definition 0.4.** $\mathcal{R}_n(F)$ *is generated by elements* $\{\infty\}, \{0\}$ *and* $\alpha(0) - \alpha(1)$ *where* $\alpha(t)$ *runs through all elements of* $\mathcal{A}_n(F(t))$.

One can show that $\delta_n \mathcal{R}_n(F) = 0$ ([G1], 1.16). So we get homomorphisms

$$\delta_n : \mathcal{B}_n(F) \longrightarrow \mathcal{B}_{n-1}(F) \otimes F^*, \quad n \geq 3; \quad \delta_2 : \mathcal{B}_2(F) \longrightarrow \Lambda^2 F^*$$

and finally the following complex $\Gamma(F, n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \Lambda^2 F^* \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathcal{B}_2 \otimes \Lambda^{n-2} F^* \xrightarrow{\delta} \Lambda^n F^*$$

where $\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \to \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i$ has degree +1 and $\mathcal{B}_n \equiv \mathcal{B}_n(F)$ placed in degree 1. One can prove that $\tilde{\mathcal{L}}_n\Big(\mathcal{R}_n(\mathbb{C})\Big)$ (see [G2] theorem 1.13).

**Conjecture 0.5.** *Let* $f(z)$ *be a measurable function such that* $\tilde{f}(\mathcal{R}_n(\mathbb{C})) = 0$. *Then* $f(z) = \lambda_0 \mathcal{L}_n(z) + \lambda_1 \mathcal{L}_{n-1}(z) \log |z| + ... + \lambda_{n-2} \mathcal{L}_2(z) \log |z|^{n-2}$, $\lambda_i \in \mathbb{C}$.

This is true for $n=2$ ([Bl]) and $n=3$ (unpublished).

Let $\gamma$ be the Adams filtration on $K_n(F)_{\mathbb{Q}}$. Hypothetically it is opposite to the rank filtration. For number fields $gr^{\gamma}_n K_m(F)_{\mathbb{Q}} = 0$ if $m \neq 2n-1$.

**Conjecture A** *a)For any field F* $H^i\Gamma(F,n)\otimes\mathbb{Q} = gr^{\gamma}_n K_{2n-i}(F)\otimes\mathbb{Q}$.

*b) The composition* $gr^{\gamma}_n K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \to H^1\Gamma(\mathbb{C},n)_{\mathbb{Q}} \to \mathbb{R}$ *is a nonzero rational multiple of the Borel regulator.*

For number fields the isomorphism $K_{2n-1}(F)_{\mathbb{Q}} = Ker_{\delta_n}$ was conjectured ( slightly differently, without the complexes $\Gamma(F,n)$) by Zagier [[Z1]).

So we get a hypothetical description of Quillen's K-groups by symbols that generalizes Milnor's approach to K-theory ($H^n\Gamma(F,n) = K^M(F)$ by definition):

$$K_m(F)_{\mathbb{Q}} \stackrel{?}{=} \oplus_n H^{2n-m}(\Gamma(F,n)\otimes\mathbb{Q}) \tag{9}$$

This suggests that $\Gamma(F,n)\otimes\mathbb{Q}$ should be the weight $n$ motivic complex conjectured by Beilinson and Lichtenbaum ([B1], [Li]). Another approach see in [Bl2].

For a compact smooth $i$-dimensional variety $X$ over $\mathbb{Q}$ Beilinson defined a regulator map to Deligne cohomology ([B2]) $r_{Be} : gr^{\gamma}_n K_{2n-i}(X) \longrightarrow H^i_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(n))$

A regular model $X_{\mathbb{Z}}$ of $X$ over $\mathbb{Z}$ defines a subgroup $gr^{\gamma}_n K_{2n-i}(X_{\mathbb{Z}}) \subset gr^{\gamma}_n K_{2n-i}(X)$. For $n > i+1$ they should coincide. Beilinson conjectured that $r_{Be}(gr^{\gamma}_n K_{2n-i}(X_{\mathbb{Z}}))$ is a lattice whose covolume with respect to the natural $\mathbb{Q}$- structure provided by $H^i_{\mathcal{D}}(X/\mathbb{R},\mathbb{Q}(n))$ up to a standard factor coincides with the value of $L$-function $L(h^i(X),s)$ at $s=i$. Unfortunately the definition of the regulator is rather implicit.

Conjecture A together with Beilinson's conjecture should give explicit formulas for special values of the $L$-functions of varieties over number fields in terms of *classical* polylogarithms. Below two examples are discussed: $\zeta$-functions of number fields and L-functions of elliptic curves.

**5. Zagier's conjecture**. Conjecture A b) and Borel's theorem [Bo2] lead to

**Conjecture 0.6.** *For any number field F there exists elements* $y_1,\dots y_{d_n} \in \mathrm{Ker}\delta_n\otimes\mathbb{Q} \subset \mathcal{B}_n(F)_{\mathbb{Q}}$ *such that*

$$\zeta_F(n) = \pi^{d_{n-1}\cdot n} d_F^{-\frac{1}{2}} \det|\tilde{\mathcal{L}}_n(\sigma_j(y_i))| \ , \quad (1\le i,j\le d_n)\ , \tag{10}$$

It was stated in [Z1]. The case $n=2$, essentially proved in [Z2], follows from the Borel theorem and the results of Bloch [Bl1] and Suslin [S2] (see s.1); a simpler proof see in s.2 of [G1]. It is not proved for $n>3$.

