Lectures on BPS states and spectral networks

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Abstract. These are notes for a lecture series on BPS states and spectral networks, delivered at Park City Mathematics Institute, July 2019. The first part is a general review of the notions of BPS state and BPS index. The second part discusses the specific case of BPS states in $\mathcal{N} = (2,2)$ supersymmetric field theories in two dimensions, and introduces the notion of spectral network as a way of computing the BPS indices in that context. The last part discusses the more general case of 2d and 4d BPS indices associated to surface defects in four-dimensional field theories of class $\mathcal{S}$.

1. Lecture 1: What is a BPS state?

BPS states appear very frequently in geometric applications of quantum field theory. The aim of this lecture is to explain rather generally what a BPS state is and some of their basic properties.

In one sentence: we'll study a representation $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ of a certain super Lie algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, and the BPS states are the ones in irreps annihilated by nontrivial subspaces of $\mathcal{A}^1$.

1.1. Quantum mechanics A time-independent quantum system involves the following data:

- A Hilbert space $\mathcal{H}$,
- A formally self-adjoint operator $\mathsf{H} : \mathcal{H} \to \mathcal{H}$.

We think of $i\mathsf{H}$ as generating the abelian Lie algebra

$$\mathrm{Lie}(\mathrm{ISO}(0,1)) = \mathrm{Lie}(\mathrm{Isom}(\mathbb{R}^{0,1})) \simeq \mathbb{R}. \tag{1.1.1}$$

Eigenvectors of H are called *bound states*; each bound state thus spans a 1-dimensional irreducible representation of $\mathrm{ISO}(0,1)$. The fact that H is formally self-adjoint implies that this representation is unitary. The eigenvalues of H are called *bound state energies*.

Example 1.1.2 (Particle on the line)**.** In your first course on quantum mechanics you study the "particle on the line." For this example you need to fix a function $V : \mathbb{R} \to \mathbb{R}$. Then there is a time-independent quantum system $T_1[\mathbb{R}, V]$ with:

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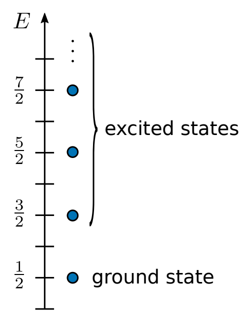
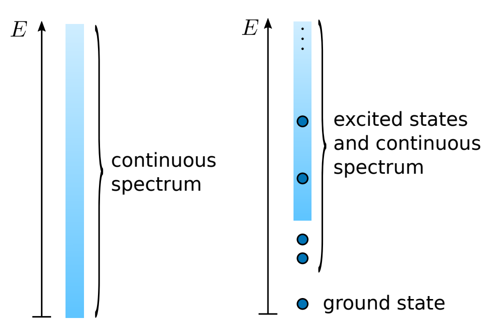
- $\mathcal{H} = L^2(\mathbb{R})$,
- $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$.

Fact 1.1.3. If $V(x)$ is smooth and $V(x) \to \infty$ as $|x| \to \infty$, then H has discrete spectrum with no accumulation points, and $\mathcal{H}$ has a (Schauder) basis consisting of eigenvectors of H. There is a unique eigenvector with smallest eigenvalue, which we call the *ground state*; we call the other eigenvectors *excited states*.

Example 1.1.4 (Harmonic oscillator)**.** If we take $V(x) = \frac{1}{2}x^2$ in the above then the ground state turns out to be $\psi(x) = e^{-\frac{1}{2}x^2}$, which has $H\psi(x) = \frac{1}{2}\psi(x)$, so that the ground state energy is $\frac{1}{2}$. The bound state energies are $n + \frac{1}{2}$, $n \in \mathbb{N}$.



Remark 1.1.5. If $V(x)$ is more general, then $\mathcal{H}$ need not have a basis consisting of eigenvectors of H. For an extreme example, if $V(x) = 0$ there are no L^2 eigenvectors at all. For some purposes the functions $\psi_p(x) = e^{ipx}$, parameterized by $p \in \mathbb{R}$, can stand in. These functions clearly have formally $H\psi_p = \frac{p^2}{2}\psi_p$. Since $\frac{p^2}{2}$ takes all nonnegative real values, one might then expect that H has continuous spectrum consisting of all nonnegative real numbers, and this is indeed true. For more general choices of $V(x)$, H could have some discrete and some continuous spectrum, as indicated in the figure below.



Next we consider replacing the domain $\mathbb{R}$ by a general Riemannian manifold:

Definition 1.1.6 (Laplace operator)**.** For M a Riemannian manifold, the Laplace operator $\Delta : L^2(M) \to L^2(M)$ is given by

$$\Delta = d^* d \tag{1.1.7}$$

where d^* is the formal adjoint of $d : L^2(M) \to \Omega^1_{L^2}(M)$.

Example 1.1.8 (Particle on a Riemannian manifold)**.** Fix a Riemannian manifold M and a function $V : M \to \mathbb{R}$. Then there is a quantum mechanical system $T_1[M, V]$, the "particle on M with potential V," with:

- $\mathcal{H} = L^2(M)$,
- $H = \frac{1}{2}\Delta + V$.

In particular, suppose M is compact and we take $V = 0$. In this case, $H = \frac{1}{2}\Delta$ has only discrete spectrum, with eigenvalues bounded below by 0, and $H\psi = 0$ iff ψ is a constant function.

1.2. The superparticle Now we come to our first supersymmetric example, following [51].

Definition 1.2.1 (Laplace operator on differential forms)**.** The Laplace operator $\Delta : \Omega^*_{L^2}(M) \to \Omega^*_{L^2}(M)$ is given by

$$\Delta = [d, d^*] = dd^* + d^*d \tag{1.2.2}$$

where d^* is the formal adjoint of $d : \Omega^*_{L^2}(M) \to \Omega^*_{L^2}(M)$.

Example 1.2.3 (Superparticle on a Riemannian manifold)**.** Again fix a Riemannian manifold M. Then there is a quantum mechanical system $T_1[M]$, the "superparticle on M", with:

- $\mathcal{H} = \Omega^*_{L^2}(M)$,
- $H = \frac{1}{2}\Delta$.

The key new feature is that in this case there are more operators around than just H: $\mathcal{H}$ is a unitary $\mathbb{Z}/2\mathbb{Z}$-graded representation of a Lie superalgebra, defined as follows.

Definition 1.2.4 ($\mathcal{N} = 2$ supersymmetry in dimension $d = 1$)**.** [1] We define a Lie superalgebra $\mathcal{A}$ by

$$\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1, \quad \mathcal{A}^0 = \mathbb{C} \cdot H, \quad \mathcal{A}^1 = \mathbb{C} \cdot Q \oplus \mathbb{C} \cdot \overline{Q}, \tag{1.2.5}$$

i.e. $\mathcal{A}$ has 2 odd generators $Q, \overline{Q}$ and one even generator H, with the brackets[2]

$$[Q, \overline{Q}] = 2H, \quad [Q, Q] = 0, \quad [\overline{Q}, \overline{Q}] = 0, \quad [Q, H] = 0, \quad [\overline{Q}, H] = 0. \tag{1.2.6}$$

A $\mathbb{Z}/2\mathbb{Z}$-*graded* representation of $\mathcal{A}$ is a representation of $\mathcal{A}$ on a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, where $\mathcal{A}^i$ maps $\mathcal{H}^j \to \mathcal{H}^{j+i}$. A *unitary* representation of $\mathcal{A}$ is a representation in which $\mathcal{H}$ is a Hilbert space, H acts by a formally self-adjoint operator, and $Q, \overline{Q}$ act by operators which are formally adjoint to one another.

[1] The $\mathcal{N} = 2$ refers to the fact that we have 2 odd generators. The $d = 1$ refers to the fact that H generates $ISO(0, 1)$, the isometries of $\mathbb{R}^{0,1}$ — it does *not* have to do with the dimension of M, which can be anything.

[2] Our convention is that $[,]$ means the *graded* bracket, i.e. for objects x, y which are in grade n_x, n_y respectively, $[x, y] = xy - (-1)^{n_x n_y} yx$. So this bracket is a commutator unless both x and y are odd, in which case it is an anticommutator.

To realize $\mathcal{H}$ as a representation of $\mathcal{A}$ we just take

$$Q = d, \qquad \overline{Q} = d^*, \qquad H = \frac{1}{2}\Delta. \tag{1.2.7}$$

Then $\mathcal{H}$ is a unitary representation of $\mathcal{A}$. It is also $\mathbb{Z}/2\mathbb{Z}$-graded:

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1, \quad \mathcal{H}^0 = \bigoplus_k \Omega^{2k}_{L^2}(M), \quad \mathcal{H}^1 = \bigoplus_k \Omega^{2k+1}_{L^2}(M). \tag{1.2.8}$$

Now let us explore a bit of the unitary representation theory of $\mathcal{A}$, following [50]. The unitarity implies that all eigenvalues of H are nonnegative, because

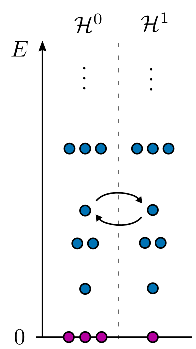
$$2\langle\psi, H\psi\rangle = \langle\psi, Q\overline{Q}\psi\rangle + \langle\psi, \overline{Q}Q\psi\rangle = \|Q\psi\|^2 + \|\overline{Q}\psi\|^2 \geqslant 0. \tag{1.2.9}$$

Moreover the norm $\|\cdot\|$ is nondegenerate, so we conclude that

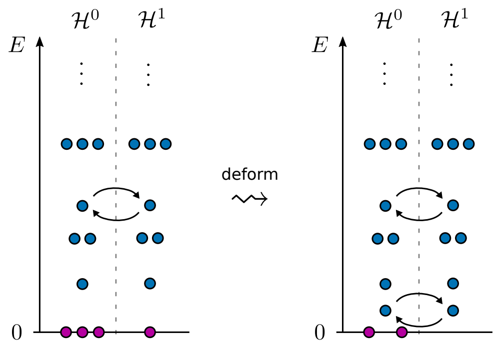
$$H\psi = 0 \quad \Leftrightarrow \quad Q\psi = 0, \overline{Q}\psi = 0. \tag{1.2.10}$$

In particular, each state with $H\psi = 0$ generates a 1-dimensional (trivial) representation of $\mathcal{A}$. We call this representation V^0_0 or V^1_0 depending whether the single state is in $\mathcal{H}^0$ or $\mathcal{H}^1$. These representations are called "short." The only other possibility for a unitary irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representation is a 2-dimensional representation, with one state in $\mathcal{H}^0$ and one in $\mathcal{H}^1$, both with $H\psi = E\psi$ for some $E > 0$; these representations are called "long."

Fact 1.2.11. If M is compact, then $\mathcal{H}$ of Example 1.2.3 is a countable orthogonal direct sum of unitary irreducible representations of $\mathcal{A}$. (We say $\mathcal{H}$ "contains only discrete spectrum.") See the figure below, where each dot represents one state; note that the states with $E > 0$ come paired up into long representations, while those with $E = 0$ are in short representations by themselves.



The short and long representations have very different character, which we see clearly if we consider deformations of the representation $\mathcal{H}$, e.g. by varying the Riemannian metric on M. As we deform $\mathcal{H}$, the nonzero eigenvalues $E > 0$ of H can change continuously: the long representations are not rigid. The eigenvalues $E = 0$ have a harder time changing, because the representations V^i_0 are rigid. This fact helps to "protect" the ground states. However, it doesn't protect them absolutely: the reducible representation $V^0_0 \oplus V^1_0$ is not rigid, since it can deform to a long representation with $E = \epsilon > 0$. See the figure below.



With this deformation process in mind we consider the following quantity.

Definition 1.2.12 (Index for representations of $\mathcal{N} = 2$ supersymmetry in $d = 1$)**.** The *index*, or (signed) *ground state degeneracy*, of a representation $\mathcal{H}$ of $\mathcal{A}$ is

$$\chi(\mathcal{H}) = (\#\text{ copies of } V_0^0 \text{ in } \mathcal{H}) - (\#\text{ copies of } V_0^1 \text{ in } \mathcal{H}). \tag{1.2.13}$$

The key property of the index is that it is invariant under deformations of $\mathcal{H}$, as long as $\mathcal{H}$ contains only discrete spectrum: if inside $\mathcal{H}$ a copy of $V_0^0 \oplus V_0^1$ deforms into a long representation, then χ changes by $1 - 1 = 0$.

The main lesson here is: while the full Hilbert space $\mathcal{H}$ depends strongly on every little detail of the system, by using a little bit of the representation theory of the supersymmetry algebra $\mathcal{A}$ — looking at representations which are particularly rigid — we are able to extract a more robust and invariant quantity. A second lesson is that the rigid representations are the ones which are smaller than usual, by virtue of being annihilated by part of $\mathcal{A}$ (in this case actually all of $\mathcal{A}$).

