Moduli of Higgs Bundles Andrew Neitzke Preliminary and incomplete draft

These are the notes for a spring 2016 course at UT Austin. The lectures are now finished but the notes are not: they are extremely incomplete, unreliable, full of mistakes and omissions, and still being updated. The latest PDF can be found at

https://gauss.math.yale.edu/~an592/exports/higgs-bundles.pdf

Please send corrections/improvements to

andrew.neitzke@yale.edu

or as pull requests to the source repository hosted at

http://github.com/neitzke/higgs-bundles

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1 Introductory motivation

Suppose given a compact Riemann surface *C* of genus $g \ge 2$ and a compact connected Lie group *G*, e.g. G = U(1), G = SU(2). Built from these data there is a moduli space

$$\mathcal{M} = \mathcal{M}^H(C, G)$$

It is *almost* a manifold: it has some singularities, but also some connected components without singularities, and at first we can focus on the parts without singularities. It can be seen in various ways:

• \mathcal{M} is the (twisted) *character variety*, i.e. moduli space of (twisted) irreducible representations $\pi_1(C) \rightarrow G_{\mathbb{C}}$. e.g. for g = 2 and G = SU(2), this means

$$\mathcal{M} = \{A_1, A_2, B_1, B_2 \in SL(2, \mathbb{C}) : A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} = \pm 1\} / \sim (1.1)$$

- *M* is the moduli space parameterizing irreducible *flat G*_C*-connections* (or complex Einstein connections) over *C*. (Certain sheaves on this moduli space are basic objects on "*B* side" of the geometric Langlands correspondence.)
- *M* is a partial compactification of *T*^{*} Bun(*C*, *G*), where Bun(*C*, *G*) is the moduli space of semistable *G*-bundles on *C*. (Lagrangian submanifolds are related to *D*-*modules* on Bun(*C*, *G*), basic objects on "*A* side" of the geometric Langlands correspondence.)



• *M* is a *complex integrable system* [1], i.e. a holomorphic symplectic space fibered over a complex base with Lagrangian fibers, generic fiber a compact complex torus.



• *M* is a noncompact *Calabi-Yau space*, i.e. a Kähler space admitting a Ricci-flat metric, in some sense a close cousin of the K3 surface; from this point of view it is a paradigmatic example of the Strominger-Yau-Zaslow philosophy [2], which says that every Calabi-Yau space arises naturally as a special Lagrangian *torus fibration*

over a complex base, and that its *mirror* can be obtained by a natural fiberwise duality operation; moreover in this case the mirror is a space of the same kind, namely $\mathcal{M}^{\vee} = \mathcal{M}^{H}(C, {}^{L}G)$ where ${}^{L}G$ is the *Langlands dual* group [3, 4]. (The mirror symmetry exchanges the two sides of the geometric Langlands correspondence.)

- *M* is a *cluster variety*, built by gluing together very simple pieces (C[×])ⁿ in an essentially *combinatorial* way. (Almost: to make this precisely true, we have to include *punctures* on *C*; but even without the punctures, some cluster-like structure seems to persist.)
- *M* is the space of solutions of an interesting PDE, *Hitchin's equations* [5], containing as special cases various sorts of harmonic maps (including *uniformization* in the case *G* = *PSU*(2)).

How can one space \mathcal{M} be so many different things at once?

A partial answer comes from another structure \mathcal{M} carries, namely the *hyperkähler* structure. This says in short that \mathcal{M} has a metric compatible with many different complex structures, fitting together in a specific way; thus \mathcal{M} gives rise to many complex manifolds which look quite different from one another, but are nevertheless canonically diffeomorphic. Loosely speaking, all these complex structures are generated by two basic ones: one of these comes from the complex structure of the Riemann surface *C*, the other comes from the complex structure of $G_{\mathbb{C}}$.

A hyperkähler structure is rather rigid and gives a lot of constraints, e.g. it implies that the metric on \mathcal{M} is Ricci-flat, and even lets us say some things about what the metric looks like (much more than we can say for "generic" Ricci-flat metrics or even Ricci-flat Kähler metrics); it also allows us to study the *topology* of \mathcal{M} , e.g. its Betti numbers.