**Theorem 0.7.** *For any number field F there is a map* $l_n : Ker\delta_n\otimes\mathbb{Q} \to K_{2n-1}(F)_{\mathbb{Q}}$ *such that for any* $\sigma : F \hookrightarrow \mathbb{C}$ *one has* $r_{\mathbb{C}}(n)(\sigma\circ l_n(y)) = \tilde{\mathcal{L}}_n(\sigma(y))$.

This was proved by Beilinson-Deligne [BD] and later de Jeu [J]. It can be deduced from the existence of the triangulated category of mixed Tate motives over $Spec(F)$ constructed by Levine [L] and Voevodsky [V].

**6. Motivic complexes for curves**. Let $K$ be a field with a discrete valuation $v$, the residue field $k_v$ and the group of units $U$. Let $u \to \bar{u}$ be the projection $U \to k_v^*$. Choose a uniformizer $\pi$. There is a homomorphism $\theta : \Lambda^n F^* \longrightarrow \Lambda^{n-1}F_v^*$ uniquely defined by the following properties ($u_i \in U$):

$$\theta\,(\pi\wedge u_1\wedge\cdots\wedge u_{n-1}) = \bar{u}_1\wedge\cdots\wedge\bar{u}_{n-1}; \qquad \theta\,(u_1\wedge\cdots\wedge u_n) = 0$$

It is clearly independent of $\pi$. Let us define a homomorphism $s_v : \mathbb{Z}[P^1_K] \longrightarrow \mathbb{Z}[P^1_{k_v}]$ setting $s_v\{x\} = \{\bar{x}\}$ if $x$ is a unit and 0 otherwise. It induces a homomorphism $s_v : \mathcal{B}_m(K) \longrightarrow \mathcal{B}_m(k_v)$. Put

$$\partial_v := s_v \otimes \theta : \mathcal{B}_m(K) \otimes \Lambda^{n-m} K^* \longrightarrow \mathcal{B}_m(k_v) \otimes \Lambda^{n-m-1} k_v^* \ .$$

It defines a morphism of complexes $\partial_v : \Gamma(K, n) \longrightarrow \Gamma(k_v, n-1)[-1]$. Let $X$ be a regular curve over a field $F$ and $F_x$ be the residue field of a point $x \in X$. Let us define the motivic complex $\Gamma(X, n)$ as follows ($\mathcal{B}_n(F(X))$ is in degree 1):

$$\begin{array}{ccccccc} \mathcal{B}_n(F(X)) & \stackrel{\delta}{\longrightarrow} & \mathcal{B}_{n-1}(F(X)) \otimes F(X)^* & \stackrel{\delta}{\longrightarrow} & \dots & \stackrel{\delta}{\longrightarrow} & \Lambda^n F(X)^* \\ & & \downarrow \coprod_x \partial_x & & & & \downarrow \coprod_x \partial_x \\ & & \coprod_{x \in X^1} \mathcal{B}_{n-1}(F_x) & \stackrel{\delta}{\longrightarrow} & \dots & \stackrel{\delta}{\longrightarrow} & \coprod_{x \in X^1} \Lambda^{n-1} F_x^* \end{array} \qquad (11)$$

**Conjecture 0.8.** *For a regular curve $X$ one has $H^i(\Gamma(X, n) \otimes \mathbb{Q}) = gr^\gamma_n K_{2n-i}(X)_{\mathbb{Q}}$.*

**7. Explicit formulas for regulators in the case of curves ([G6])**. Let me recall that for a curve $X$ over $\mathbb{R}$ and $n > 1$ $H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) = H^2(X, \mathbb{R}(n-1))^+$ where "+" means invariants of the complex conjugation acting both on $X(\mathbb{C})$ and $\mathbb{R}(n-1)$. Beilinson's regulator for curves over $\mathbb{Q}$ is a homomorphism

$$r_{Be}(n) : K_{2n-2}(X)_{\mathbb{Q}} \longrightarrow H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$$

Cup product with $\omega \in \Omega^1(\bar{X})$ identifies $H^1(\bar{X}, \mathbb{R}(n-1))$ with $H^0(\bar{X}, \Omega^1)^\vee$. So we will view elements of $H^2_{\mathcal{D}}(\bar{X}/\mathbb{R}, \mathbb{R}(n))$ as functionals on $H^0(\bar{X}, \Omega^1)^\vee$.

Set $\alpha(f, g) := \log|f| d\log|g| - \log|g| d\log|f|$.

**Theorem 0.9.** *Let $X$ be a curve over $\mathbb{Q}$. Then for each element $\gamma_{2n-2} \in K_{2n-2}(X)$, $n = 3, 4$, there are rational functions $f_i, g_i \in \mathbb{Q}(X)^*$ such that $\sum_i \{f_i\}_{n-1} \otimes g_i$ is a 2-cocycle in (11) and for any $\omega \in \Omega^1(X)$ one has ($a_n, b_n \in \mathbb{Q}^*$):*

$$\int_{X(\mathbb{C})} r_{Be}(n)(\gamma_{2n-2}) \wedge \omega = a_n \cdot \sum_i \int_{X(\mathbb{C})} \mathcal{L}_{n-1}(f_i) d\log|g_i| \wedge \omega = \qquad (12)$$

$$b_n \cdot \sum_i \int_{X(\mathbb{C})} \log|g_i| \log^{n-3}|f_i| \alpha(1-f_i, f_i) \wedge \omega$$

For $n = 2$ this is the famous symbole modéré of Beilinson and Deligne. Hypothetically (12) should be true for all $n$.