1.3. Q**-cohomology** Here is another viewpoint on the ground states. In any representation of $\mathcal{A}$, say $W = W^0 \oplus W^1$, we can define the Q-cohomology,

$$H_Q^i(W) = \ker Q|_{W^i} / \operatorname{Im} Q|_{W^{i-1}}. \tag{1.3.1}$$

In the case of Example 1.2.3 the Q-cohomology is something well known: it is the de Rham cohomology of M (with the $\mathbb{Z}$-grading collapsed to a $\mathbb{Z}/2\mathbb{Z}$-grading),

$$H_Q^i(\mathcal{H}) = H_{dR}^i(M) = \ker d|_{W^i} / \operatorname{Im} d|_{W^{i-1}}. \tag{1.3.2}$$

Now $H_Q^i(\bigoplus_\alpha W_\alpha) \simeq \bigoplus_\alpha H_Q^i(W_\alpha)$, so to compute $H_Q^i(W)$ it's enough to understand the cohomology of each irreducible constituent of W. When W is irreducible,

$$H_Q^i(W) = \begin{cases} W & \text{if } W \simeq V_0^i, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.3}$$

Thus only the short representations in the decomposition of W contribute to $H_Q^i(W)$, and each contributes a 1-dimensional space. (This is a $\mathbb{Z}/2\mathbb{Z}$-graded version of the Hodge theorem for compact Riemannian manifolds.)

In particular, the deformation invariant index χ which we discussed has a simple interpretation:

$$\chi(\mathcal{H}) = \dim H^0_Q(\mathcal{H}) - \dim H^1_Q(\mathcal{H}) = \dim H^0_{dR}(M) - \dim H^1_{dR}(M) \tag{1.3.4}$$

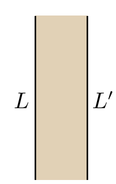
i.e. it is the Euler characteristic of M!

This supersymmetric-quantum-mechanics viewpoint on the cohomology of M was introduced by Witten in [51]. There he also considered various deformations of the story, which we won't have time to treat here. One modification involves adding a Morse function on M (this becomes especially interesting in the limit where the Morse function is very large; one gets a description of the cohomology of M in terms of critical points of the Morse function and gradient flows between them.) Another involves passing to *equivariant* cohomology for a G-action on M.

1.4. Richer examples So far we've discussed how cohomology of a compact Riemannian manifold M arises as a space of supersymmetric ground states of a time-independent quantum system $T_1[M]$. This was essentially a reinterpretation of the usual story of Hodge theory of compact Riemannian manifolds.

We briefly mention a few richer examples of the same structure, to make contact with other lectures at this school. These are not rigorous statements, but rather accounts of what is claimed in the physics literature.

Physics Example 1.4.1 (Lagrangian Floer homology)**.** Suppose M is a Kähler manifold. There is a 2-dimensional quantum field theory $T_2[M]$ known as the supersymmetric sigma model into M, with $\mathcal{N} = (2, 2)$ supersymmetry. A Lagrangian submanifold $L \subset M$ gives a supersymmetric boundary condition in $T_2[M]$. Now consider $T_2[M]$ in the 2-dimensional spacetime $X = [0, 1] \times \mathbb{R}$, with boundary conditions at the two sides induced by L and L'.



This system is effectively 1-dimensional: in particular its geometric symmetry is just $ISO(0, 1)$. It gives a time-independent quantum system $T_1[M, L, L']$ which is $\mathcal{N} = 2$ supersymmetric, so that its Hilbert space $\mathcal{H}$ is a representation of $\mathcal{A}$. The space of supersymmetric ground states is expected to be isomorphic to (some appropriate version of) Lagrangian Floer homology $HF(L, L')$.

Physics Example 1.4.2 (Khovanov-Rozansky homology)**.** Let $\mathfrak{g}$ be a Lie algebra of ADE type (eg $\mathfrak{g} = \mathfrak{sl}(N)$.) Let L be a link in $\mathbb{R}^3$, and R an irreducible finite-dimensional representation of $\mathfrak{g}$. There is a 6-dimensional quantum field theory $T_6[\mathfrak{g}]$, the $(2, 0)$ superconformal field theory of type $\mathfrak{g}$, which admits 2-dimensional defects of various types, one for each R. Theory $T_6[\mathfrak{g}]$ can be formulated on the spacetime $\mathbb{R}^{5,1}$ with a defect inserted along a 2-dimensional submanifold $S \subset \mathbb{R}^{5,1}$:

$$\mathbb{R}^{5,1} = \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{0,1} \tag{1.4.3}$$

$$S = L \times \{pt\} \times \mathbb{R}^{0,1} \tag{1.4.4}$$

The geometric symmetry remaining is $\mathrm{ISO}(0,1) \times \mathrm{SO}(2)$. This gives a time-independent quantum mechanical system $T_1[\mathfrak{g}, R, L]$ which is $\mathcal{N} = 2$ supersymmetric,[3] so that its Hilbert space $\mathcal{H}$ is a representation of $\mathcal{A}$.

The full $\mathcal{H}$ probably depends on every little detail of L, but optimistically, the space of supersymmetric ground states should be well defined, finite-dimensional and depend only on the isotopy class of L; in fact it should be isomorphic to the Khovanov-Rozansky homology of L. (This is more or less conjectured in [56] Section 5.1.6, in turn a reinterpretation of a proposal in [26].)

In this case there are two $U(1)$ symmetries in the theory, so the Hilbert space $\mathcal{H}$ is $\mathbb{Z} \times \mathbb{Z}$-graded, as the Khovanov-Rozansky homology is. One of these symmetries comes from the rotational $\mathrm{SO}(2)$ mentioned above, the other comes from an internal "R-symmetry" of the $(2,0)$ theory.[4]

1.5. Field theory In the remainder of these lectures we will be interested in a more complicated class of examples, which arise from quantum field theory (QFT) in Minkowski space $\mathbb{R}^{d-1,1}$.

Such a QFT is an example of a time-independent quantum system, and so it determines a Hilbert space $\mathcal{H}$, as before. However, if we study this Hilbert space the way we did before, we will find that everything is unpleasantly infinite: in particular, H has no discrete spectrum except maybe the vacuum state. To make further progress we need to organize $\mathcal{H}$ better. We use the fact that instead of $\mathrm{ISO}(0,1)$ which appeared before, now $\mathcal{H}$ is a unitary representation of a larger group, the Poincare group

$$\mathrm{ISO}(d-1,1) = \mathrm{SO}(d-1,1) \ltimes \mathbb{R}^{d-1,1} \subset \mathrm{Isom}(\mathbb{R}^{d-1,1}) \tag{1.5.1}$$

or more precisely its universal cover $\mathrm{ISpin}(d-1,1)$. Thus, we need to know a little bit about the unitary representations of $\mathrm{ISpin}(d-1,1)$.

The Lie algebra of $\mathrm{ISpin}(d-1,1)$ is spanned by $\mathfrak{so}(d-1,1)$ plus translation generators P^i, $i \in \{0,1,\dots,d-1\}$, with $H = P^0$. In the universal enveloping algebra there is a quadratic Casimir operator

$$\rho = (P^0)^2 - (P^1)^2 - (P^2)^2 - \cdots - (P^{d-1})^2 \tag{1.5.2}$$

which thus acts by a scalar in every irreducible representation. We will only be interested in representations where this scalar is nonnegative. It will be convenient to write M for a nonnegative square root of this scalar, i.e. $M = \sqrt{\rho}$.

[3]For physicists: a cheap way of getting this counting is to remember from [57] that in $\mathcal{N} = 4$ super Yang-Mills a Wilson loop constrained to a 3-dimensional subspace is $\frac{1}{8}$-BPS, i.e. it preserves 2 of the 16 supercharges.

[4]The $(2,0)$ theory itself has $\mathrm{SO}(5)$ R-symmetry, but the surface defect on $L \times \mathbb{R}^{0,1}$ breaks that down to $U(1)$.

A key device for understanding unitary representations of $\mathrm{ISpin}(d-1,1)$ is to first diagonalize the translation generators, then focus on a single eigenspace, as follows.

Definition 1.5.3 (Rest subspace)**.** Given a representation V of $\mathrm{ISpin}(d-1,1)$, in which ρ acts by a scalar, the *rest subspace* of V is

$$V^{\mathrm{rest}} = \{\psi \in V : P^0\psi = M\psi,\ P^i\psi = 0 \text{ for } i \in \{1,\dots,d-1\}\}. \tag{1.5.4}$$

Given V, the subspace V^{rest} is a representation of the subgroup

$$G^{\mathrm{rest}} = \mathrm{Spin}(d-1) \ltimes \mathbb{R}^{d-1,1}. \tag{1.5.5}$$

The action of the translation group $\mathbb{R}^{d-1,1}$ on V^{rest} is by the fixed character $(M,0,\dots,0)$. In contrast, the structure of V^{rest} as a representation of $\mathrm{Spin}(d-1)$ can be arbitrary (though unitary). Conversely, suppose given an irreducible representation S of $\mathrm{Spin}(d-1)$ and a scalar $M > 0$; then we may extend S to a representation V^{rest} of G^{rest} by letting the translations act by the character $(M,0,\dots,0)$, and there is a canonical procedure ("induction") for extending V^{rest} to a unitary irreducible representation V of the full $\mathrm{ISpin}(d-1,1)$. Thus we have:

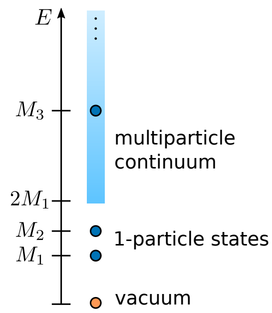
Fact 1.5.6 (1-particle representations of Poincare group)**.** Given a unitary irreducible representation S of the compact group $\mathrm{Spin}(d-1)$ and a scalar $M > 0$, there is a corresponding unitary irreducible representation $V_{S,M}$ of $\mathrm{ISpin}(d-1,1)$, such that $V^{\mathrm{rest}}_{S,M}$ is identical to S as a representation of $\mathrm{Spin}(d-1)$.

Physically $V_{S,M}$ should be thought of as representing the space of states of a massive particle propagating in $\mathbb{R}^{d-1,1}$, with rest mass M, and spin governed by the representation S.

Our main focus in what follows will be theories with a mass gap. In such theories a typical structure for the full Hilbert space $\mathcal{H}$ is as follows:

- $\mathcal{H}$ contains one copy of the trivial representation of $\mathrm{ISpin}(d-1,1)$; this represents the quantum vacuum of the theory.
- $\mathcal{H}$ contains a collection of direct summands, of the form $V_{S,M}$, with masses M_n where $0 < M_1 \leqslant M_2 \leqslant M_3 \leqslant \cdots$. These summands represent 1-particle states. M_1 is called the mass gap.
- The full $\mathcal{H}$ is not an orthogonal direct sum of irreducible representations of $\mathrm{ISpin}(d-1,1)$. In particular, the operator M has continuous spectrum $[2M_1,\infty)$. The continuous spectrum reflects the contributions of multiparticle states.

See the figure for a schematic picture of the spectrum of M.



1.6. Supersymmetric field theory Now let's consider the Hilbert spaces of supersymmetric field theories. These will be representations of a super Lie algebra $\mathcal{A}$ extending $\mathrm{Lie}(\mathrm{ISpin}(d-1,1))$, in parallel to Example 1.2.3 above where we met a super extension of $\mathrm{Lie}(\mathrm{ISpin}(0,1))$.

The story has a different flavor depending on the dimension. We will focus on two examples.

Example 1.6.1 ($\mathcal{N} = (2,2)$ supersymmetry in $d = 2$)**.** The supersymmetry algebra $\mathcal{A}$ which we will consider in this case has 4 odd generators,

$$Q^{\pm}, \quad \overline{Q}^{\pm}, \tag{1.6.2}$$

and 6 even generators,

$$P^0, \quad P^1, \quad B, \quad Z, \quad \overline{Z}, \quad F. \tag{1.6.3}$$

Here B is a generator of $\mathfrak{so}(1,1)$. Z and $\overline{Z}$ are central generators, known as central charges. The nonvanishing commutation relations of the odd generators are

$$[Q^+, \overline{Q}^+] = P^+, \quad [Q^+, Q^-] = Z, \tag{1.6.4}$$

$$[Q^-, \overline{Q}^-] = P^-, \quad [\overline{Q}^+, \overline{Q}^-] = \overline{Z}, \tag{1.6.5}$$

where we defined

$$P^{\pm} = P^0 \pm P^1 = H \pm P^1. \tag{1.6.6}$$

The generator F, "fermion number," obeys $[F, Q^{\pm}] = -Q^{\pm}$ and $[F, \overline{Q}^{\pm}] = \overline{Q}^{\pm}$.[5] The only other nonvanishing commutation relations are those between the $\mathfrak{so}(1,1)$ generator B and the translations P (which transform in the fundamental representation) and the odd generators (which are spinors).

Now we want to study unitary $\mathbb{Z}/2\mathbb{Z}$-graded representations of $\mathcal{A}$. As before we start by considering the subalgebra $\mathcal{A}^{\mathrm{rest}}$, generated by $Q^{\pm}$, $\overline{Q}^{\pm}$, P^0, P^1, F, Z, $\overline{Z}$. We fix some $M > 0$ and $Z \in \mathbb{C}$, and consider a representation of $\mathcal{A}^{\mathrm{rest}}$ where $P = (P^0, P^1)$ acts by the character $(M, 0)$ for some $M > 0$, and Z, $\overline{Z}$ act by the scalars Z, $\overline{Z}$. Then the brackets of odd generators become

$$[Q^+, \overline{Q}^+] = M, \quad [Q^+, Q^-] = Z, \tag{1.6.7}$$

$$[Q^-, \overline{Q}^-] = M, \quad [\overline{Q}^+, \overline{Q}^-] = \overline{Z}. \tag{1.6.8}$$

[5] Other variants are possible; for what we do here, all that is really important is that F behaves as "fermion number" in the sense that each odd generator Q shifts it by ± 1.