Our first major aim is to understand this structure — first we will study some simpler "baby" examples of hyperkähler geometry, then we will study $\mathcal{M}(C, G)$ for G = U(1), finally we will come to $\mathcal{M}(C, G)$ for general G.

(A fuller answer should come from the way M fits into supersymmetric quantum field theory; but this is mostly beyond the scope of this course.)

2 Local symplectic, complex and Kähler geometry: a quick review

This section is only intended as a review, to fix notation, and to give references for some facts we will need: we will not give complete proofs here.

There are many references for the material on complex and Kähler geometry: one good one is [6].

In this section "manifold" will mean a finite-dimensional manifold. (Later we will need to talk about infinite-dimensional Banach manifolds, but then we will always say "Banach manifold" instead of just "manifold.")

2.1 Quotients

Definition 2.1 (Free action). Suppose *X* is a manifold with a Lie group *G* acting. We say the action is *free* if the stabilizers of all $x \in X$ are trivial.

Definition 2.2 (Proper action). Suppose *X* is a manifold with a Lie group *G* acting. We say the action is *proper* if the map

$$G \times X \to X \times X$$
 (2.1)

$$(g, x) \mapsto (x, gx) \tag{2.2}$$

is a proper map, i.e. inverse images of compact sets in $X \times X$ are compact in $G \times X$.

Proposition 2.3 (Compact group actions are proper). If *G* is compact, then any action of *G* on a manifold is proper.

Definition 2.4 (Slice). Suppose *X* is a manifold with a Lie group *G* acting freely, and $x \in X$. A *slice at x* for this action is a submanifold $S_x \subset X$, with $x \in S_x$, such that for every $g \in G$ we have $gS_x \cap S_x = \emptyset$, and the action map $G \times S_x \to X$ is a diffeomorphism onto some neighborhood of *x*.



Proposition 2.5 (Free proper quotients are manifolds). Suppose *X* is a manifold with a Lie group *G* acting properly and freely. Let g = Lie G and let

$$\rho: \mathfrak{g} \to \operatorname{Vect}(X) \tag{2.3}$$

be the infinitesimal action. Then:

- The quotient *X*/*G* (the set of *G*-orbits on *X*) has a natural structure of manifold, compatible with the quotient topology.
- The differential of the projection $X \rightarrow X/G$ is surjective and gives an isomorphism

$$TX/\rho(\mathfrak{g}) \simeq T(X/G).$$
 (2.4)

We do not review the proof here, but only comment that it relies on the existence of a slice S_x around any $x \in X$. If *G* is compact then this is relatively easy (choose a *G*-invariant Riemannian metric on *X* and then let S_x be the exponential of a small disc in the

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normal bundle to the orbit Gx at x). If G is non-compact but still acts properly then it is more difficult, but proven in [7].

Note that the requirement that the *G*-action be proper is important. One can easily construct non-Hausdorff quotients by letting *G* be say an infinite discrete group. For a spectacularly bad example, let *M* be the torus $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$, and consider the action of \mathbb{R} by translations in an irrational direction, e.g. let $t \in \mathbb{R}$ act by $(x, y) \mapsto (x + t, y + \sqrt{2}t)$. Then the quotient T^2/\mathbb{R} has the indiscrete topology, so it is definitely not a manifold.

2.2 Symplectic manifolds

Definition 2.6 (Nondegenerate skew pairing). Suppose *V* is a vector space over \mathbb{R} or \mathbb{C} . We say $\omega \in \wedge^2(V)$ is *nondegenerate* if the map

$$V \to V^*$$
 (2.5)

$$v \mapsto \iota_v \omega = \omega(v, \cdot) \tag{2.6}$$

is an isomorphism.