**Example**. For $n = 3$ the condition "$\sum_i \{f_i\}_2 \otimes g_i$ is a 2-cocycle in (11)" means that $\sum_i (1-f_i) \wedge f_i \wedge f_i = 0$ in $\Lambda^3 \mathbb{Q}(X)^*$ and $\sum_i v_x(g_i)\{f_i(x)\}_2 = 0$ in $\mathcal{B}_2(\bar{\mathbb{Q}})$ for any $x \in X(\bar{\mathbb{Q}})$. Here $v_x$ is the valuation defined by a point $x$.

**8. Special values of $L$-functions of elliptic curves ([G6])**. Let $E$ be an elliptic curve $/\mathbb{Q}$ and $\Gamma := H_1(E(\mathbb{C}), \mathbb{Z})$. A holomorphic 1-form $\omega$ defines an embedding $\Gamma \hookrightarrow \mathbb{C}$ together with an isomorphism $E(\mathbb{C}) = \mathbb{C}/\Gamma = \Gamma \otimes \mathbb{R}/\Gamma$. The intersection pairing $\Gamma \times \Gamma \to \mathbb{Z}(1)$ provides a pairing $(\cdot, \cdot) : E(\mathbb{C}) \times \Gamma \longrightarrow U(1) \subset \mathbb{C}^*$. If $\Gamma = \mathbb{Z}u + \mathbb{Z}v \subset \mathbb{C}$ with $Im(u/v) > 0$ then $(z, \gamma) = \exp A(\Gamma)^{-1}(z\bar{\gamma} - \bar{z}\gamma)$ where $A(\Gamma) = \frac{1}{2\pi i}(\bar{u}v - u\bar{v})$. Consider the generalized Eisenstein-Kronecker series ($\gamma_i \in \Gamma$)

$$K_n(x, y, z) := \sum_{\gamma_1 + \ldots + \gamma_n = 0}{}' \frac{(x, \gamma_1)(y, \gamma_2 + \ldots + \gamma_{n-1})(z, \gamma_n)(\bar{\gamma}_n - \bar{\gamma}_{n-1})}{|\gamma_1|^2 |\gamma_2|^2 \ldots |\gamma_n|^2}, \quad n \geq 3$$

They are invariant under the shift $(x,y,z) \to (x+t, y+t, z+t)$ and so live actually on $E(\mathbb{C}) \times E(\mathbb{C})$. For n = 2 put $K_2(x,y,z) := \sum'_\gamma \frac{(x-z,\gamma)}{|\gamma|^2\gamma}$.

Let $\omega \in H^0(E, \Omega^1_{E/\mathbb{Q}})$ and $\Omega = \int_{E(\mathbb{R})} \omega$ be the real period of $E$.

**Conjecture 0.10.** *a) Let $E$ be an elliptic curve / $\mathbb{Q}$ and $n \geq 3$. Then there exist functions $f_i, g_i \in \mathbb{Q}(E)^*$ such that $\sum_i \{f_i\}_{n-1} \otimes g_i$ is a 2-cocycle in (11) and*

$$q \cdot L(E,n) = \Big(\frac{2\pi A(\Gamma)}{f_E}\Big)^{n-1} \Omega \cdot \sum_i K_n(x_i, y_i, z_i) \tag{13}$$

*where $x_i, y_i, z_i$ are divisors of $f_i, g_i, 1-f_i$ and $q \in \mathbb{Q}^*$.*

*b) For any $f_i, g_i \in \mathbb{Q}(E)^*$ as above formula (13) holds with (possibly 0) $q \in \mathbb{Q}$.*

For n=2 (13) is Bloch's conjecture [Bl1] and for n=3 it was conjectured (slightly differently, using Massey products) by Deninger [Den]. A conjecture for any elliptic curve over a number field involves determinants with entries $K_n(x,y,z)$.

**Theorem 0.11.** *Conjecture 0.10 holds for modular elliptic curves over $\mathbb{Q}$ in the cases $n = 3$ and $n = 4$.*

The proof uses theorem 0.3, a similar result in weight 4, theorem 0.9 and weak Beilinson's conjecture for modular curves proved in [B3]. For example for $n = 3$ we get the formula

$$L(E,3) \sim \Big(\frac{2\pi A(\Gamma)}{f_E}\Big)^2 \Omega \cdot \sum_i \sum'_{\gamma_1+\gamma_2+\gamma_3=0} \frac{(x_i,\gamma_1)(y_i,\gamma_2)(z_i,\gamma_3)}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2}$$

In a similar conjecture about $L(S^nE, n+1)$ appears determinants whose entries are the classical Kronecker-Eisenstein series $\sum_{\gamma\in\Gamma} \frac{(x-y,\gamma)}{\gamma^a\bar{\gamma}^b}$ $(a+b = 2n+1)$. Their motivic interpretation was given in [BL]. One should have it also for functions $K_n(x,y,z)$, and, more generally, for functions needed to compute $L(S^nE, m)$.