Now consider the generator

$$Q_\vartheta = \frac{1}{\sqrt{2}}\left(e^{i\vartheta/2}Q^+ + e^{-i\vartheta/2}\overline{Q}^-\right). \tag{1.6.9}$$

This has

$$[Q_\vartheta, \overline{Q}_\vartheta] = M + \frac{1}{2}(e^{i\vartheta}Z + e^{-i\vartheta}\overline{Z}). \tag{1.6.10}$$

We optimize by choosing $\vartheta = \pi - \arg Z$. Then

$$[Q_\vartheta, \overline{Q}_\vartheta] = M - |Z|. \tag{1.6.11}$$

But now

$$(M - |Z|)\|\psi\|^2 = \langle\psi, [Q_\vartheta, \overline{Q}_\vartheta]\psi\rangle = \|Q_\vartheta\psi\|^2 + \|\overline{Q}_\vartheta\psi\|^2 \geqslant 0 \tag{1.6.12}$$

implies the bound (sometimes called BPS bound)

$$M \geqslant |Z|, \tag{1.6.13}$$

which moreover is saturated if and only if

$$Q_\vartheta\psi = \overline{Q}_\vartheta\psi = 0. \tag{1.6.14}$$

The nature of the representations depends on whether the BPS bound (1.6.13) is saturated. Indeed, the relations (1.6.7)-(1.6.8) say that the four Q's generate a Clifford algebra:

- When $M > |Z|$ this Clifford algebra is nondegenerate, and its irreducible representations have dimension $2^{4/2} = 4$.[6]
- When $M = |Z|$ we've seen that Q_ϑ and $\overline{Q}_\vartheta$ act identically as zero, and the Clifford algebra is degenerate: its irreducible representations have only dimension $2^{2/2} = 2$.

By induction as before, these representations of $\mathcal{A}^{\text{rest}}$ can be extended to representations of the whole algebra $\mathcal{A}$, with the following result:

Fact 1.6.15 (1-particle representations of $\mathcal{N} = (2,2)$ supersymmetry in $d = 2$)**.** Given a complex number $Z \neq 0$, a real scalar $M \geqslant |Z|$, and a real scalar f, there is a unique irreducible unitary representation $V^f_{Z,M}$ of $\mathcal{A}$, in which the Casimir operator ρ acts as M^2, the generators $Z, \overline{Z} \in \mathcal{A}$ act as the complex numbers Z, $\overline{Z}$, and the smallest eigenvalue of F is f.

The representations $V^f_{Z,M}$ with $M = |Z|$ are called *short* or *BPS*, while the $V^f_{Z,M}$ with $M > |Z|$ are called *long*. The short representations are separately rigid, but $V^f_{Z,M} \oplus V^{f+1}_{Z,M}$ can deform to a long representation $V^f_{Z,M'}$, in parallel to what we saw earlier in supersymmetric quantum mechanics.

Now let us restrict attention to the case where the eigenvalues of F are all integers.[7] Then we define a new index which counts short representations with signs:

---

[6]Recall that, when n is even, the nondegenerate complex Clifford algebra on a vector space of dimension n has a unique irreducible representation, which has dimension $2^{n/2}$.

[7]In examples which occur in nature one sometimes gets a canonically defined F which does not have integer eigenvalues, but then one can define some other operator $F' = F + \delta$, where δ acts as a scalar

Definition 1.6.16 (Index for representations of $\mathcal{N} = (2,2)$ supersymmetry in $d = 2$)**.**

$$\mu(\mathcal{H}) = \sum_{f\in\mathbb{Z}} (-1)^f (\text{\# copies of } V^f_{Z,M=|Z|} \text{ in } \mathcal{H}) \tag{1.6.17}$$

$$= \mathrm{Tr}_{\mathcal{H}^{1,\mathrm{BPS,rest}}} (-1)^F F. \tag{1.6.18}$$

Remark 1.6.19 (The phase of the central charge)**.** Note that the phase of the complex number Z plays a critical role in this story: it determines *which* 2-dimensional subspace of the 4-dimensional $\mathcal{A}^1$ annihilates the BPS states.

Remark 1.6.20 (Charge gradings)**.** We will typically consider situations in which the Hilbert space $\mathcal{H}$ has a further grading by "charges"

$$\mathcal{H} = \bigoplus_a \mathcal{H}_a \tag{1.6.21}$$

with an additive structure, such that in each sector $\mathcal{H}_a$ the central charge acts by a constant $Z_a \in \mathbb{C}$, and $Z_{a+b} = Z_a + Z_b$. In this situation we can define $\mu(\mathcal{H}_a)$ for each sector separately.

Remark 1.6.22 (Deformation invariance and wall-crossing)**.** Formally it looks like the indices $\mu(\mathcal{H}_a)$ deserve to be invariant under deformations of $\mathcal{H}$, as was the index in Definition 1.2.12, but there is a hitch: as we noted before, in QFT $\mathcal{H}$ always contains continuous spectrum, and this could spoil our arguments. As long as the continuum stays bounded away from $M = |Z|$ we will be safe (we can just restrict to the part of the Hilbert space below the continuum), but if it touches $M = |Z|$ we may be in trouble.

As a rough diagnostic for whether this will happen, we use our earlier remark that the continuous spectrum arises from multiparticle states. Thus let us consider a 2-particle state: take two particles in a common rest frame, and imagine we can neglect their interaction (say by placing them very far apart). This state will have total $M = M_1 + M_2$ and $Z = Z_1 + Z_2$. Now,

$$M = M_1 + M_2 \geqslant |Z_1| + |Z_2| \geqslant |Z_1 + Z_2| = |Z| \tag{1.6.23}$$

with equality iff both particles are BPS *and* $\arg Z_1 = \arg Z_2$. Thus the 2-particle states usually are safely separated from the 1-particle BPS states: the only exception is a state of 2 particles which are themselves BPS and have $\arg Z_1 = \arg Z_2$.

This leads to the expectation that $\mu(\mathcal{H}_a)$ will be invariant under deformations, *except* that it may jump when the central charges Z_b and Z_c of two BPS particles become aligned, where $a = b + c$. This is indeed what one finds in examples (and we will see it in the following lectures). Note that this kind of alignment is a codimension-1 phenomenon in parameter-space; thus what we will find is that the quantities $\mu(\mathcal{H}_a)$ are piecewise constant, but jump at some codimension-1 "walls" in parameter-space.

in each $\mathcal{H}_\alpha$, such that F' does have integer eigenvalues. We will not try to discuss this subtle issue here.

Example 1.6.24 ($\mathcal{N}=2$ supersymmetry in $d=4$)**.** Now we consider 4-dimensional theories with $\mathcal{N}=2$ supersymmetry.

In this case the supersymmetry algebra $\mathcal{A}$ extending Lie(ISpin(3,1)) has 8 odd generators, described as follows. $\mathrm{Spin}(3,1)$ has 2 inequivalent spin representations $S^{\pm}$, both complex and 2-dimensional. Each of $S^{\pm}$ is equipped with an invariant pairing $\cdot$, and there is an intertwiner $\Gamma : S^{+}\otimes S^{-}\to V$, with $V=\mathbb{R}^{3,1}$ the vector representation. The odd generators of $\mathcal{A}$ are Q^1, Q^2 valued in S^+ and $\overline{Q}^1$, $\overline{Q}^2$ valued in S^-.[8] The odd bracket relations are

$$[Q^I(s),Q^J(s')]=(s\cdot s')\epsilon^{IJ}Z,\quad [\overline{Q}^I(s),\overline{Q}^J(s')]=(s\cdot s')\epsilon^{IJ}\overline{Z}, \tag{1.6.25}$$

$$[Q^I(s),\overline{Q}^J(s')]=\delta^{IJ}\Gamma(s,s')P, \tag{1.6.26}$$

Suppressing the spinor and vector indices, we write these more schematically as

$$[Q^I,Q^J]=\epsilon^{IJ}Z,\qquad [\overline{Q}^I,\overline{Q}^J]=\epsilon^{IJ}\overline{Z},\qquad [Q^I,\overline{Q}^J]=\delta^{IJ}P. \tag{1.6.27}$$

The analysis of representations runs parallel to what we did in Example 1.6.1, as follows. We consider the "little algebra" $\mathcal{A}^{\mathrm{rest}}$ generated by Q^I, $\overline{Q}^I$, Z, P and generators of $\mathrm{Spin}(3)\subset\mathrm{Spin}(3,1)$, acting on a unitary representation in which P acts by the character $(M,0,0,0)$ for some $M>0$, and Z acts by some scalar $Z\in\mathbb{C}$.

$\mathrm{Spin}(3)\simeq SU(2)$ has only one spin representation S, which is complex and 2-dimensional. So $S^+\simeq S^-\simeq S$ when considered as representations of Spin(3). After fixing an isomorphism, we could write the odd brackets in $\mathcal{A}^{\mathrm{rest}}$ as

$$[Q^I,Q^J]=\epsilon^{IJ}Z,\qquad [\overline{Q}^I,\overline{Q}^J]=\epsilon^{IJ}\overline{Z},\qquad [Q^I,\overline{Q}^J]=\delta^{IJ}M. \tag{1.6.28}$$

Now, we consider the generators (cf. (1.6.9))

$$Q_\vartheta=\frac{1}{\sqrt{2}}(e^{i\vartheta/2}Q^1+e^{-i\vartheta/2}\overline{Q}^2). \tag{1.6.29}$$

By considering $[Q_\vartheta,\overline{Q_\vartheta}]$, in parallel to what we did in the two-dimensional case, we conclude that $M\geqslant|Z|$ in unitary representations, and if $M=|Z|$, then a 4-dimensional subspace of the 8-dimensional space of odd supercharges acts trivially. Then using the representation theory of Clifford algebras as before, we arrive at the following:

Fact 1.6.30 (1-particle representations of $\mathcal{N}=2$ supersymmetry in $d=4$)**.** Given a complex scalar $Z\neq0$, a real scalar $M\geqslant|Z|$, and an irreducible representation S of $\mathrm{Spin}(3)\simeq SU(2)$, there is a corresponding unitary irreducible representation $V_{Z,M,S}$ of $\mathcal{A}$. The representations with $M=|Z|$ are short or "BPS" while those with $M>|Z|$ are long.

[8] What we mean by "Q is valued in S^+" is that for any $s\in S^+$, there is a corresponding operator $Q(s)$, depending linearly on s. $S^+\oplus S^-$ admits a conjugate-linear involution exchanging the factors, so given $s\in S^{\pm}$ we can write $\overline{s}\in S^{\mp}$; then the adjointness condition means that in unitary representations $Q^I(s)$ is adjoint to $\overline{Q}^I(\overline{s})$, i.e. $\overline{Q(s)}=\overline{Q}(\overline{s})$.

As before, the short representations are individually rigid, but can combine to make long representations. Thus as before we introduce an index which is insensitive to long representations. The only index available is the "second helicity supertrace,"

$$\Omega(\mathcal{H}) = \sum_{n \geqslant 1} (-1)^{n+1} n (\#\text{ copies of } V_{Z,M=|Z|,S_n} \text{ in } \mathcal{H}) \tag{1.6.31}$$

$$= -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}^{1,\mathrm{BPS,rest}}} (-1)^{2J_3} (2J_3)^2 \tag{1.6.32}$$

where S_n denotes the (unique up to isomorphism) irreducible n-dimensional representation of Spin(3), and $J_3 \in \mathfrak{so}(3,1)$ denotes the generator of rotations around the x^3-axis. For example,

$$\Omega(V_{Z,|Z|,S_1}) = +1, \tag{1.6.33}$$

$$\Omega(V_{Z,|Z|,S_2}) = -2. \tag{1.6.34}$$

These two representations are referred to as the massive hypermultiplet and the massive vector multiplet, respectively. We will see a geometric realization of states in these representations in the final lecture.

Just as in the 2-dimensional case, we will study examples in which $\mathcal{H}$ is graded, $\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\gamma$; then the indices $\Omega(\gamma)$ are formally deformation invariant, but they can suffer from wall-crossing when the central charges Z_{γ_1} and Z_{γ_2} of two BPS particles become aligned, with $\gamma = \gamma_1 + \gamma_2$.

2. Lectures 2-3: 2d theories, tt^* geometry and Stokes phenomenon

Now we focus on $\mathcal{N} = (2,2)$ supersymmetric theories in $d = 2$. Such theories come in families parameterized by complex manifolds.[9] These parameter spaces carry a rich geometric structure, revealed in a large body of literature: pioneering early work is Witten [52], Dijkgraaf-Verlinde-Verlinde [7], Cecotti-Vafa [5,6], Dubrovin [12].

2.1. Landau-Ginzburg models We begin by introducing a specific class of $\mathcal{N} = (2,2)$ theories.

Physics Fact 2.1.1. Suppose given a Kähler manifold M and a holomorphic function W on M. There is a 2-dimensional quantum field theory $T_2[M, W]$ called a supersymmetric Landau-Ginzburg model on M, with $\mathcal{N} = (2,2)$ supersymmetry. W is called the superpotential.