Proposition 2.7 (Standard basis for a nondegenerate skew pairing). If *V* is a finite-dimensional vector space over \mathbb{R} or \mathbb{C} , and $\omega \in \wedge^2(V)$ is nondegenerate, then *V* has dimension 2n for some *n*, and there exists a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for *V* such that

$$\omega(e_i, f_j) = \delta_{ij},\tag{2.7}$$

$$\omega(e_i, e_j) = 0, \tag{2.8}$$

$$\omega(f_i, f_j) = 0. \tag{2.9}$$

Definition 2.8 (Symplectic manifold). A *symplectic manifold* is a pair (X, ω) where X is a manifold and $\omega \in \Omega^2(X)$, such that

$$\mathrm{d}\omega = 0 \tag{2.10}$$

and $\omega(x)$ is nondegenerate for every $x \in X$.

Definition 2.9 (Exact symplectic manifold). An *exact symplectic manifold* is a tuple (X, ω, λ) where (X, ω) is a symplectic manifold and $\lambda \in \Omega^1(X)$ has $d\lambda = \omega$.

Example 2.10 (Cotangent bundle is an exact symplectic manifold). If *X* is any manifold and $Y = T^*X$, then *Y* carries a canonical 1-form ("Liouville form"), $\lambda \in \Omega^1(Y)$, defined as follows:

$$\lambda(x,p) \cdot v = p \cdot \pi_* v \qquad x \in X, p \in T_x^* X, v \in TY.$$
(2.11)

Then there is a canonical symplectic form on Y given by

$$\omega = \mathrm{d}\lambda. \tag{2.12}$$

Exercise 2.1. Show that, in the canonical coordinate system (p_i, q_i) on T^*X induced by a coordinate system (q_i) on X, we have $\lambda = \sum_{i=1}^{n} p_i dq_i$, and $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.

2.3 Symplectic quotients

Definition 2.11 (Moment map). Suppose *X* is a symplectic manifold, with symplectic form ω , acted on by a real Lie group *G*. Let $\mathfrak{g} = \text{Lie } G$ and let

$$\rho: \mathfrak{g} \to \operatorname{Vect}(X) \tag{2.13}$$

be the infinitesimal action. Suppose given a function

$$\mu: X \to \mathfrak{g}^* \tag{2.14}$$

and for $Z \in \mathfrak{g}$ write $\mu_Z = \mu \cdot Z$. We say μ is a *moment map* for the *G*-action if for all $Z \in \mathfrak{g}$ we have

$$\iota_{\rho(Z)}\omega = \mathrm{d}\mu_Z,\tag{2.15}$$

and in addition the map (2.14) is *G*-equivariant (for the *G*-action on *X* and the coadjoint *G*-action on g^*).

In particular, the moment map μ *determines* the *G*-action.

Note that moment maps do not always exist. At the very least, the existence of a moment map requires that $\iota_{\rho(Z)}\omega$ is closed for all $Z \in \mathfrak{g}$, by (2.15). Using Cartan's "magic formula"

$$\mathcal{L}_{v}\omega = \mathbf{d}(\iota_{v}\omega) + \iota_{v}(\mathbf{d}\omega) \tag{2.16}$$

and the fact that $d\omega = 0$, this is equivalent to requiring $\mathcal{L}_{\rho(Z)}\omega = 0$, i.e. the *G*-action preserves ω infinitesimally. But even if the *G*-action preserves ω , a moment map still may not exist.¹

Conversely, if a moment map μ does exist and \mathfrak{g} has nontrivial center, we can get another moment map by taking $\mu' = \mu + c$, where *c* is fixed by the *coadjoint* action of *G*, i.e. $c \in [\mathfrak{g}, \mathfrak{g}]^{\perp} \subset \mathfrak{g}^*$.