**9. Motivic Lie algebra $L(F)_\bullet$ ([G2]).** Beilinson conjectured ([B1], [BD2]) that for any filed $F$ should exist a Tannakian (i.e. abelian, tensor, ...) category $\mathcal{M}_T(F)$ of mixed Tate motives over $F$ . It is generated (as tensor category) by an invertible object $\mathbb{Q}(1)_\mathcal{M}$ (Tate motive). Set $\mathbb{Q}(n)_\mathcal{M} := \mathbb{Q}(1)^{\otimes n}_\mathcal{M}$, $n \in \mathbb{Z}$. The crucial axiom is:

$$\mathrm{Ext}^i_{\mathcal{M}_T(F)}(\mathbb{Q}(0)_\mathcal{M}, \mathbb{Q}(n)_\mathcal{M}) \overset{?}{\cong} gr^n_\gamma K_{2n-i}(F)_\mathbb{Q} \tag{14}$$

Any object $M$ of this category carries canonical increasing weight filtration $W_\bullet M$ such that $gr^W_{2k}M = \oplus\mathbb{Q}(-k)_\mathcal{M}$ and $gr^W_{2k-1}M = 0$. There is canonical fiber functor $\omega$ from $\mathcal{M}_T(F)$ to the category of finite dimensional graded $\mathbb{Q}$-vector spaces: $\omega(M) := \oplus Hom(\mathbb{Q}(-k)_\mathcal{M}, gr^W_{2k}M)$. Let $U(F)_\bullet := End\omega$ be the space of all endomorphisms of the functor $\omega$. It is a graded (pro) Hopf algebra over $\mathbb{Q}$.

Let $L(F)_\bullet$ be the Lie algebra of all derivations of $\omega$. It is naturally graded: $L(F)_\bullet = \oplus_{n\geq 1} L(F)_{-n}$ and $U(F)_\bullet$ is its universal enveloping algebra. The functor $\omega$ is an equivalence of the category $\mathcal{M}_T(F)$ with the category of finite dimensional graded modules over $L(F)_\bullet$.

The degree $n$ part of the cochain complex $(\Lambda^\bullet(L(F)^\vee_\bullet),\partial)$ of the Lie algebra $L(F)_\bullet$ forms a subcomplex (here $V^\vee$ is dual to $V$, and $L^\vee_{-n}$ is in degree 1):

$$L^\vee_{-n}\stackrel{\partial}{\longrightarrow}\ldots\stackrel{\partial}{\longrightarrow}L^\vee_{-2}\otimes\Lambda^{n-2}L^\vee_{-1}\stackrel{\partial}{\longrightarrow}\Lambda^n L^\vee_{-1} \qquad (15)$$

Its cohomology is predicted by formula (14). Moreover it should be quasiisomorphic to the weight n motivic complex for $Spec(F)$: (14) provides its key property. So conjecture A suggests that it should be quasiisomorphic to our complex $\Gamma(F,n)$.

One should have canonical injective homomorphisms $l_n:\mathcal{B}_n(F)\hookrightarrow L(F)^\vee_{-n}$ ( see s.12 below). But already for $n=4$ in degree 2 of (15) appears $\Lambda^2 L^\vee_{-2}(F)\stackrel{?}{=}\Lambda^2\mathcal{B}_2(F)$ which is absent in $\Gamma(F,4)$. So complex (15) is bigger then $\Gamma(F,n)$

Set $I_\bullet:=\oplus_{n=2}^\infty L(F)_{-n}$ and let $H^1_{(n)}(I(F)_\bullet)$ be the degree $n$ part of $H^1(I(F)_\bullet)$. Conjecture A is essentially equivalent to the following one about the structure of the Lie algebra $L(F)_\bullet$:

**Conjecture B**. a) $I(F)_\bullet$ *is a free graded pro-Lie algebra.*

b) $H^1_{(n)}(I(F)_\bullet)=\mathcal{B}_n(F)_\mathbb{Q}$ for $n\geq 2$, i.e. $I(F)_\bullet$ *is generated as a graded pro-Lie algebra by the spaces* $\mathcal{B}_n(F)^\vee$ of degree $-n$.

c) *The action of* $L_\bullet/I_\bullet=F^{*\vee}_\mathbb{Q}$ on $H_1^{(n)}(I(F)_\bullet)=\mathcal{B}_n(F)^\vee_\mathbb{Q}$ *coming from the extension* $0\to H_1(I_\bullet)\to L_\bullet/[I_\bullet,I_\bullet]\to L_\bullet/I_\bullet\to 0$ *is described by the homomorphism dual to* $\delta_n:\mathcal{B}_n(F)_\mathbb{Q}\to\mathcal{B}_{n-1}(F)_\mathbb{Q}\otimes F^*$.

Assuming conjecture B it is easy to see that the Hochshild-Serre spectral sequence for $H^*_{(n)}(L(F)_\bullet)$ with respect to the ideal $I_\bullet$ reduces exactly to the complex $\Gamma(F,n)$. Indeed thanks to a) and b) we have

$$E_1^{p,q}=C^p(L_\bullet/I_\bullet,H^q_{(n-p)}(I_\bullet))=\begin{cases}\Lambda^pF^*_\mathbb{Q}\otimes\mathcal{B}_{n-p}(F)_\mathbb{Q}: & q=1\\ \Lambda^nF^*_\mathbb{Q} & :q=0,n=p\\ 0 & :\text{otherwise}\end{cases}$$

and the differentials coincide with the ones in $\Gamma(F,n)$ because of c) .

**10. Framed mixed Tate motives and $U(F)_\bullet$ ([BMS],[BGSV])**. A mixed $\mathbb{Q}$ - Hodge structure $H$ is called a Hodge-Tate structure if all the quotients $gr^W_\bullet H$ are of Hodge type $(p,p)$. It is an $n$-framed Hodge-Tate structure if supplied with nonzero vectors $v\in gr^W_{2n}H$ and $f\in(gr^W_0 H)^*$.