In these lectures we will just consider the case where $M = \mathbb{C}^k$ (usually $k = 1$), with its standard metric, and W is a polynomial. We will mainly discuss two examples:

[9] Actually the structure can be more general, but we'll simplify by considering only the "chiral" deformations and not the "twisted chiral" ones.

Example 2.1.2 (Superpotentials for the cubic Landau-Ginzburg model)**.** We take $C = \mathbb{C}$ and for any $z \in C$:

$$W_z(x) = \frac{1}{3}x^3 - zx. \tag{2.1.3}$$

Example 2.1.4 (Superpotentials for the quartic Landau-Ginzburg model)**.** We take $C = \mathbb{C}^2$ and for any $z = (z_1, z_2) \in C$:

$$W_z(x) = \frac{1}{4}x^4 - \frac{1}{2}z_1 x^2 - z_2 x. \tag{2.1.5}$$

We will be interested in how the Landau-Ginzburg model varies as we vary the couplings $z \in C$.

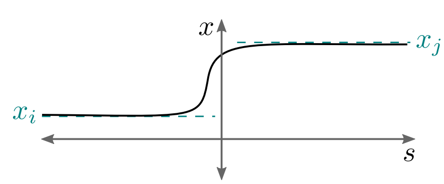
2.2. BPS solitons We now discuss the theory of BPS solitons in the supersymmetric Landau-Ginzburg model, as developed in [6].

Definition 2.2.1 (Classical BPS solitons in Landau-Ginzburg models)**.** Suppose given a superpotential $W(x^1, \dots, x^k)$ which is a holomorphic Morse function (ie all critical points of W are nondegenerate). A *classical BPS soliton with phase* ϑ is a map $x : \mathbb{R} \to \mathbb{C}^k$ obeying

$$\frac{dx^j}{ds} = e^{i\vartheta}\overline{\partial_j W(x(s))}\,. \tag{2.2.2}$$

Given two critical points x_i, x_j of W, an ij-*soliton* is a BPS soliton with

$$\lim_{s\to-\infty} x(s) = x_i, \quad \lim_{s\to+\infty} x(s) = x_j. \tag{2.2.3}$$



Remark 2.2.4 (Motivation for BPS soliton equations)**.** Equation (2.2.2) arises from the classical field theory counterpart of the Landau-Ginzburg model $T_2[\mathbb{C}^k, W]$. When spacetime is taken to be $\mathbb{R}^{1,1}$, the bosonic fields in this classical field theory are maps $x : \mathbb{R}^{1,1} \to \mathbb{C}^k$. Considering time-independent configurations reduces us to maps $x : \mathbb{R} \to \mathbb{C}^k$, now with s interpreted as the spatial coordinate. The full space of fields carries an action of the superalgebra $\mathcal{A}$ from Example 1.6.1, and requiring that the configuration is invariant under the action of half of the odd generators of $\mathcal{A}$ leads to (2.2.2). This is the classical counterpart of the condition that a state in the Hilbert space of the theory is annihilated by half of the odd generators of $\mathcal{A}$, which we recall defines the BPS states. Indeed, classical BPS solitons give rise to BPS states after quantization.

If (2.2.2) is satisfied we have $\frac{dW(x(s))}{ds} = e^{i\vartheta}|\nabla W(x(s))|^2$. Thus composing with the map W projects a classical BPS soliton with phase ϑ to a line segment in $\mathbb{C}$, inclined by an angle ϑ from the positive real axis. An ij-soliton projects to a segment running from $W(x_i)$ to $W(x_j)$. In particular, for an ij-soliton the angle ϑ is determined:

$$e^{i\vartheta} = \frac{W(x_j) - W(x_i)}{|W(x_j) - W(x_i)|}.$$

To construct solutions of (2.2.2) explicitly is difficult, but if we don't keep track of the parameterization it becomes easier, as follows. Fix some w on the segment connecting $W(x_i)$ to $W(x_j)$, and consider the set of "half-solitons," i.e. solutions of (2.2.2) on the half-line $(-\infty, 0]$ which have

$$\lim_{s \to -\infty} x(s) = x_i, \quad W(x(0)) = w. \tag{2.2.5}$$

Now we consider the possible values of $x(0)$ for such half-solitons. One can show that these values sweep out a cycle $V_i(w) \subset \mathbb{C}^k$ homeomorphic to S^{k-1}. As $w \to w_i$, $V_i(w)$ collapses to the point x_i; thus $V_i(w)$ is called a *vanishing cycle*. A parallel construction for solutions on $[0, \infty)$ with $\lim_{s \to +\infty} x(s) = x_j$ gives another vanishing cycle $V_j(w)$.

This description of the ij-solitons motivates a topological approach to counting them, as follows. Both vanishing cycles have real dimension $k-1$ and lie in the fiber $W^{-1}(w)$, which has complex dimension $k-1$. Thus we are in the correct dimension for the topological intersection number of $V_i(w)$ and $V_j(w)$ to be defined. We choose orientations of $V_i(w)$ and $V_j(w)$, and make the following definition:

Definition 2.2.6 (Soliton degeneracies in Landau-Ginzburg models)**.** Given two critical points x_i, x_j of W, we define $\mu(i,j)$ to be the topological intersection number between $V_i(w)$ and $V_j(w)$ in $W^{-1}(w)$, for any w in the interval connecting $W(x_i)$ to $W(x_j)$.

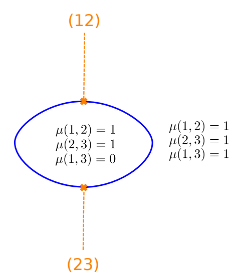
The ij-solitons are in 1-1 correspondence with points of the setwise intersection $V_i(w) \cap V_j(w)$. The topological intersection number $\mu(i,j)$ gives a signed count of these points. Moreover, as was argued in [6], this signed count turns out to agree with the signed count one would obtain by quantizing the classical solutions and taking the BPS index:

Physics Fact 2.2.7. The $\mu(i,j)$ are BPS indices in the $\mathcal{N} = (2,2)$ supersymmetric Landau-Ginzburg model $T_2[\mathbb{C}^k, W]$. (More precisely: the Hilbert space in this theory is divided up into sectors $\mathcal{H} = \bigoplus_{i,j} \mathcal{H}_{ij}$, and $\mu(i,j) = \mu(\mathcal{H}_{ij})$ in the sense of (1.6.17) above.)

So far we have neglected a subtlety: how to choose the orientations of the vanishing cycles $V_i(w)$? In general there is no completely natural way to do it, and thus there is no completely natural definition of an integer $\mu(i,j)$; only $|\mu(i,j)|$ is independent of the choice of orientations and thus canonical. This difficulty in defining the overall sign of $\mu(i,j)$ is related to the subtlety we mentioned in the first lecture, that in general there is no canonical definition of the fermion number F; indeed, shifting $F \to F + 1$ reverses the sign of the BPS index. I will not delve into this here, nor will I try to construct the actual Hilbert spaces of solitons (as opposed to counts of their multiplicities.) A good reference for that material, with vastly more detail than I can give here, is [25].[10]

Example 2.2.8 (Soliton degeneracies in the cubic Landau-Ginzburg model)**.** In this case the structure is very simple: V_i and V_j are distinct 2-element subsets of a 3-element set, so $|V_i \cap V_j|$ is always 1, i.e. for any $z \neq 0$ we have $\mu(1,2;z) = \pm 1$ and $\mu(2,1;z) = \mp 1$.

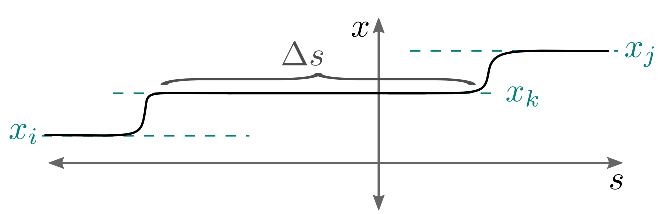
Example 2.2.9 (Soliton degeneracies in the quartic Landau-Ginzburg model)**.** This case is already more interesting: as we change z_1 and z_2 the $\mu(i,j;z_1,z_2)$ can change. The change occurs when three critical values of W become collinear. In one region there are two solitons, in another region there are three; e.g. in the slice $z_1 = -1$, the picture in the z_2-plane has the shape shown in the figure below.



The boundary between the two regions, indicated in blue, is called a *wall of marginal stability*. In this figure we have chosen a specific labeling of the critical points as x_1, x_2, x_3 on the complement of two branch cuts, indicated by orange dashed lines; each cut is labeled by a transposition of the labels.

This is our first concrete example of the wall-crossing phenomenon we mentioned in the first lecture.

What happens to the 13-soliton that disappears at the wall? As we approach the wall from the outside, this 13-soliton starts to look more and more like a combination of a 12-soliton and a 23-soliton separated by a large distance Δs, as indicated in the figure below. This large distance $\Delta s \to \infty$ as z approaches the wall.



[10]To be exact, for the construction of the vector spaces, see Section 12.3, page 266. The basic idea: realize solutions of (2.2.2) as critical points of a certain Morse function in infinite dimensions, then promote them to generators of a Morse complex with differentials given by gradient flows. cf. a simpler example: critical points of a Morse function on a finite-dimensional manifold can be promoted to cohomology vector spaces by the same construction.

2.3. Wall-crossing formula and spectral networks for 2d $\mathcal{N} = (2,2)$ theories The heuristic picture above suggests that the wall-crossing behavior should be governed by a universal law, as follows. Let

$$Z_{ij} = W(x_i) - W(x_j). \tag{2.3.1}$$

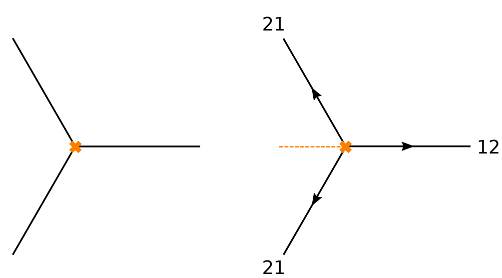
When we cross a locus in parameter-space where Z_{ij} and Z_{jk} become collinear, $\mu(i,k)$ should change by a jump of the form

$$\mu(i,k) \to \mu(i,k) \pm \mu(i,j)\mu(j,k). \tag{2.3.2}$$

This law turns out to be true, not only in supersymmetric Landau-Ginzburg models but in $\mathcal{N} = (2,2)$ theories much more generally; it is the Cecotti-Vafa wall-crossing formula, first derived in [6]. We are going to sketch a route to proving it, but first let's explore how it lets us determine μ in practice.

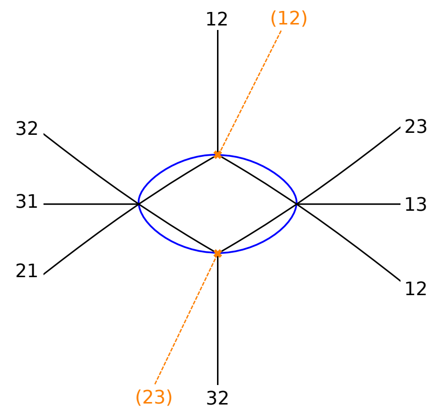
Definition 2.3.3 (Spectral network for 2d $\mathcal{N} = (2,2)$ theory)**.** For any ϑ, the spectral network $\mathrm{SN}(\vartheta) \subset C$ is the set of all $z \in C$ such that there exist some critical points x_i, x_j with $\mu(i,j;z) \neq 0$ and $\arg Z_{ij} = \vartheta$.

Example 2.3.4 (Spectral network for the cubic Landau-Ginzburg model)**.** In this case the critical values are $W(x_i) = \pm\frac{2}{3}z^{\frac{3}{2}}$. There will be a soliton with phase $\vartheta = 0$ just if the two critical values have the same imaginary part, i.e. just when $z^{\frac{3}{2}}$ is real. This gives the picture of $\mathrm{SN}(\vartheta = 0)$:



In the second picture we chose a branch cut for the function $z^{\frac{3}{2}}$ and labeled the two critical values as x_1, x_2. The paths are labeled by the ij type of the soliton with phase $\vartheta = 0$. The orientation of each wall shows the direction in which the mass $M = |W(x_i) - W(x_j)| = \frac{4}{3}|z|^{\frac{3}{2}}$ of the soliton increases.

Example 2.3.5 (Spectral network for the quartic Landau-Ginzburg model)**.** In this case the full spectral network $\mathrm{SN}(\vartheta)$ would be drawn on $C = \mathbb{C}^2$. The figure below shows a slice of $\mathrm{SN}(\vartheta = \frac{\pi}{2})$ as the z_2-plane, with $z_1 = -1$ held fixed:



This figure is drawn in the same plane as the figure in Example 2.2.9. (This spectral network and many similar ones were first drawn in [33].)

In the figure above there is an interesting new feature: some trajectories are "born" at intersection points of other trajectories. This is a manifestation of wall-crossing; indeed the intersection point lies on the wall of marginal stability. If we vary the parameter ϑ, the intersection point moves, sweeping out the whole wall of marginal stability.