Exercise 2.2. Suppose $X = \mathbb{R}^2$ with $\omega = dx_1 \wedge dx_2$, and $G = SO(2) = U(1) = \{e^{i\alpha} : \alpha \in \mathbb{R}\}$. Then $\mathfrak{u}(1)$ is 1-dimensional, spanned by ∂_{α} . Show that the counterclockwise rotation action of U(1) on X, given by the matrices

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \tag{2.17}$$

has a moment map $\mu : \mathbb{R}^2 \to \mathfrak{u}(1)^*$, given by

$$\mu(x_1, x_2) \cdot \partial_{\alpha} = -\frac{1}{2}(x_1^2 + x_2^2).$$
(2.18)

Thus if we identify $\mathfrak{u}(1) \simeq \mathbb{R}$ using the generator ∂_{α} , we can think of μ just as an \mathbb{R} -valued function on X,

$$\mu(x_1, x_2) = -\frac{1}{2}(x_1^2 + x_2^2).$$
(2.19)

¹When *G* is compact, moment maps exist at least *locally* on *X*, though maybe not globally. When *G* is not compact there can even be a local obstruction.

Exercise 2.3. Suppose *X* is any manifold, with a compact group *G* acting. Then T^*X is a symplectic manifold which also has a canonical action of *G*. Verify that

$$\mu_Z(x,p) = -p \cdot (\rho(Z)(x)) \qquad x \in X, p \in T_x^* X$$
(2.20)

gives a moment map for this action.

Definition 2.12 (Symplectic quotient). [8] Suppose *X* is a symplectic manifold, with a Lie group *G* acting on it, with moment map μ . Then the *symplectic quotient* is

$$X//G = \mu^{-1}(0)/G.$$
 (2.21)

More generally, for $c \in [\mathfrak{g}, \mathfrak{g}]^{\perp}$ we can define

$$X //_{c} G = \mu^{-1}(c) / G.$$
(2.22)

This is equivalent to changing our choice of μ to $\mu' = \mu + c$, so there is no essential loss of generality in always taking c = 0, but sometimes one or the other description is more convenient.²

Proposition 2.13 (Symplectic quotient is symplectic). Suppose *X* is a symplectic manifold, with a compact group *G* acting on it, with moment map μ . If *G* acts freely on $\mu^{-1}(0)$, then *X*//*G* is a manifold,

$$\dim(X//G) = \dim X - 2\dim G, \qquad (2.24)$$

and there is a symplectic form $\omega_{X//G}$ on X//G, with the property

$$\pi^* \omega_{X//G} = \iota^* \omega \tag{2.25}$$

where $\iota : \mu^{-1}(0) \hookrightarrow X$ is the inclusion.

Proof. Let $Y = \mu^{-1}(0)$. We want to show that 0 is a regular value of μ , i.e. that $d\mu : T_x X \to \mathfrak{g}^*$ is surjective whenever $x \in Y$. This is equivalent to saying that for every $Z \in \mathfrak{g}$ we have $d\mu_Z(x) \neq 0$. But by (2.15) this just means that $\iota_{\rho(Z)}\omega \neq 0$, which is true since $\rho(Z) \neq 0$ (*G* acts freely) and ω is nondegenerate. Thus *Y* is a submanifold of *X*, with dim $Y = \dim X - \dim G$. *G* is a compact group acting freely on *Y*. Then by Proposition 2.5, X//G = Y/G is also a manifold, of the desired dimension, and $T(X//G) = T(Y/G) = TY/\rho(\mathfrak{g})$.

It remains to check that ω descends to a symplectic form $\omega_{X//G}$. According to (2.25) we should define $\omega_{X//G}(v, w) = \omega(\tilde{v}, \tilde{w})$ where $\pi_* \tilde{v} = v$, $\pi_* \tilde{w} = w$, and $\tilde{v}, \tilde{w} \in TY$. Then:

• $\omega_{X//G}$ is well defined on T(X//G) = T(Y/G): that means we want $\omega(v, w) = 0$ when $v = \rho(Z)$ for some *Z*, and $w \in TY$. This follows directly from (2.15).

$$X//_{c}G = \mu^{-1}(c)/G_{c}$$
(2.23)

²Some authors define more generally for arbitrary $c \in \mathfrak{g}^*$

where $G_c \subset G$ is the stabilizer of *c*. This construction gives rise to many interesting symplectic manifolds; especially, applying this to the *G*-action on T^*G itself gives the symplectic structures on coadjoint orbits [8].