Consider the coarsest equivalence relation on the set of all $n$-framed Hodge-Tate structures for which $H_1\sim H_2$ if there is a morphism of mixed Hodge structures $H_1\to H_2$ respecting the frames. Let $\mathcal{H}_n$ be the set of equivalence classes. It has an abelian group structure: $(H;v,f)\oplus(H';v',f'):=(H\oplus H';(v,v'),f+f')$. Set $\mathcal{H}_0:=\mathbb{Z}$. The tensor product of mixed Hodge structures induces the commutative multiplication $\mu:\mathcal{H}_k\otimes\mathcal{H}_\ell\to\mathcal{H}_{k+\ell}$. A comultiplication $\nu=\bigoplus_k\nu_{k,n-k}:\mathcal{H}_n\to\bigoplus_k\mathcal{H}_k\otimes\mathcal{H}_{n-k}$ is defined as follows. Let $\{e_j\}$ and $\{e^j\}$ be dual bases in $gr^W_{2k}H_\mathbb{Q}$ and $gr^W_{2k}H^*_\mathbb{Q}$. Set $\nu_{k,n-k}((H;v,f)):=\sum_j(H;v,e^j)\otimes(H;e_j,f)$.

Then $\mathcal{H}_\bullet:=\oplus\mathcal{H}_n$ is a commutative graded Hopf algebra.

Similary the equivalence classes of $n$-framed objects in the category $\mathcal{M}_T(F)$ form a commutative graded Hopf algebra $\mathcal{M}_\bullet$. It maps to $U(F)^\vee_\bullet$: the value of the functional defined by $(\omega(M),v,f)$ on $A\in End\omega$ is $<f,Av>$. *This map is an* *isomorphism of Hopf algebras.* In particulary

$$Ker\Big(U(F)^{\vee}_{-n} \stackrel{\Delta}{\longrightarrow} \oplus_k U(F)^{\vee}_{-(n-k)} \otimes U(F)^{\vee}_{-k}\Big) \stackrel{?}{\cong} gr^{\gamma}_{n} K_{2n-1}(F)_{\mathbb{Q}} \qquad (16)$$

It seems that any example of variation of framed mixed Tate motives should be of great interest; the corresponding Hodge periods deserve to be called polylogarithms (don't confuse them with the *classical* polylogarithms!). Below I discuss two such examples where periods are volumes of non-euclidian geodesic simplices and hyperlogarithms. Another example see in [BGSV].

**10. Hyperbolic geometry ([G4]).**

**Theorem 0.12.** *Let $V^5$ be a 5-dimensional complete hyperbolic manifold of finite volume. Then there are algebraic numbers $z_i \in \bar{\mathbb{Q}}^*$ such that*

$$\sum_i \{z_i\}_2 \otimes z_i = 0 \text{ in } B_2(\bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}^* \quad \text{and} \quad \mathrm{vol}(V^5) = \sum_i \mathcal{L}_3(z_i)$$

**Conjecture 0.13.** *Let $V^{2n-1}$ be an $(2n-1)$-dimensional complete hyperbolic manifold of finite volume. Then there are algebraic numbers $z_i \in \bar{\mathbb{Q}} \subset \mathbb{C}$ such that $(n \geq 3)$ $\delta_n(\sum_i \{z_i\}_n) = 0$ and $\mathrm{vol}(V^{2n-1}) = \sum_i \mathcal{L}_n(z_i)$.*

A geodesic simplex $M$ in the hyperbolic space $\mathcal{H}^m$ define a mixed Tate motive. Indeed, in the Klein model $\mathcal{H}^m$ is the interior of a ball in $\mathbb{R}^m$ and geodesics are straight lines. So a geodesic simplex is the usual one inside the absolute: sphere $Q$.

After complexification and compactification we get $\mathbb{CP}^m$ together with a quadric $Q$ ( the absolute) and a collection of hyperplanes $M = (M_1, \dots, M_{m+1})$ ( $(n-1)$-faces of a geodesic simplex). $H(Q, M) := H^m(\mathbb{C}P^m \backslash Q, M)$ is a Hodge-Tate structure.

Let $m = 2n-1$ and $\tilde{Q}(x) = 0$ be a quadratic equation of $Q$. Set

$$\omega_Q := \pm \frac{\sqrt{det\tilde{Q}}}{(2\pi i)^n} \frac{\sum_{i=o}^{2n-1} (-1)^i x_i dx_0 \wedge ...\hat{dx_i} \wedge dx_{2n-1}}{\tilde{Q}(x)^n}$$

The sign depends on the choise of a generator in the primitive part of $H^{n-1}(Q, \mathbb{Z})$. It is provided by an orientation of $\mathcal{H}^{2n-1}$. The simplex $M$ defines a chain $\Delta_M$ representing a generator in $H_{2n-1}(\mathbb{C}P^{2n-1}, M)$. Then $vol(M) = \int_{\Delta_M} \omega_Q$.

The scissor congruence group $\mathcal{P}(\mathcal{H}^m)$ is an abelian group generated by pairs $[M, \alpha]$ where $M$ is an oriented geodesic simplex and $\alpha$ is an orientation of $\mathcal{H}^m$.The relations are: $[M, \alpha] = [M_1, \alpha] + [M_2, \alpha]$ if $M = M_1 \cup M_2$; $[M, \alpha]$ changes sign if we change orientation of $M$ or $\alpha$, and $[M, \alpha] = [gM, g\alpha]$ for any $g \in O(m, 1)$. The spherical scissor congruence groups $\mathcal{P}(S^m)$ are defined similary. $\mathcal{P}(\mathcal{S}^{2k}) = 0$.