How was this picture made? In principle it could have been done by directly analyzing the BPS solitons. In fact, it was done in a simpler way, as follows. After restriction to the one-dimensional slice $z_1 = -1$, one can see directly from Definition 2.3.3 that the paths making up $\mathrm{SN}(\vartheta)$ are solutions of a first-order ODE:

$$dZ_{ij}/dt = e^{i\vartheta}. \tag{2.3.6}$$

We consider the locus where some pair of critical points collide, say $W(x_i) = W(x_j)$. Near this locus, we will have $\mu(i,j;z) = \pm 1$ as in (2.3.4) above. Thus the local structure of $\mathrm{SN}(\vartheta)$ around each such point is 3-pronged as in that example. So first we draw these 3 paths coming out of each such point, and then we use the ODE (2.3.6) to continue the paths from there. When an ij-path and a jk-path in $\mathrm{SN}(\vartheta)$ cross one another, the point of intersection becomes the initial point of an ik-path which is also included in $\mathrm{SN}(\vartheta)$, in the manner indicated in the figure above. On a generic one-dimensional slice of C, this algorithm can be used to determine $\mathrm{SN}(\vartheta)$ for any ϑ, and thus determine all the $\mu(i,j;z)$.

This algorithm is a slight variant of one given in [24] in the context of surface defects in 4-dimensional theories, which we will discuss in the last lecture.

2.4. Chiral rings and vacua

Physics Fact 2.4.1 (Chiral ring)**.** Suppose C is the parameter space of chiral deformations of an $\mathcal{N} = (2,2)$ supersymmetric field theory. Then C carries:

- a holomorphic vector bundle of commutative algebras E over C (chiral rings),
- a holomorphic map of vector bundles $q : TC \to E$.

(The origin of this structure is as follows: E is a certain subspace of the space of local operators in the theory; the map q arises because deformations of the theory are constructed by perturbing the action by local operators.)

Remark 2.4.2 (Vacua)**.** Fix some $z \in C$. Dually, sometimes it is convenient to consider the spectrum Σ_z of the commutative algebra E_z instead of the algebra itself. The points of Σ_z are in 1-1 correspondence with the *vacua* of the theory; as z varies, they sweep out a space $\Sigma \to C$ (in our examples it will be a branched covering).

Remark 2.4.3 (Chiral ring gives a Higgs bundle)**.** If we define $\varphi : TC \to \mathrm{End}(E)$ by $(\varphi(v))(w) = q(v) \cdot w$ then the pair (E, φ) define a Higgs bundle over C[11]. Σ is then also known as the *spectral curve* of this Higgs bundle.

Example 2.4.4 (Chiral rings in Landau-Ginzburg models)**.** Suppose given a family of complex polynomials $W_z(x^1, \dots, x^k)$ parameterized by a space C.[12]

Then the Landau-Ginzburg model gives the following structures:

- E_z is the Jacobian ring,

$$E_z = \mathbb{C}[x^1, \dots, x^k]/\langle \partial_{x^1} W_z, \dots, \partial_{x^k} W_z \rangle. \tag{2.4.5}$$

- The map q takes a vector $\partial_z \in TC$ to $\partial_z W_z \in E$.

Σ_z consists of the critical points of W_z, ie solutions of $\partial_x W_z = 0$.

Example 2.4.6 (Chiral rings in the cubic Landau-Ginzburg model)**.** We consider the Landau-Ginzburg model with a family of potentials parameterized by $C = \mathbb{C}$:

$$W(x) = \frac{1}{3}x^3 - zx. \tag{2.4.7}$$

Then E has a basis $\{1, x\}$ and the algebra structure is given by the relation $x^2 = z$, from which we obtain

$$\varphi(\partial_z) = (-x\cdot) = -\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}. \tag{2.4.8}$$

The vacua make a double cover $\Sigma \to \mathbb{C}$, branched over $z = 0$.

Example 2.4.9 (Chiral rings in the quartic Landau-Ginzburg model)**.** We consider the Landau-Ginzburg model with the potentials parameterized by $C = \mathbb{C}^2$:

$$W(x) = \frac{1}{4}x^4 - \frac{1}{2}z_1 x^2 - z_2 x. \tag{2.4.10}$$

---

[11]This is a Higgs bundle in the same sense as in Laura Schaposnik's lectures, except that here we allow C to be higher-dimensional; then the definition of Higgs bundle requires the extra condition $[\varphi, \varphi] = 0$, which is satisfied here because the algebra structure on E is commutative.

[12]I'm not sure of the precise conditions one should put on this family of polynomials.

Now E has a basis $\{1, x, x^2\}$ and the relation $x^3 = z_1 x + z_2$, so

(2.4.11)

$$\varphi(\partial_{z_1}) = \left(-\frac{1}{2}x^2\cdot\right) = -\frac{1}{2}\begin{pmatrix}0 & z_2 & 0\\ 0 & z_1 & z_2\\ 1 & 0 & z_1\end{pmatrix}, \qquad \varphi(\partial_{z_2}) = (-x\cdot) = -\begin{pmatrix}0 & 0 & z_2\\ 1 & 0 & z_1\\ 0 & 1 & 0\end{pmatrix}.$$

The vacua make a triple cover $\Sigma \to \mathbb{C}^2$, branched over the curve $4z_1^3 = 27z_2^2$.

2.5. The topological connection

Physics Fact 2.5.1 (Topological connection)**.** In addition to the chiral ring structure just discussed, a family of $\mathcal{N} = (2, 2)$ theories also has more (ultimately derived from the possibility of making a family of 2d TFTs by topological twisting): there are

- a symmetric bilinear pairing $\langle , \rangle$ on E such that $\langle a, bc\rangle = \langle ab, c\rangle$,
- a pencil of flat connections $\nabla^\hbar$ in E, of the form

$$\nabla^\hbar = \nabla^\infty + \hbar^{-1}\varphi, \tag{2.5.2}$$

such that $\langle , \rangle$ is compatible with ∇^∞.

This structure is closely related to the notion of Frobenius manifold, but slightly different since we are not identifying E with the tangent bundle of C: q need not be an isomorphism.

Example 2.5.3 (Topological connection in the cubic Landau-Ginzburg model)**.** In the cubic Landau-Ginzburg model, in the basis $\{1, x\}$, the pairing and connections are

$$\langle , \rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \nabla_z^\hbar = \partial_z - \hbar^{-1}\begin{pmatrix}0 & z\\ 1 & 0\end{pmatrix}. \tag{2.5.4}$$

The covariantly constant sections can be written in terms of Airy functions; indeed $\nabla^\hbar$ can be obtained from the Airy equation $(\hbar^2\partial_z^2 + z)\psi(z) = 0$ by the standard device of writing a second-order equation as a matrix first-order equation.

Example 2.5.5 (Topological connection in the quartic Landau-Ginzburg model)**.** In the quartic Landau-Ginzburg model, in the basis $\{1, x, x^2\}$, the pairing and connections are

$$\langle , \rangle = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & z_1\end{pmatrix}, \tag{2.5.6}$$

(2.5.7)

$$\nabla_{z_1}^\hbar = \partial_{z_1} + \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} - \frac{\hbar^{-1}}{2}\begin{pmatrix}0 & z_2 & 0\\ 0 & z_1 & z_2\\ 1 & 0 & z_1\end{pmatrix}, \qquad \nabla_{z_2}^\hbar = \partial_{z_2} - \hbar^{-1}\begin{pmatrix}0 & 0 & z_2\\ 1 & 0 & z_1\\ 0 & 1 & 0\end{pmatrix}.$$

Note that in this example $\nabla^\infty_{z_1} \neq \partial_{z_1}$, i.e., the basis $\{1, x, x^2\}$ is not covariantly constant for ∇^∞.

2.6. Contour integrals A general construction of the desired pairings $\langle \cdot, \cdot \rangle$ and connections $\nabla^\hbar$ in Landau-Ginzburg models exists, going back to work of Saito; see [5, 12, 28, 45] for accounts of aspects of this construction. Here we cannot describe the construction completely, but just touch on a few key features.

The flat connection $\nabla^\hbar$ can be produced by first writing a basis of sections $\psi_i(z)$ of E, then defining $\nabla^\hbar$ to be the connection for which each $\psi_i(z)$ is covariantly constant. The $\psi_i(z)$ in turn can be characterized by their pairing with a general $\alpha \in E_z$. In our examples, this pairing is given by an explicit contour integral in the x-plane,

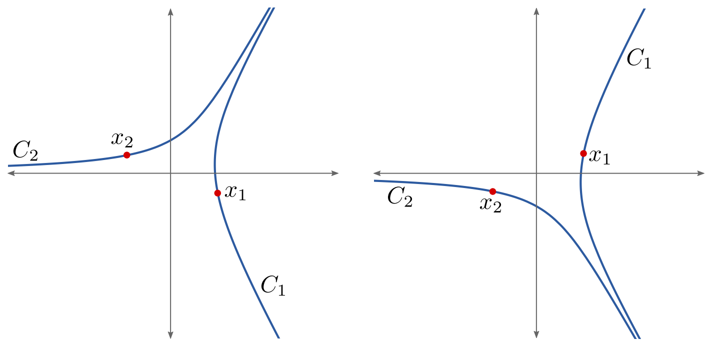
$$\langle \psi_i(z), \alpha \rangle = \int_{C_i} dx \, e^{\hbar^{-1} W_z(x)} \, \alpha(x). \tag{2.6.1}$$

Here $\alpha(x)$ denotes the unique representative for the class α which is a polynomial of degree $\leqslant (\deg W) - 2$ (this choice really matters: the integral (2.6.1) is not invariant under shifting α by a multiple of W_z.) The resulting $\nabla^\hbar$ is indeed of the form (2.5.2); verifying this requires some investigation of the asymptotics of the integrals (2.6.1) as $\hbar \to 0$ and $\hbar \to \infty$.

In this construction we need C_i to have no boundary, but we also need the integral (2.6.1) to be convergent. A systematic way to get a contour C_i is to start from a critical point x_i of W and follow the paths of steepest descent for the integrand $e^{\hbar^{-1} W(x)}$, i.e. paths along which $\mathrm{Im}(\hbar^{-1} W)$ is constant while $\mathrm{Re}(\hbar^{-1} W) \to -\infty$. These are simple examples of Lefschetz thimbles.[13] This procedure yields a well defined contour C_i unless the path of steepest descent runs into another critical point x_j. The latter can happen only if there is a soliton connecting x_i to x_j, with phase $\vartheta = \arg \hbar$. As we vary z across a value where such a soliton exists, the contour C_i jumps.

For example, the figure below shows the contours C_i obtained by steepest descent in the case of the cubic Landau-Ginzburg model with $z = 1 - i$ (left) and $z = 1 + i$ (right), when $\arg \hbar = 0$.

[13] cf. Pavel Putrov's lectures at this school, which involved much more sophisticated examples of Lefschetz thimbles: there the superpotential W was defined on the infinite-dimensional space of connections on a 3-manifold M, and the critical points x_i were flat connections. To connect the two stories directly one should try to find a natural construction of a 2-dimensional $\mathcal{N} = (2,2)$ supersymmetric theory with this W; it could likely be done by starting with 5-dimensional supersymmetric Yang-Mills theory and compactifying it appropriately on M [55].



As we vary z along a straight segment from $1-i$ to $1+i$, the contours C_i deform continuously, except at the moment when z crosses the real axis at $z=1$; at that moment there is a soliton connecting x_1 to x_2 with phase $\vartheta = 0$, and the contour C_2 jumps by the addition of $\pm C_1$ (with the sign depending on how we orient the contours), while C_1 varies continuously. Thus, when z crosses the real axis, the local solutions ψ_i are transformed as $\psi_1 \to \psi_1, \psi_2 \to \psi_2 \pm \psi_1$. This transformation can alternatively be expressed as a unipotent matrix:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.6.2}$$

The upper-right entry of this matrix should be interpreted as the soliton count $\pm\mu(1,2;z=1)$.

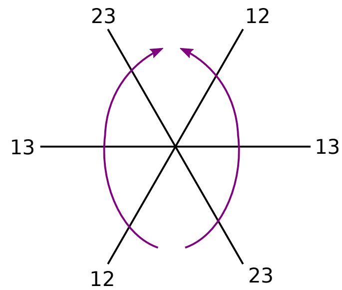
2.7. Spectral network as jumping locus for covariantly constant sections We have arrived at the following picture. In each domain on the complement of the spectral network $SN(\vartheta = \arg\hbar)$ we have a distinguished basis of covariantly constant sections $(\psi_i(z))_{i=1}^N$ for $\nabla^\hbar$. This basis can change when z crosses $SN(\vartheta = \arg\hbar)$.

More precisely, as we move from one domain to another across a wall with label ij, the basis $(\psi_i(z))_{i=1}^N$ is transformed by a unipotent change-of-basis matrix S of the form

$$S = \mathbf{Id} \pm \mu(i,j;z)e_{ij}, \tag{2.7.1}$$

where e_{ij} is the elementary matrix. One can establish (2.7.1) by studying how the contours (Lefschetz thimbles) change as z varies, as we did above in the example of the cubic Landau-Ginzburg model.

2.8. Wall-crossing formula via the spectral network Assuming (2.7.1), we now revisit the wall-crossing formula (2.3.2) for the soliton counts $\mu(i,j;z)$. We consider the changes of basis induced by traveling two different paths in C, as shown in the figure below.