- $\omega_{X//G}$ is nondegenerate on T(X//G) = T(Y/G): the symplectic orthogonal complement of *TY* has dimension (dim *G*), and contains $\rho(\mathfrak{g})$ by (2.15), so it must be equal to $\rho(\mathfrak{g})$. But this means that any vector which annihilates all of *TY* is zero in T(Y/G), i.e. ω is nondegenerate on T(Y/G).
- $\omega_{X//G}$ is closed on Y/G: letting $\pi : Y \to Y/G$ be the quotient map, on Y we have $\iota^* \omega = \pi^* \omega_{X//G}$, so that $d(\pi^* \omega_{X//G}) = 0$, ie $\pi^* d\omega_{X//G} = 0$. Since π is a submersion this implies $d\omega_{X//G} = 0$ as desired.

The next exercise gives (perhaps) some motivation for the notion of symplectic quotient:

Exercise 2.4. Suppose *X* is a manifold with a compact group *G* acting freely. Let μ be the moment map of Exercise 2.3. Show that *G* acts freely on $\mu^{-1}(0)$ and

$$T^*X//G \simeq T^*(X/G)$$
 (2.26)

as symplectic manifolds.

2.4 Complex manifolds

In this section *X* is a manifold.

Definition 2.14 (Almost complex structure). An *almost complex structure* on *X* is a smooth section *I* of End(*TX*) with $I^2 = -1$. An *almost complex manifold* is a pair (*X*, *I*) where *I* is an almost complex structure. If *X* has real dimension 2n, an almost complex structure *I* equips *TX* with the structure of *complex* vector bundle over *X*, of rank *n*, and we say the *complex dimension* dim_C *X* is *n*.

Example 2.15 (Flat complex space). \mathbb{C}^n has a canonical almost complex structure *I*, as follows. Each tangent space $T_p\mathbb{C}^n \simeq \mathbb{C}^n$ canonically; *I* is multiplication by i, thought of as an endomorphism of the underlying 2n-dimensional real vector space. Writing $z_i = x_i + iy_i$, and taking the coordinate basis $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_n}\}$ for for $T_p\mathbb{C}^n$, *I* is represented by the matrix

$$I = \begin{pmatrix} \mathbf{0}_{n \times n} & -\mathbf{1}_{n \times n} \\ \mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix}.$$
 (2.27)

Definition 2.16 (Holomorphic maps). If (X, I_X) and (Y, I_Y) are almost complex manifolds, a *holomorphic map* $\phi : X \to Y$ is one obeying

$$I_Y \circ \mathrm{d}\phi = \mathrm{d}\phi \circ I_X. \tag{2.28}$$

Exercise 2.5. Show that, if both (X, I_X) and (Y, I_Y) are \mathbb{C} with its canonical almost complex structure, Definition 2.16 becomes the standard definition of holomorphic function (Cauchy-Riemann equations).

Definition 2.17 (Antiholomorphic maps). If (X, I_X) and (Y, I_Y) are almost complex manifolds, an *antiholomorphic map* $\phi : X \to Y$ is a holomorphic map $(X, I_X) \to (Y, -I_Y)$.

Definition 2.18 (Complex structures). An almost complex structure *I* on *X* is *integrable*, or a *complex structure*, if there is a covering of *X* by open sets U_{α} with holomorphic diffeomorphisms $\phi_{\alpha} : U_{\alpha} \to V_{\alpha} \subset \mathbb{C}^n$ (where on \mathbb{C}^n we take the canonical almost complex structure.) A *complex manifold* is an almost complex manifold (*X*, *I*) with *I* integrable.

Exercise 2.6. Show that Definition 2.18 is equivalent to the usual definition of a complex manifold as a space *X* with a covering by charts $\phi_{\alpha} : U_{\alpha} \to \mathbb{C}^{n}$, where the transition maps are holomorphic (obey Cauchy-Riemann equations).