The volume provides homomorphisms $\mathcal{P}(\mathcal{H}^m) \to \mathbb{R}$ and $\mathcal{P}(S^m) \to \mathbb{R}/\mathbb{Z}$.

We have a vector $[\omega_Q]$ in $H^{2n-1}(\mathbb{C}P^{2n-1} \backslash Q) = gr^W_{2n} H(Q, M)$ and a functional $[\Delta_M]$ on $H^{2n-1}(\mathbb{C}P^{2n-1}, M) = gr^W_0 H(Q, M)$. So we get an n-framed Hodge-Tate structure associated with $[M, \alpha]$. This construction defines a homomorphism of groups $\mathcal{P}(\mathcal{H}^{2n-1}) \to \mathcal{H}_n$ and similary $\mathcal{P}(S^{2n-1}) \to \mathcal{H}_n$.

Let us define the Dehn invariant $\mathcal{P}(\mathcal{H}^{2n-1}) \stackrel{D^h_n}{\longrightarrow} \oplus_k \mathcal{P}(\mathcal{H}^{2k-1}) \otimes \mathcal{P}(S^{2(n-k)-1})$. Each $(2k-1)$-face $A$ of $M$ is a hyperbolic simplex $h(A)$. In the orthogonal plane $A^{\perp}$ $M$ cuts a spherical simplex

$s(A)$. Choose orientations $\alpha_A$ and $\beta_A$ of $A$ and $A^\perp$ such that $\alpha_A \otimes \beta_B = \alpha$. Then $D^h_n([M,\alpha]) := \sum_A [h(A),\alpha_A] \otimes [s(A),\beta_A]$.

**Theorem 0.14.** *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{P}(\mathcal{H}^{2n-1}) & \stackrel{D^h_n}{\longrightarrow} & \oplus_k \mathcal{P}(\mathcal{H}^{2k-1}) \otimes \mathcal{P}(S^{2(n-k)-1}) \\ \downarrow & & \downarrow \\ \mathcal{H}_n & \stackrel{\nu}{\longrightarrow} & \oplus_k \mathcal{H}_k \otimes \mathcal{H}_{n-k} \end{array}$$

A similar motivic interpretation has the spherical Dehn invariant $D^s_n : \mathcal{P}(S^{2n-1}) \longrightarrow \oplus_k \mathcal{P}(\mathcal{S}^{2k-1}) \otimes \mathcal{P}(S^{2(n-k)-1})$. So (16) leads to

**Conjecture 0.15.** *There are canonical injective homomorphisms*

$$KerD^h_n \otimes \mathbb{Q} \hookrightarrow [gr^\gamma_n K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}(n)]^- \qquad KerD^s_n \otimes \mathbb{Q} \hookrightarrow [gr^\gamma_n K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}(n)]^+$$

*whose composition with Beilinson's regulator coincide with the volume homomorphisms.*

If $n = 2$ they exist and are isomorphisms by the results of [D], [DS], [S1].

Each complete hyperbolic $(2n-1)$-manifold can be cuted on geodesic simplices and so produces an element in $\mathcal{P}(\mathcal{H}^{2n-1})$. Its Dehn invariant is equal to zero. So conjecture 0.13 follows from conjectures 0.15 and A.

**11. Hyperlogarithms ([G5]).** They where considered by Kummer ([Ku]), Poincare, Lappo-Danilevsky, .... We define them as the following iterated integrals:

$$\Psi_{m_1,...,m_l}(a_1,...,a_l) := \int_0^1 \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ ... \circ \frac{dt}{t}}_{m_1 \text{ times}} \circ ... \circ \underbrace{\frac{dt}{t-a_l} \circ \frac{dt}{t} \circ ... \frac{dt}{t}}_{m_l \text{ times}}$$

This formula means the following. Let $n := m_1 + ... + m_l$ and

$$\Delta := \{(t_1,...,t_n) \subset \mathbb{R}^n | 0 \le t_1 - a_1 \le t_2 \le ... \le t_{m_1} \le t_{m_1+1} - a_2 \le t_{m_1+2} ... \le t_{m_l}\}$$

Let $L$ be a coordinate simplex in $\mathbb{C}P^n$ related to coordinates $(t_0 : ... : t_n)$ and $\omega_L := \frac{dt_1}{t_1} \wedge ... \wedge \frac{dt_n}{t_n}$. Then $\Psi_{m_1,...,m_l}(a_1,...,a_l) = \int_\Delta \omega_L$.

Let $M$ be collection of all the hyperplanes corresponding to codimension 1 faces of $\Delta$. Then $H(L,M) := H^n(\mathbb{C}P^n \backslash L, M)$ is a Hodge-Tate structure. It has canonical $n$-framing: $[\omega_L]$ is a vector in $H^n(\mathbb{C}P^n \backslash L) = gr^W_{2n} H(L,M)$ and $\Delta$ produces a class $[\Delta] \in H_n(\mathbb{C}P^n, M) = gr^W_0 H(L,M)$. So we get an element $\Psi^{\mathcal{H}}_{m_1,...,m_l}(a_1,...,a_l) \in \mathcal{H}_n$. According to the general philosophy *a mixed Hodge structure in the cohomology of a (simplicial) variety is a realisation of a mixed motive.* So we should have an $n$-framed mixed Tate motive $\Psi^{\mathcal{M}}_{m_1,...,m_l}(a_1,...,a_l)$.

More generally, if $F$ is a field and $a_i \in F^*$ one should also have an $n$-framed mixed Tate motive $\Psi^{\mathcal{M}}_{m_1,...,m_l}(a_1,...,a_l)$ related to $H^n(P^n_F \backslash L, M)$.