Requiring that these two changes of basis are equal leads to the following relation among the change-of-basis matrices:

$$\begin{pmatrix} 1 & \mu(1,2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mu(1,3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu(2,3) \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu'(2,3) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mu'(1,3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu'(1,2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.8.1}$$

where $\mu(i,j)$ and $\mu'(i,j)$ refer respectively to the values of $\mu(i,j;z)$ for z on the left and right sides of the figure. This equation holds just if

$$\mu'(1,2) = \mu(1,2), \quad \mu'(2,3) = \mu(2,3), \quad \mu'(1,3) = \mu(1,3) + \mu(1,2)\mu(2,3), \tag{2.8.2}$$

which is the Cecotti-Vafa wall-crossing formula (2.3.2), now interpreted as a consequence of the existence of the sections $\psi_i(z)$.

Remark 2.8.3 (Stokes phenomenon)**.** The construction of the $\psi_i(z)$ by contour integration allows one to determine their complete asymptotic series expansion as $\hbar \to 0$ with $\arg \hbar = \vartheta$, by the method of steepest descent (eg the leading behavior is like $\exp(W(x_i)/\hbar)$.) This asymptotic series only depends on what happens very near the critical point; in particular, it does not change when the contours C_i jump. (Another way of saying this is that when the jump occurs, it shifts the contour by adding the contribution of a contour through a subdominant critical point, which thus doesn't change the leading asymptotic.)

Thus the local solutions $\psi_i(z)$ have the nice property that they have a simple uniform asymptotic expansion as $\hbar \to 0$ with $\arg \hbar = \vartheta$. This is very valuable if one wants to use them to study the parallel transport of $\nabla^\hbar$. On the other hand, it turns out that this property alone is already enough to imply that the $\psi_i(z)$ cannot be global analytic objects: they have to jump somewhere. As we saw, they do jump, at the locus $\mathrm{SN}(\vartheta)$. This is (a manifestation of) the Stokes phenomenon, which might loosely be described as a conflict between analyticity and uniform asymptotics. For this reason, $\mathrm{SN}(\vartheta)$ is sometimes called the *Stokes graph*.

2.9. tt^* **geometry** Another crucial discovery of Cecotti-Vafa [5] is that $\mathcal{N} = (2,2)$ theories have a canonical metric structure, obeying a highly constrained system of PDEs. This structure has been extensively studied in both the physics and mathematics literature, under various names; see e.g. [6,11,29,44]. Our treatment here will be very brief.

Definition 2.9.1 (tt^* structure)**.** A tt^* *structure* over C is a chiral ring structure plus a Hermitian metric h in E, obeying the tt^* *equations*,

$$F_{D_h} + [\varphi, \varphi^{\dagger_h}] = 0, \qquad \partial_{D_h}\varphi = 0. \tag{2.9.2}$$

Here D_h is the Chern connection uniquely determined by compatibility with both the holomorphic structure and the Hermitian metric in E.

Remark 2.9.3 (tt^* equations and Hitchin's equation)**.** The second equation in (2.9.2) expresses the vanishing of a $(2,0)$-form on C. When C has complex dimension 1, this equation holds automatically, and so (2.9.2) reduces to

$$F_{D_h} + [\varphi, \varphi^{\dagger_h}] = 0, \tag{2.9.4}$$

which is also called *Hitchin's equation* (the same equation that appeared in Laura Schaposnik's lectures at this school). More generally, we can view the tt^* equations (2.9.2) as a natural higher-dimensional extension of Hitchin's equation.

Physics Fact 2.9.5 (Existence of tt^* structures [5])**.** Suppose C is the parameter space of chiral deformations of an $\mathcal{N} = (2,2)$ supersymmetric field theory. Then C carries a tt^* structure.

Example 2.9.6 (The tt^* structure for the cubic Landau-Ginzburg model)**.** In the cubic Landau-Ginzburg model of Example 2.1.2 there is a tt^* structure for which the metric h is diagonal: making the ansatz

$$h = \begin{pmatrix} e^{-u} & 0 \\ 0 & e^{u} \end{pmatrix} \tag{2.9.7}$$

the tt^* equations reduce to

$$\partial_z \partial_{\bar{z}} u - (e^{2u} - e^{-2u}|z|^2) = 0. \tag{2.9.8}$$

With an appropriate boundary condition as $|z| \to \infty$, this equation has a unique smooth solution $u(|z|)$, expressible in terms of Painleve transcendents [5]. This turns out to be a very useful "model solution" for studying the limiting behavior of solutions of Hitchin's equation when one goes to infinity in moduli space (see e.g. [20,38].)

Definition 2.9.9 (The improved connections)**.** We define a family of connections ∇^ζ in the bundle E over C by

$$\nabla^\zeta = D_h + \zeta^{-1}\varphi + \zeta\varphi^{\dagger_h}. \tag{2.9.10}$$

The tt^* equations imply that ∇^ζ is *flat* for every $\zeta \in \mathbb{C}^\times$.

The picture we described above for the topological connections $\nabla^{\hbar}$ holds equally for the connections ∇^{ζ}: there are covariantly constant sections in each domain, given by a formula like (2.6.1),

$$\langle \psi_i(z), \alpha \rangle = \int_{C_i} dx\, e^{\zeta^{-1} W_z(x) + \zeta \overline{W_z(x)}} \alpha(x), \tag{2.9.11}$$

but in this case the representatives $\alpha(x)$ we choose don't have a simple explicit expression. Nevertheless the formal structure is the same; in particular, the basis sections $\psi_i(z)$ also transform by (2.7.1) at walls.

Although the $\psi_i(z)$ do not admit a simple explicit formula, they can be determined from the data of $W_z(x)$ together with the soliton counts $\mu(i,j;z)$. The idea is that these data are sufficient to determine the analytic properties of $\psi_i(z)$ in the ζ-plane, and these properties in turn identify $\psi_i(z)$ as the solution of a certain integral equation. Once the $\psi_i(z)$ are determined, they determine the connections ∇^{ζ}, and finally these connections determine the metric h; thus one obtains a solution of the tt^* equations. This strategy is developed in [6, 11].

3. Lecture 4: 4d theories and spectral networks

We have discussed some rich structure which appears on C when C is a moduli space of $\mathcal{N} = (2,2)$ supersymmetric field theories: in particular C carries

- a Higgs bundle (E, φ),
- two distinguished families of flat connections $\nabla^{\hbar}$ and ∇^{ζ} in E,
- BPS state indices and a spectral network, which control the Stokes phenomena of these connections.

The kind of C we met so far were very simple: just vector spaces. However, it turns out that this kind of structure exists far more generally: for example, we can take C to be any compact Riemann surface. Thus we will obtain a physical construction which leads naturally to Higgs bundles and flat connections over Riemann surfaces.

One of the payoffs of this kind of construction is that it gives new tools to study the moduli space of Higgs bundles on C; still, for this lecture we mainly focus on one Higgs bundle at a time.

3.1. Class S and surface defects The physical underpinnings of the construction depend on facts we mentioned in Example 1.4.2, also mentioned in Pavel Putrov's last lecture at this school:

Physics Fact 3.1.1. There is a 6-dimensional supersymmetric QFT $T_6[\mathfrak{g}]$, depending only on the data of an ADE Lie algebra $\mathfrak{g}$ (see e.g. [46, 53, 54] for some discussion of this theory). This theory supports supersymmetric 2-dimensional defects, labeled by irreducible finite-dimensional representations R of $\mathfrak{g}$.

Theory $T_6[\mathfrak{g}]$ is sometimes referred to as "the six-dimensional $(2,0)$ superconformal field theory."

Now suppose given a Riemann surface C. We consider the theory $T_6[\mathfrak{g}]$ on the spacetime $C \times \mathbb{R}^{3,1}$, in the limit where the metric on C is rescaled to be extremely small. In this way we obtain a 4-dimensional QFT $T_4[\mathfrak{g}, C]$, described in [17, 20]. These theories are known as "theories of class S."

From now on we will specialize to $\mathfrak{g} = \mathfrak{sl}_N$ or $\mathfrak{g} = \mathfrak{gl}_N$. Next we place a surface defect of the theory $T_6[\mathfrak{g}]$ on the subspace $\{z\} \times \mathbb{R}^{1,1} \subset C \times \mathbb{R}^{3,1}$, taking R to be the fundamental (N-dimensional) representation of $\mathfrak{g}$. In this way we obtain a 2-dimensional surface defect in the theory $T_4[\mathfrak{g}, C]$, depending on the parameter $z \in C$. We call the combined 2d-4d system $T_{4+2}[\mathfrak{g}, C, z]$.

As far as the symmetry algebra goes, the 2d-4d system $T_{4+2}[\mathfrak{g}, C, z]$ behaves just like an ordinary 2d $\mathcal{N} = (2, 2)$ supersymmetric theory; the introduction of the surface defect reduces the $ISO(3, 1)$ symmetry to $ISO(1, 1)$, and also reduces the 8 supercharges to 4. Thus we have achieved our goal of realizing C as a moduli space of 2d $\mathcal{N} = (2, 2)$ supersymmetric QFTs.

3.2. Chiral rings The chiral ring of $T_{4+2}[\mathfrak{g}, C, z]$ is infinite-dimensional, essentially because it is a module over the infinite-dimensional chiral ring of the bulk theory $T_4[\mathfrak{g}, C]$. The latter is identified with the space of functions on the Coulomb branch of $T_4[\mathfrak{g}, C]$. This Coulomb branch can be identified with the base of Hitchin's integrable system (see e.g. [20]):

Definition 3.2.1 (Hitchin base)**.** Fix a Riemann surface C and a Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$ or $\mathfrak{g} = \mathfrak{sl}_N$. The *Hitchin base* $\mathcal{B}(C, \mathfrak{g})$ is

$$\mathcal{B}(C, \mathfrak{g}) = \begin{cases} \bigoplus_{n=1}^{N} H^0(C, K_C^n) & \text{for } \mathfrak{g} = \mathfrak{gl}_N, \\ \bigoplus_{n=2}^{N} H^0(C, K_C^n) & \text{for } \mathfrak{g} = \mathfrak{sl}_N. \end{cases} \tag{3.2.2}$$

Thus, a point of $\mathcal{B}(C, \mathfrak{g})$ is a tuple $(\phi_1, \phi_2, \ldots, \phi_N)$ where ϕ_n is a holomorphic n-differential on C (locally $\phi_n = f(z)dz^n$), and $\phi_1 = 0$ when $\mathfrak{g} = \mathfrak{sl}_N$.

Instead of considering the full chiral ring we consider a quotient, obtained by specializing to a point $u \in \mathcal{B}(C, \mathfrak{g})$. In physical terms this corresponds to fixing a vacuum of the theory $T_4[\mathfrak{g}, C]$ on its Coulomb branch. This reduced chiral ring is finite-dimensional and will be a close analog of the chiral rings we had in the purely 2-dimensional case.

Physics Fact 3.2.3 (Reduced chiral ring for surface defect in class S theory)**.** Fix C, $\mathfrak{g}$, $z \in C$, and $u \in \mathcal{B}(C, \mathfrak{g})$. The reduced chiral ring R_z is (see e.g. [18, 22])

$$R_z = \mathrm{Sym}^*(T_z C)/I \tag{3.2.4}$$

where I is an ideal depending on z and u, generated by $\sigma^N(1 + \phi_1 + \cdots + \phi_N)$, where σ is any element of degree 1 and thus $\sigma^N \phi_i$ is an element of degree $N - i$. In local coordinates, we would represent this as follows: write each $\phi_i = P_i(z)dz^i$, trivialize $T_z C$ by the generator $\sigma = \partial_z$, and then

$$R_z = \mathbb{C}[\sigma]/\langle \sigma^N + P_1 \sigma^{N-1} + \cdots + P_N \rangle. \tag{3.2.5}$$

The map $q : T_z C \to R_z$ is then the obvious embedding, $\partial_z \mapsto \sigma$.

The spectrum of R_z consists generically of N points, which are identified as the vacua of the theory living on the surface defect. As z varies over C, these vacua give a branched N-fold covering of C, explicitly

$$\Sigma_u = \{\lambda^N + \phi_1 \lambda^{N-1} + \cdots + \phi_N = 0\} \subset T^*C. \tag{3.2.6}$$

Also as in the 2d case, q induces a Higgs field φ valued in $\mathrm{End}(R) \otimes K_C$, so that R is naturally a Higgs bundle; then Σ_u is the spectral curve.

As a holomorphic bundle, R is $\mathcal{O} \oplus TC \oplus TC^2 \oplus \cdots \oplus TC^{N-1}$. There is a closely related bundle E over C, consisting of the ground states of the surface defect. That one differs from R by a global twist:

$$E = R \otimes K_C^{\frac{N-1}{2}} = K_C^{-\frac{N-1}{2}} \oplus \cdots \oplus K_C^{\frac{N-1}{2}}. \tag{3.2.7}$$

Then E is an $\mathfrak{sl}_N$-Higgs bundle, lying in the "Hitchin section" described in Laura Schaposnik's lectures.

3.3. BPS solitons As in the pure 2d case, we are interested in understanding the physics of BPS particles living on the surface defect. Unlike the Landau-Ginzburg model, here we won't give a direct description of the BPS particles in terms of solving some equation along the string. Nevertheless we continue to use the term "soliton" for these BPS particles.