Example 2.19 (Complex structure on \mathbb{C}^n **).** A tautological example is $X = \mathbb{C}^n$ itself with its canonical almost complex structure: just take a single open set $U = \mathbb{C}^n$, and $\phi : U \to \mathbb{C}^n$ to be the identity map. So the canonical almost complex structure on \mathbb{C}^n is, tautologically, a complex structure.

There are various equivalent ways of formulating the integrability condition. One which will be useful for us is:

Proposition 2.20 (Integrability means vanishing of Nijenhuis tensor). Define the *Nijenhuis tensor* $N_I \in \Omega^0(\wedge^2 T^*X \otimes TX)$ as a map $T^*X \otimes T^*X \to TX$ by

$$N_I(v,w) = [\tilde{v},\tilde{w}] + I[\tilde{v},I\tilde{w}] + I[I\tilde{v},\tilde{w}] - [I\tilde{v},I\tilde{w}], \qquad (2.29)$$

where \tilde{v} and \tilde{w} are any vector fields extending v, w. Then I is integrable if and only if

$$N_I = 0.$$
 (2.30)

Proof. To show that integrability implies $N_I = 0$ is straightforward by directly computing in a local holomorphic coordinate chart. The other direction is much harder — it is the content of the Newlander-Nirenberg theorem.

2.5 Type decompositions

Suppose (X, I) is an almost complex manifold. We have a decomposition of $T_{\mathbb{C}}X = TX \otimes_{\mathbb{R}} \mathbb{C}$,

$$T_{\rm C}X = T^{1,0}X \oplus T^{0,1}X \tag{2.31}$$

where $T^{1,0}X$ and $T^{0,1}X$ are respectively the +i and -i eigenspaces of *I*. Both *TX* and $T^{1,0}X$ are complex vector bundles of rank *n*; it is sometimes convenient to identify them, by projection on the (1,0) part.

Exercise 2.7. Show that this projection $\pi : TX \to T^{1,0}X$ indeed is an isomorphism of complex vector bundles. (This reduces essentially to a question of linear algebra, concerning a vector space *V* with complex structure *I*, and its complexification *V*_C.) If $X = \mathbb{C}$, what are $\pi(\partial_x)$ and $\pi(\partial_y)$?

$$T^*_{\mathbb{C}}X = (T^*)^{1,0}X \oplus (T^*)^{0,1}X, \qquad (2.32)$$

where $(T^*)^{1,0}X$ is the annihilator of $T^{0,1}X$, and $(T^*)^{0,1}X$ is the annihilator of $T^{1,0}X$. This decomposition induces

$$\wedge^* T^*_{\mathbb{C}} X = \bigoplus_{p,q=0}^n \wedge^{p,q} T^* X, \qquad \Omega^*_{\mathbb{C}} X = \bigoplus_{p,q=0}^n \Omega^{p,q}(X).$$
(2.33)

Proposition 2.21 (Integrability versus type decompositions). Suppose (X, I) is an almost complex manifold. The following are equivalent:

- *I* is integrable.
- There is a decomposition

$$\mathbf{d} = \partial + \bar{\partial}, \quad \partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \quad \bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X).$$
(2.34)

• The distribution $T^{0,1}X$ is integrable: if v, w are sections of $T^{0,1}X$ then [v, w] is also a section of $T^{0,1}X$.

Complex conjugation is an \mathbb{R} -linear map $\Omega^{p,q}(X) \to \Omega^{q,p}(X)$; thus it maps $\Omega^{p,p}(X)$ to itself; we let $\Omega^{p,p}_{\mathbb{R}}(X)$ denote the fixed subspace.

2.6 Holomorphic vector bundles

In this section (X, I) is always a complex manifold.