There is a remarkable power series expansion of the hyperlogarithms. Namely, consider *multiple* polylogarithms

$$\Phi_{m_1,...,m_l}(x_1,...,x_l) := \quad (-1)^l \sum_{0<k_1<k_2<...<k_l} \frac{x_1^{k_1} x_2^{k_2} ... x_l^{k_l}}{k_1^{m_1} k_2^{m_2} ... k_l^{m_l}}$$

**Theorem 0.16.** *([G5]) Suppose* $|a_i/a_{i-1}| < 1$. *Then*

$$\Psi_{m_1,...,m_l}(a_1,...,a_l) = \Phi_{m_1,...,m_l}(\frac{a_2}{a_1}, \frac{a_3}{a_2}, ..., \frac{1}{a_l})$$

In particulary $\zeta(m_1,...,m_l) := \Psi_{m_1,...,m_l}(1,1,...,1)$ are the multiple zeta values of Euler [E], rediscovered and studied by Zagier [Z3], see also [Dr] and [Ko].

**Conjecture 0.17.** . *a) Any n-framed mixed Tate motive over F is a sum of hyperlogarithmic ones* $\Psi^{\mathcal{M}}_{m_1,...,m_l}(a_1,...,a_l)$, *where* $n = m_1 + ... + m_l$; $a_i \in F^*$.

*b)Any n-framed mixed Tate motive over Spec*($\mathbb{Z}$) *is a sum of motivic multiple zeta's* $\zeta^{\mathcal{M}}(m_1,...,m_l)$

The first part of the conjecture is motivated by the following

**Proposition 0.18.** *(***Universality of hyperlogarithms***) Any iterated integral* $F(z) = \int_x^z \omega_1 \circ ... \circ \omega_n$ *of rational 1-forms* $\omega_i$ *on a rational variety X is a sum of hyperlogarithms, i.e. there exist* $f_j^{(i)}(z) \in \mathbb{C}(X)^*$ *such that*

$$F(z) = \sum_i \Psi_{m_1^{(i)},...,m_{l^{(i)}}^{(i)}}(f_1^{(i)}(z),...,f_l^{(i)}(z)) \quad + \quad C \qquad (C \text{ is a constant})$$

**12. Motivic interpretation of the "weak" part of conjecture A**. For any $a \in F^*$ the $n$-framed mixed Tate motive $\Psi_n^{\mathcal{M}}(a^{-1})$ ( corresponding to $Li_n(a)$) provides a homomorphism $\tilde{l}_n : \mathbb{Z}[F^*] \to U(F)^\vee_{-n}$. Denote by $l_n$ its composition with the canonical projection $U(F)^\vee_{-n} \to L(F)^\vee_{-n}$.

One should have $l_n(\mathcal{R}_n(F)) = 0$, so $l_n : \mathcal{B}_n(F) \to L(F)^\vee_{-n}$. It turns out that $\partial(l_n\{a\}) = l_{n-1}\{a\} \wedge a$ (we identified $L(F)^\vee_{-1}$ with $F^*_{\mathbb{Q}}$), Therefore *homomorphisms* $\{l_i\}$ *provide a canonical homomorphism of the complex* $\Gamma(F,n)$ *to the complex (15).* Using (14) we get canonical maps $H^i(\Gamma(F,n)\otimes\mathbb{Q}) \to gr_n^\gamma K_{2n-i}(F)_{\mathbb{Q}}$.

**13. The quantum dilogarithm ([FK])**. Mixed Tate motives give the best explanation *all* of the different appearances of the dilogarithm discussed above. However recently the dilogarithm appeared in conformal field theory and exactly solvable problems of statistical mechanics. Here is one example.

Let $\Psi(x) := \prod_{n=1}^\infty (1 - xq^n), \quad |q| < 1$. Then for $q = exp(\epsilon)$, $Im(\epsilon) < 0$

$$\Psi(x) = \frac{1}{\sqrt{1-x}} exp(Li_2(x)/\epsilon)(1 + O(\epsilon)), \quad \epsilon \to 0$$

**Theorem 0.19.** *([FK]) Suppose* $\hat{U}$ *and* $\hat{V}$ *satisfies* $\hat{U}\hat{V} = q\hat{V}\hat{U}$. *Then*

$$\Psi(\hat{V})\Psi(\hat{U}) = \Psi(\hat{U})\Psi(-\hat{U}\hat{V})\Psi(\hat{V})$$

*and in the classical limit we get the 5-term relation for the Rogers dilogarithm.*

**Acnowledgement** I am grateful to A.Borel for many useful remarks about a preliminary version of the paper.

## References


[B] Beilinson A.A, *Height pairings between algebraic cycles*, Lect. Notes in Math. 1289, (1987), 1–26.

[B2] Beilinson A.A, *Higher regulators and values of L-functions*, VINITI, 24 (1984), 181–238.

[B3] Beilinson A.A, *Higher regulators for modular curves*, Cont. Math., v.55, (1987), 1–34.

[BD1] Beilinson A.A., Deligne P, *Interpr etation motivique de la conjecture de Zagier* in Symp. in Pure Math., v. 55, part 2, 1994, ?-??

[BD2] Beilinson A.A., Deligne P, *Polylogarithms and regulators*, (In preparation)

[BL] Beilinson A.A., Levin A., *Elliptic polylogarithms* in Symp. in Pure Math., v. 55, part 2, 1994, 101-156.

[BMS] Beilinson A.A., MacPherson R., Schechtman, V.V, *Notes on motivic cohomology*, Duke Math. J. 54 (1987), 679–710.