3.4. Spectral networks One can compute the spectrum of BPS solitons in the surface defect system using a spectral network, in parallel to what we did in the Landau-Ginzburg model. Here we just summarize what the spectral network looks like, formulating it as a definition; for the physical background, the precise interpretation in terms of solitons, and the computation of BPS soliton counts $\mu(i, j, a)$ generalizing the $\mu(i, j)$ from above, see [24].[14]

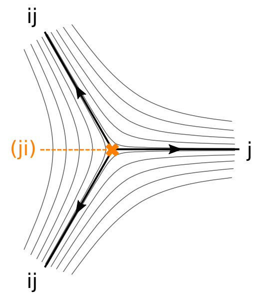
Definition 3.4.1 (Spectral network for class S theory)**.** Fix C, $\mathfrak{g} = \mathfrak{gl}_N$ or $\mathfrak{sl}_N$, and $u \in \mathcal{B}(C, \mathfrak{g})$. Label the sheets of the spectral cover $\Sigma_u \to C$ by a (locally defined) index $i = 1, \ldots, N$; they correspond to holomorphic 1-forms λ_i on C. Define an ij*-trajectory with phase* ϑ on C to be a path along which

$$e^{-i\vartheta} \int \lambda_i - \lambda_j \in \mathbb{R}. \tag{3.4.2}$$

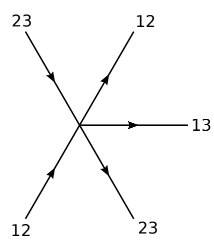
Locally the ij-trajectories give a foliation of C.

For generic enough u, the spectral curve Σ_u has only simple branch points. In this case the spectral network $\mathrm{SN}(\vartheta)$ on C can be constructed by the same algorithm we used in the pure 2d case, as follows. Around each simple branch point, the foliation by ij-trajectories has a 3-pronged singularity, just like the 3-pronged singularities we encountered in the spectral networks for Landau-Ginzburg models; see the figure below.

[14]The charges of solitons in the 2d-4d setting are a bit more elaborate than they were in the pure 2d case. In the pure 2d case, the only data specifying the soliton charge was the pair (i, j) of vacua. In the surface defect case, the data is (i, j, a), where $i, j \in \Sigma_u$ are two vacua of the surface defect i.e. preimages of $z \in C$, and a is (roughly) a path from i to j on Σ_u.



We shoot three ij-trajectories from each ij-branch point, and evolve them following the differential equation (3.4.2). When an ij-trajectory crosses a jk-trajectory we create a new ik-trajectory born from the intersection point, and include it in $SN(\vartheta)$ as well, as indicated in the figure below (again following the same pattern we saw in the 2d case, eg in Example 2.3.5.)



If Σ_u has non-simple branch points, then in order to construct $SN(\vartheta)$ we need to somehow determine the local structure around each branch point. Various examples have been studied, e.g. in [24, 27, 32, 33], but as far as I know, a completely systematic picture has not yet been developed. At any rate, once we have somehow determined the local structure around the branch points, the rest of the construction can proceed as described above.

Remark 3.4.3 (Spectral networks in the $\mathfrak{sl}_2$ case)**.** When $\mathfrak{g} = \mathfrak{sl}_2$, all this becomes simpler:

- The data of u reduces to a holomorphic quadratic differential ϕ_2.
- The branch points of Σ_u are the zeroes of ϕ_2.
- The ij-trajectories are paths along which $e^{-i\vartheta}\sqrt{\phi_2}$ is real. These are also called ϑ-trajectories of ϕ_2. They make up a single, global, singular foliation of the curve C. They are well-studied objects; e.g. they are the main subject of the book [49].

In this case $SN(\vartheta)$ is also known as the critical graph of $e^{-2i\vartheta}\phi_2$.

3.5. Adding punctures If you try to apply the above construction of $SN(\vartheta)$ in an example, say taking C to be a compact genus 2 surface, you run into a problem immediately: a typical ij-trajectory will wind around the surface with dense image, and thus $SN(\vartheta)$ will be dense on C. In principle one expects that all our analysis can be made to work even in this setting, but it is not yet fully understood (see [14] for some work on spectral networks in this setting).

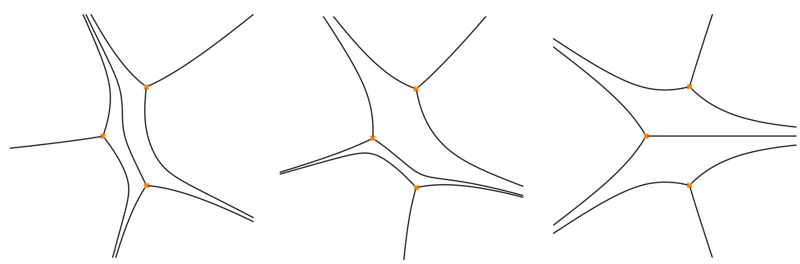
To avoid this problem we now add one extra ingredient: we allow C to have punctures, and make the differentials ϕ_n meromorphic instead of holomorphic, with poles at the punctures. The definition of the theory then includes additional parameters controlling the singular behavior of the ϕ_n at the punctures. For us, the main virtue of including these punctures is that if the poles have sufficiently high order, then the punctures will generically attract the ij-trajectories, making the pictures and the analysis much simpler.

In the physical context, adding a puncture at $p \in C$ corresponds to adding a 4-dimensional defect of theory $T_6[\mathfrak{g}]$ along the locus $\{p\} \times \mathbb{R}^{3,1} \subset C \times \mathbb{R}^{3,1}$.

Example 3.5.1 (Spectral networks in Argyres-Douglas theory)**.** We consider the case[15] where $\mathfrak{g} = \mathfrak{sl}_2$, $C = \mathbb{CP}^1$ with a puncture at $p = \infty$, and

$$\phi_2 = (z^3 + \Lambda z + u)\, dz^2. \tag{3.5.2}$$

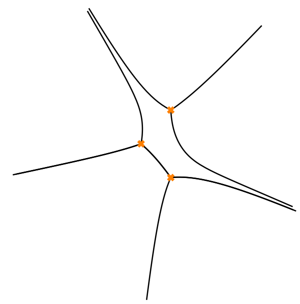
The figure below shows $SN(\vartheta)$ when we fix $\Lambda = 1$ and $u = 1$, and $\vartheta = \frac{\pi}{10}$ (left), $\vartheta = \frac{\pi}{5}$ (middle), $\vartheta = \frac{\pi}{2}$ (right).



Remark 3.5.3 (Role of parameters)**.** The parameters Λ and u in Example 3.5.1 above play different roles. Λ is a parameter which enters the definition of the theory; it should be thought of as like a complex structure modulus associated to the puncture at $z = \infty$, despite the fact that naively a once-punctured $\mathbb{CP}^1$ has no moduli. (This point of view has been emphasized by Boalch, e.g. [2].) The Hitchin base in this case depends on Λ; we write it as $\mathcal{B}(C, \Lambda, \mathfrak{sl}_2)$. The parameter u is then a (global) coordinate on $\mathcal{B}(C, \Lambda, \mathfrak{sl}_2)$.

3.6. BPS indices in the $\mathfrak{sl}_2$ case Comparing the pictures of spectral networks $SN(\vartheta)$ in the figure above, one notices that the topology of $SN(\vartheta)$ abruptly changes at certain critical phases — e.g. by comparing the left and middle panes one sees that there must be such a phase between $\vartheta = \frac{\pi}{10}$ and $\vartheta = \frac{\pi}{5}$. Looking closer one finds that the critical phase is $\vartheta_c \approx \frac{4\pi}{25}$. $SN(\vartheta)$ for $\vartheta \approx \vartheta_c$ is shown in the figure below:

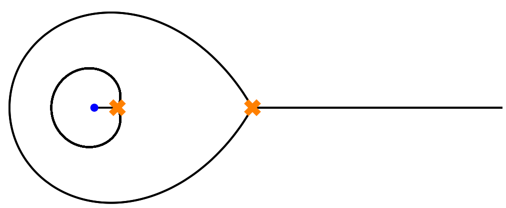
[15] More generally, a very nice class of examples arises from spectral curves obtained by perturbing the equation $y^n = z^m$ by lower-degree terms; physically these are associated with a QFT called the Argyres-Douglas theory of type (A_{n-1}, A_{m-1}). The original Argyres-Douglas theory, studied in [1], is the (A_1, A_2) example. The $(\mathfrak{g}, \mathfrak{g}')$ taxonomy was introduced in [4].



At $\vartheta = \vartheta_c$, $\mathrm{SN}(\vartheta)$ contains a saddle connection:

Definition 3.6.1 (Saddle connection)**.** Given a holomorphic quadratic differential ϕ_2, a *saddle connection of* ϕ_2 *with phase* ϑ is a ϑ-trajectory of ϕ_2, with its two ends on two distinct zeroes of ϕ_2.

Another kind of ϑ-trajectory which could occur in $\mathrm{SN}(\vartheta)$ at special phases ϑ is a loop, which begins and ends at the same branch point. See the figure below, which shows $\mathrm{SN}(\vartheta = 0)$ for the quadratic differential $\phi_2 = \left(z^{-3} - 3z^{-2} + z^{-1}\right) dz^2$.



In this figure we see two loops, which are the boundaries of a ring domain:

Definition 3.6.2 (Ring domain)**.** Given a holomorphic quadratic differential ϕ_2, a *ring domain of* ϕ_2 *with phase* ϑ is an annulus or punctured disc on C foliated by ϑ-trajectories of ϕ_2.

This phenomenon occurs generally: whenever we have a closed loop it is the boundary of a ring domain. As with saddle connections, the existence of a ring domain with phase ϑ implies a topology change for $\mathrm{SN}(\vartheta)$ at that phase, but one of a more complicated kind; see [20] for a discussion of what happens near such a phase, and some examples.

Now, how should we interpret these topology changes? Recall that $\mathrm{SN}(\vartheta)$ captures counts of BPS soliton states. It follows that discontinuous changes in $\mathrm{SN}(\vartheta)$ are related to processes where counts of BPS soliton states change. These processes are hard to interpret in purely 2-dimensional terms: unlike the wall-crossing we discussed in pure 2d theories above, they are not related to decay of one soliton into other solitons, or the reverse. Rather, it turns out that they are processes where the soliton decays into another soliton plus a BPS particle in the ambient four-dimensional theory $T_4[\mathfrak{g}, C]$, or the reverse.[16] Indeed, saddle connections and ring domains are associated with BPS particles in the 4d theory $T_4[\mathfrak{sl}_2, C]$:

Physics Fact 3.6.3 (BPS particles in theories $T_4[\mathfrak{sl}_2, C]$)**.** Consider theory $T_4[\mathfrak{sl}_2, C]$, at a point u of its Coulomb branch corresponding to the quadratic differential ϕ_2.

- Saddle connections of ϕ_2 correspond to massive hypermultiplets of this theory, which thus contribute $+1$ to the BPS index, as in (1.6.33).
- Ring domains of ϕ_2 correspond to massive vectormultiplets of this theory, which thus contribute -2 to the BPS index, as in (1.6.34).

This observation was first made in [35], and further developed in various works, e.g. [20, 40, 41, 47].

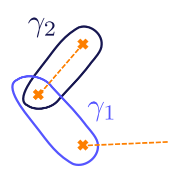
Each of these 4d BPS particles carries an electromagnetic (and perhaps flavor) charge, which we now describe:

Physics Fact 3.6.4 (Electromagnetic/flavor charge lattice)**.** Let

$$\Gamma = H_1(\Sigma_u, \mathbb{Z}). \tag{3.6.5}$$

Γ is the lattice of electromagnetic and flavor charges in the theory $T_4[\mathfrak{g} = \mathfrak{gl}_N]$ at the point u of its Coulomb branch.

Example 3.6.6 (Charge lattice in the Argyres-Douglas (A_1, A_2) theory)**.** In the example of Example 3.5.1, at $u = 1$ and $\Lambda = 1$, the charge lattice is generated by cycles γ_1 and γ_2 shown in the figure below.



Definition 3.6.7 (Charge of a saddle connection or ring domain)**.** For each saddle connection with phase ϑ one can define a class $\gamma \in \Gamma$, represented by a lift to Σ_u of a loop around the saddle connection. The orientation of γ is fixed by the condition that $e^{-i\vartheta} \oint_\gamma \lambda$ is a negative real number. We call γ the *charge* of the saddle connection. Likewise for each ring domain of ϕ_2 with phase ϑ there is a charge $\gamma \in \Gamma$ given by the sum of the two lifts of a generic curve in the ring, with opposite orientations, chosen such that $e^{-i\vartheta} \oint_\gamma \lambda$ is negative.

Physics Fact 3.6.8 (Central charges of 4d particles)**.** A particle with electromagnetic charge $\gamma \in \Gamma$ has central charge given by

$$Z_\gamma = \oint_\gamma \lambda, \tag{3.6.9}$$

[16] These processes are captured by a more elaborate wall-crossing formula, the "2d-4d wall-crossing formula" described in [22], which is a hybrid between the Cecotti-Vafa [6] and Kontsevich-Soibelman [36] formulas.

where λ denotes the standard Liouville 1-form "$p\,dq$" on T^*C.