Definition 2.22 (Holomorphic vector bundle). A *holomorphic vector bundle* over X is a complex vector bundle *E* over X, equipped with an operator

$$\bar{\partial}_E: \Omega^{p,q}(E) \to \Omega^{p,q+1}(E) \tag{2.35}$$

obeying

$$\bar{\partial}_E(\alpha\psi) = (\bar{\partial}\alpha)\psi + (-1)^{|\alpha|}\alpha \wedge \bar{\partial}_E\psi \qquad \alpha \in \Omega^*(X), \quad \psi \in \Omega^0(E)$$
(2.36)

and the integrability condition

$$\bar{\partial}_E^2 = 0. \tag{2.37}$$

It is useful to think of $\bar{\partial}_E$ as a kind of partially-defined flat connection, which allows us to differentiate only in the (0,1) "directions." The structure of holomorphic vector bundle is much more rigid than that of a merely complex vector bundle. We emphasize that this structure makes sense only when *X* is a complex manifold, while complex vector bundles make sense over any *X*.

Proposition 2.23 (Equivalence of definitions of holomorphic vector bundle). A structure of holomorphic vector bundle on *E* is equivalent to a maximal atlas of preferred trivializations of *E*, such that the transition maps $U_{\alpha} \cap U_{\beta} \rightarrow GL(r, \mathbb{C})$ are holomorphic.

Proof. This is a sort of linear analogue of the Newlander-Nirenberg theorem; for a proof see [9] Theorem 2.1.53, proven in Section 2.2. \Box

Example 2.24 (Tangent bundle as a holomorphic bundle). The tangent bundle *TX* carries a canonical structure of holomorphic vector bundle. Indeed, the holomorphic charts $\phi_{\alpha} = (z_1, \ldots, z_n)$ give rise to preferred trivializations corresponding to the bases $\{\partial_{z_1}, \ldots, \partial_{z_n}\}$ for $T^{1,0}X \simeq TX$, and the transition maps are given by the Jacobian matrices $(\partial z'_i / \partial z_j)^n_{i,j=1}$, which are holomorphic.

Example 2.25 (Canonical bundle). The canonical line bundle over X is defined by

$$K_X = \wedge^n T X. \tag{2.38}$$

It inherits a holomorphic structure from that of *TX*.

Definition 2.26 (Connection compatible with holomorphic structure). If *E* is a holomorphic vector bundle over *X*, a connection *D* in *E* is *compatible with the holomorphic structure* if, for all $\psi \in \Omega^0(E)$, the (0, 1) part of $D\psi$ is $\bar{\partial}_E \psi$.

Definition 2.27 (Chern connection). If *E* is a holomorphic vector bundle over *X* with a Hermitian metric *h*, the *Chern connection* in *E* is the unique connection which is *h*-unitary and compatible with the holomorphic structure.

Exercise 2.8. Show that Definition 2.27 makes sense, i.e. that there indeed *is* a unique connection in *E* with the claimed properties. Relative to a local holomorphic trivialization, show that D = d + A, where

$$A = h^{-1}\partial h. \tag{2.39}$$

2.7 Hermitian and Kähler metrics

Throughout this section (X, I) is an almost complex manifold.

Definition 2.28 (Hermitian metric on complex manifold). A Hermitian metric on (X, I) is a Riemannian metric *g* obeying

$$g(v,w) = g(Iv, Iw).$$

Equivalently, with respect to the decomposition

$$\operatorname{Sym}^{2}(T_{\mathbb{C}}X) = \operatorname{Sym}^{2,0} TX \oplus \operatorname{Sym}^{1,1} TX \oplus \operatorname{Sym}^{0,2} TX, \qquad (2.40)$$

we have $g \in \text{Sym}^{1,1} TX$, i.e. g is of "type (1, 1)."

Definition 2.29 (Fundamental form). If *g* is a Hermitian metric on (*X*, *I*), the *fundamental form* $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$ is

$$\omega(v,w) = g(Iv,w). \tag{2.41}$$

Definition 2.30 (Positive form). If $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$, ω is *positive* if the symmetric pairing

$$g(v,w) = \omega(v,Iw) \tag{2.42}$$

is positive definite.