[BGSV] Beilinson A.A., Goncharov A.B., Schechtman V.V, Varchenko A.N, *Projective geometry and algebraic $K$-theory*, Algebra and Analysis (1990) n 3, 78–131.

[Bl1] Bloch S, *Higher regulators, algebraic $K$- theory and zeta functions of elliptic curves*, Lect. Notes U.C. Irvine, (1977).

[Bl2] Bloch S, *Algebraic cycles and higher K-theory*, Adv. in Math., (1986), v. 61, 267–304.

[BK] Bloch S. Kriz I, *Mixed Tate motives*, Preprint 1992.

[Bo1] Borel A, *Stable real cohomology of arithmetic groups*, Ann. Sci. Ec. Norm. Super., (4) 7 (1974), 235–272

[Bo2] Borel A, *Cohomologie de $SL_n$ et valeurs de fonctions zêta aux points entiers*, Ann. Sc. Norm. Sup. Pisa 4 (1977), 613–636.

[Den] Deninger C, *Higher order operations in Deligne cohomology*, Schriftenreihe des Mathematischen Instituts der Universitat Munster. (1991), 1–27.

[Dr] Drinfeld V.G, *On quasi-triangular quasi-Hopf algebras and some group related to* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Algebra and Analysis ( 1991).

[D] Dupont J. *Homology of flag complexes* Math. J. 1982

[DS] Dupont J., Sah C.-H, *Scissors congruences II*, J. Pure Appl. Algebra, v. 25, (1982), 159–195.

[FK] Faddeev L.D., Kashaev R.M. :Quantum dilogarithm, Preprint 1993.

[E] L. Euler: "Opera Omnia," Ser. 1, Vol XV, Teubner, Berlin 1917, 217-267.

[GGL] Gabrielov A.M., Gelfand I.M., Losic M.V, *Combinatorial computation of characteristic classes*, Funct. Anal. and its Appl. v. 9, n 2 (1975) 5–26.

[GM] Gelfand I.M., MacPherson R, *Geometry in Grassmannians and a generalisation of the dilogarithm*, Adv. in Math., 44 (1982) 279–312.

[G1] Goncharov A.B,*Geometry of configurations, polylogarithms and motivic cohomology*, Adv. in Math., 1995 (to appear).

[G2] Goncharov A.B., *Polylogarithms and motivic Galois groups* in Symp. in Pure Math., v. 55, part 2, 1994, 43 - 97.

[G3] Goncharov A.B., *Explicit construction of characteristic classes*, Adv. in Sov. Math., 1993, v. 16, p. 169-210

[G4] Goncharov A.B., *Volumes of hyperbolic manifolds*, Preprint 1993.

[G5] Goncharov A.B., *Multiple $\zeta$-values, hyperlogarithms and mixed Tate motives*, Preprint 1993.

[G6] Goncharov A.B., *Special values of Hasse-Weil L-functions and generalized Eisenstein-Kronecker series*, Preprint 1994.

[J] De Jeu R., *Zagier's conjecture and wedge complexes in algebraic K-theory*, Preprint Utrecht Univercity, 1993.

[HM] Hain R, MacPherson R, *Higher Logarithms*, Ill. J. of Math,, vol. 34, (199O) N2, 392–475.

[HaM] Hanamura M, MacPherson R, *Geometric construction of polylogarithms*, Duke Math. J. 1993

[K] Kummer E.E, *Uber die Transcendeten, welche aus wiederholten Integrationen rationaler Formeln entstehen*, J. reine ang. Math. 21 (1840), pp. 74–90.

[Ko] Kontsevich M., Oral communication.

[Le] Levine M, *The derived motivic category*, Preprint

[L] Lewin L, *Dilogarithms and associated functions*, North Holland, 1981.

[Lei] Gerhardt C.I. (ed). G.W.Leibniz Mathematisch Schriften III/1 pp. 336-339 Georg Olms Verlag, Hildesheim. New York 1971.

[Li] Lichtenbaum S, *Values of zeta functions at non-negative integers*, Lect. Notes in Math., 1068, Springer Verlag, 1984, 127-138.

[NZ] Neumann W., Zagier D, *Volumes of hyperbolic three- manifolds*, Topology 24 (1985), 307-331.

[S1] Suslin A.A,$K_3$ *of a field and Bloch's group*, Proceedings of the Steklov Institute of Mathematics 1991, Issue 4.

[S2] Suslin A.A,*Homology of* $GL_n$*, characteristic classes and Milnor's* $K$*-theory.* Lect. Notes in Math. 1046 (1989), 357–375.

[V] Voevodsky V. *Triangulated categories of motives over a fileld*, Preprint 1994.

[Z1] Zagier D,*Polylogarithms, Dedekind zeta functions and the algebraic* $K$*-theory of fields*, Arithmetic Algebraic Geometry (G.v.d.Geer, F.Oort, J.Steenbrink, eds.), Prog. Math., Vol 89, Birkhauser, Boston, 1991, pp. 391–430.

[Z2] Zagier D,*Hyperbolic manifolds and special values of Dedekind zeta functions*, Inventiones Math. 83 (1986), 285–301.

[Z3] Zagier D, *Special values of L-functions*, Proceedins of the First European Congress of Mathematicians in Paris. (1992),

[Y] Yang R., Ph.D.Thesis, Univ. of Washington, Seattle (1991).

Department of Mathematics, MIT, Cambridge MA 02139 U. S. A.; sashagon@math.mit.edu