Note that a saddle connection with charge γ always has phase $\vartheta = \arg(-Z_\gamma)$. Physics Fact 3.6.3 above motivates the following definition:

Definition 3.6.10 (BPS indices in theories $T_4[\mathfrak{sl}_2, C]$)**.** For $\mathfrak{g} = \mathfrak{sl}_2$, fixed C and ϕ_2 we define

$$\Omega(\gamma) = (\#\text{ saddle conn. of charge } \gamma) - 2(\#\text{ ring domains of charge } \gamma). \tag{3.6.11}$$

Example 3.6.12 (BPS indices in Argyres-Douglas theory)**.** In Example 3.5.1, the BPS indices at $u = 1$, $\Lambda = 1$ are [20,47]

$$\Omega(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \{\gamma_1, \gamma_2, \gamma_1+\gamma_2, -\gamma_1, -\gamma_2, -\gamma_1-\gamma_2\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.6.13}$$

while at $u = 0$, $\Lambda = 1$ they are

$$\Omega(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6.14}$$

This is another example of a wall-crossing phenomenon, where the index that jumps is the 4d index $\Omega(\gamma)$ as a function of u. These jumps are governed by the Kontsevich-Soibelman formula [36] (in this case and also in general for 4d $\mathcal{N} = 2$ theories; see e.g. [8,20,23,37] for more about this).

Remark 3.6.15 (Charge conjugation symmetry)**.** Any saddle connection with phase ϑ is also a saddle connection with phase $\vartheta + \pi$, and similarly for ring domains. This fact implies a symmetry

$$\Omega(\gamma) = \Omega(-\gamma). \tag{3.6.16}$$

Remark 3.6.17 (BPS indices as Donaldson-Thomas invariants)**.** BPS indices in $\mathcal{N} = 2$ supersymmetric QFTs have various structural properties in common with generalized Donaldson-Thomas invariants associated to 3-Calabi-Yau categories — in particular, they obey the same wall-crossing formula. It is natural to conjecture that for each $\mathcal{N} = 2$ theory there is some associated 3-Calabi-Yau category, so that the BPS indices are literally equal to the generalized Donaldson-Thomas invariants. This proposal is particularly natural in cases where the $\mathcal{N} = 2$ theory can be realized by compactifying Type IIA/B string theory on a noncompact Calabi-Yau threefold X; then the realization of BPS states as D-branes leads to the idea that the category in question is some "compactly supported" piece of the derived category of coherent sheaves on X (for Type IIA), or the Fukaya category of X (for Type IIB).

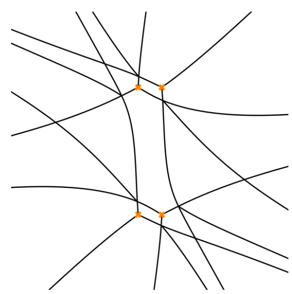
In general this proposal has not been proven, even in cases where the BPS indices can be mathematically defined. However, for the theories $T_4[\mathfrak{sl}_2, C]$ where C has punctures, the situation is much better, thanks to work of Bridgeland-Smith [3], which shows that the invariants defined in Definition 3.6.10 really are generalized Donaldson-Thomas invariants for a certain 3-Calabi-Yau category. Moreover, Smith showed [48] that this category is an appropriate piece of the Fukaya category of the expected Calabi-Yau threefold.

3.7. BPS particles in higher rank cases So far we have focused on $\mathfrak{g} = \mathfrak{sl}_2$. For more general $\mathfrak{g}$ the formal structure is very similar to the $\mathfrak{sl}_2$ case, but exploration of examples reveals that the actual phenomena which occur are much richer, and much harder to classify. Here we very briefly discuss some examples of the new phenomena which can occur in this case.

Example 3.7.1 (Argyres-Douglas theories of type (A_2, A_1))**.** We consider the case $C = \mathbb{CP}^1$, $\mathfrak{g} = \mathfrak{sl}_3$, and

$$\phi_2 = \Lambda \, dz^2, \qquad \phi_3 = (z^2 + u) \, dz^3. \tag{3.7.2}$$

The figure below shows an example of $SN(\vartheta)$, at $u = 1$, $\Lambda = 1$, $\vartheta = \frac{7\pi}{25}$.



In this case, varying the phase ϑ we again see topology changes in the network $SN(\vartheta)$, but these are in general not associated with saddle connections or ring domains: rather they are sometimes associated with more general webs, built from ij-trajectories for several different values of i, j. These more general webs correspond to BPS particles in the theory $T_4[\mathfrak{g}, C]$, just as did the saddle connections and ring domains in the $\mathfrak{g} = \mathfrak{sl}_2$ case. See [24] for examples.

In general, the kinds of degenerations that can occur in $SN(\vartheta)$ for $\mathfrak{g} \neq \mathfrak{sl}_2$ are rather complicated, and the definition of the BPS index $\Omega(\gamma)$ is not as simple as Definition 3.6.10 above. An algorithm for computing $\Omega(\gamma)$ in the general case $\mathfrak{g} = \mathfrak{sl}_N$ is given in [24].

3.8. Families of flat connections Just as in the pure 2d case, the Higgs bundle E over C carries a Hermitian metric h solving the tt^* equations, which now reduce to the Hitchin equations since we consider one-dimensional C. Thus one has a family of flat connections ∇^{ζ} ($\zeta \in \mathbb{C}^{\times}$) in E, given as before by (2.9.10). These connections are hard to write down explicitly since we do not have an explicit form for the metric h.

There is also a family $\nabla^{\hbar}$ of flat connections which are the analogue of the topological connections in the pure 2d case; these connections are also known as opers over C. Unlike ∇^{ζ}, we can sometimes write $\nabla^{\hbar}$ in completely explicit terms.

For example, in Example 3.5.1 above, we have

$$\nabla^{\hbar} = \partial - \hbar^{-1} \begin{pmatrix} 0 & P_2 \\ 1 & 0 \end{pmatrix} dz, \qquad P_2(z) = z^3 + \Lambda z + u. \tag{3.8.1}$$

In this case the $\nabla^{\hbar}$-flatness equation is equivalent to a (complexified) Schrödinger equation,

$$(\hbar^2 \partial_z^2 + P_2(z))\psi(z) = 0. \tag{3.8.2}$$

This is a close analogue of the way the Airy equation arose as the topological connection in the cubic Landau-Ginzburg model (2.5.3).

3.9. Abelianization One can analyze the flat connections $\nabla^{\hbar}$, ∇^{ζ} by a method which is parallel to what we described in the 2d case, but a bit subtler: in the 2d case we described a single distinguished basis of covariantly constant sections in each domain of $C \setminus \mathcal{W}(\vartheta)$; in the 2d-4d case we only expect to have a basis of covariantly constant sections up to a rescaling ambiguity. This new ambiguity ultimately has to do with the nontriviality of $H_1(\Sigma_u, \mathbb{Z})$, or more physically, with the contributions from the 4d BPS particles we discussed above.

One way to formalize the situation is as follows (see e.g. [24, 30, 31]):

Definition 3.9.1 (Almost-flat connections)**.** Suppose u is generic, so that the spectral curve Σ_u is smooth. Let $\Delta \subset \Sigma_u$ be the branch locus of the projection $\pi : \Sigma_u \to C$, and $\Sigma'_u = \Sigma_u \setminus \Delta$. An *almost-flat* connection over Σ_u is a flat connection over Σ'_u, such that the holonomy around any branch point is -1.

Conjecture 3.9.2 (Abelianization for $\nabla^{\hbar}$)**.** *Suppose u is generic. There exists a family of almost-flat $\mathrm{GL}(1)$-connections $\nabla^{\hbar}_{\mathrm{ab}}$ over the spectral cover Σ_u which "abelianize" $\nabla^{\hbar}$, in the following sense:*

(1) *On $C \setminus \mathrm{SN}(\vartheta = \arg \hbar)$, there is an isomorphism $\iota : \nabla^{\hbar} \to \pi_* \nabla^{\hbar}_{\mathrm{ab}}$.*
(2) *At a wall of $\mathrm{SN}(\vartheta = \arg \hbar)$ of type ij, the isomorphism ι jumps by a unipotent endomorphism of $\pi_* \nabla^{\hbar}_{\mathrm{ab}}$, whose only off-diagonal entry is in the ij position.*

If we define $\mathcal{X}_\gamma(\hbar)$ to be the holonomy of $\nabla^{\hbar}_{\mathrm{ab}}$ around a cycle $\gamma \in H_1(\Sigma'_u, \mathbb{Z})$, then:

(1) *$\mathcal{X}_\gamma(\hbar) \sim \pm \exp(Z_\gamma / \hbar)$ as $\hbar \to 0$,*
(2) *$\mathcal{X}_\gamma(\hbar)$ depends on $\hbar$ in a piecewise holomorphic way, with possible jumping when $\arg \hbar = \arg Z_\gamma$ for some γ with $\Omega(\gamma) \neq 0$.*

Thus, if U is a simply connected domain in $C \setminus \mathrm{SN}(\vartheta = \arg \hbar)$, choosing a basis of covariantly constant sections of $\nabla^{\hbar}_{\mathrm{ab}}|_{\pi^{-1}(U)}$ induces a basis of covariantly constant sections ψ_i of $\nabla^{\hbar}|_U$; this should be understood as an analog of the bases which we discussed in the 2-dimensional case.

Remark 3.9.3 (Nonabelianization)**.** The process of abelianization is in a sense reversible: given the holonomies $\mathcal{X}_\gamma(\hbar)$ one can reconstruct the isomorphism class of the connection $\nabla^{\hbar}$, and one can also compute invariants of the connection such as its monodromy or Stokes data.

Remark 3.9.4 ($\mathcal{X}_\gamma(\hbar)$ as cluster coordinates)**.** In many examples, the quantities $\mathcal{X}_\gamma(\hbar)$ have a simple concrete interpretation: they are cluster coordinates of the flat connection $\nabla^\hbar$, in the sense of [15]. In particular, this happens for generic u when $\mathfrak{g} = \mathfrak{sl}_2$ and C has punctures [20, 24], and for some special u when $\mathfrak{g} = \mathfrak{sl}_N$ [21]. When the topology of $\mathrm{SN}(\vartheta)$ changes, the functions $\mathcal{X}_\gamma(\hbar)$ change discontinuously; in many cases the transformation they undergo is a cluster transformation. Studying these transformations gives one route to explaining the Kontsevich-Soibelman wall-crossing formula obeyed by the $\Omega(\gamma)$, as described in [20,23,24].

Remark 3.9.5 (Abelianization of $\nabla^\hbar$ as exact WKB, when $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{gl}_2$)**.** In the case $N = 2$, when C has punctures, Conjecture 3.9.2 is a theorem; it can be thought of as a reinterpretation and extension of the celebrated exact WKB method for analysis of Schrödinger equations. The local solutions in this case can be obtained by Borel resummation of the formal series contemplated in WKB analysis. An account of this in the language of abelianization is given in [30]; for a more direct description of the link between exact WKB and cluster algebras see [34].

Remark 3.9.6 (Computation of $\mathcal{X}_\gamma(\hbar)$)**.** One practical use of abelianization is as a way of getting concrete information about the monodromy or Stokes data of $\nabla^\hbar$. Indeed, according to a conjecture of [19], knowing the central charges Z_γ and the BPS indices $\Omega(\gamma)$ — both computable from the point $u \in \mathcal{B}(C, \mathfrak{g})$ — is sufficient to write down an integral equation which determines the $\mathcal{X}_\gamma(\hbar)$ and thus the monodromy or Stokes data. This conjecture generalizes the "ODE/IM correspondence" which had been discovered earlier [9,10].

Finally let us briefly discuss the more interesting and difficult case of the connections ∇^ζ. The expected picture, from [20,24], has a formal structure almost completely parallel to that for $\nabla^\hbar$:

Conjecture 3.9.7 (Abelianization for ∇^ζ)**.** *Suppose u is generic. There exists a family of almost-flat $\mathrm{GL}(1)$-connections ∇^ζ_{ab} over the spectral cover Σ_u which "abelianize" ∇^ζ, in the following sense:*

(1) *On $C \setminus \mathrm{SN}(\vartheta = \arg \zeta)$, there is an isomorphism $\iota : \nabla^\zeta \to \pi_* \nabla^\zeta_{ab}$.*
(2) *At a wall of $\mathrm{SN}(\vartheta = \arg \zeta)$ of type ij, the isomorphism ι jumps by a unipotent endomorphism of $\pi_* \nabla^\zeta_{ab}$, whose only off-diagonal entry is in the ij position.*

If we define $\mathcal{X}_\gamma(\zeta)$ to be the holonomy of ∇^ζ_{ab} around a cycle γ, then:

(1) *$\mathcal{X}_\gamma(\zeta) \sim \exp(Z_\gamma/\zeta)$ as $\zeta \to 0$,*
(2) *$\mathcal{X}_\gamma(\zeta)$ depends on ζ in a piecewise holomorphic way, with possible jumping when $\arg \zeta = \arg Z_\gamma$ for some γ with $\Omega(\gamma) \neq 0$.*
(3) *$\mathcal{X}_\gamma(-1/\bar{\zeta}) = \overline{\mathcal{X}_\gamma(\zeta)}^{-1}$.*

This picture leads to a new scheme for computing the monodromy and Stokes data of ∇^ζ, again in terms of systems of integral equations determining the functions $\mathcal{X}_\gamma(\zeta)$; for some self-contained examples of predictions which follow from this scheme, see [43]. In turn, the monodromy and Stokes data of ∇^ζ are simply related to the holomorphic symplectic structures on the moduli space of flat connections, and thus (by standard tricks in hyperkähler geometry) to the hyperkähler metric on the moduli space of Higgs bundles. Thus one obtains new predictions for what that hyperkähler metric should look like; see [42] for a review of the proposed construction. Some aspects of these predictions have recently been confirmed by other means, e.g. [13,16,38,39].

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