Exercise 2.9. If *g* is a Hermitian metric on (X, I) check that

$$\operatorname{vol}_g = \frac{\omega^n}{n!} \tag{2.43}$$

The term "Hermitian" might seem confusing here since g is just an ordinary realvalued metric on the real vector bundle TX. The following should help:

Exercise 2.10. If g is a Hermitian metric on (X, I), verify that

$$h = g - i\omega \tag{2.44}$$

defines a Hermitian metric on the complex vector bundle TX. (Our convention is that Hermitian metrics are \mathbb{C} -linear in the *first* slot.)

Let ∇ denote the Levi-Civita connection on *TX* induced by the metric *g*.

Definition 2.31 (Kähler metric). If *g* is a Hermitian metric on (*X*, *I*), we say *g* is *Kähler* if

$$\nabla I = 0. \tag{2.45}$$

Then (X, g, I) is a Kähler manifold, and ω is the Kähler form.

Example 2.32 (Standard metric on \mathbb{C} **is Kähler).** Take $X = \mathbb{C}$ with coordinate z = x + iy, I the standard complex structure, and g the standard Riemannian metric, $g = dx^2 + dy^2$. Evidently $\nabla I = 0$, so this is a Kähler metric. The Kähler form is

$$\omega = \mathrm{d}x \wedge \mathrm{d}y = \frac{\mathrm{i}}{2} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.46}$$

The Kähler property has various useful alternative characterizations:

Proposition 2.33 (Characterizations of Kähler metrics). If *g* is a Hermitian metric on (X, I), with fundamental form ω , then the following are equivalent:

- 1. g is Kähler,
- 2. $\nabla I = 0$,
- 3. $\nabla \omega = 0$,
- 4. *I* is integrable and ∇ agrees with the Chern connection on *TX*, when we view *TX* as a holomorphic vector bundle, with the induced Hermitian metric *h* of Exercise 2.10,
- 5. *I* is integrable and $d\omega = 0$.

Proof. As we defined Kähler we have automatically $(1) \Leftrightarrow (2)$. Using $\nabla g = 0$ we easily obtain $(2) \Leftrightarrow (3)$. So all of (1)-(3) are equivalent.

Now we consider (4). To show (2) implies integrability of *I*, note that for *v*, *w* sections of $T^{0,1}X$ we have (using the torsion-free property of ∇ and $\nabla I = 0$)

$$I[v,w] = I(\nabla_v w - \nabla_w v) = \nabla_v (Iw) - \nabla_w (Iv) = -i(\nabla_v w - \nabla_w v) = -i[v,w], \quad (2.47)$$

so by Proposition 2.21 *I* is integrable. Also (2) implies that for $v \in TX$ and *w* a section of $T^{1,0}X$ we have $I(\nabla_v w) = \nabla_v(Iw) = i\nabla_v w$, so that $\nabla_v w$ is also a section of $T^{1,0}X$; this means ∇ is compatible with the holomorphic structure. Finally (2) + (3) implies $\nabla h = 0$. So we have shown that (2) \Rightarrow (4). Conversely, we have easily (4) \Rightarrow (3), since ω is the imaginary part of *h*. Thus, all of (1)-(4) are equivalent.

Finally we consider (5). We already showed that (2) implies integrability of *I*. Also (3) immediately implies $d\omega = 0$. Thus we have $(2) + (3) \Rightarrow (5)$. All that remains is to see that $(5) \Rightarrow (4)$, which is the most interesting part. This amounts to verifying that the Chern connection is torsion-free (then it will have to agree with ∇ , since ∇ is the unique connection in *TX* which is torsion-free and has $\nabla g = 0$.) [...]

In particular Proposition 2.33 implies that any complex submanifold of a Kähler manifold is again Kähler. Combining this with the fact that \mathbb{CP}^n admits a Kähler metric (Fubini-Study), we obtain a huge supply of examples.

Proposition 2.34 (Ricci form for Kähler manifold is curvature of canonical bundle). If (X, I, g) is Kähler, the Ricci form R(v, w) = Ric(Iv, w) is equal to the curvature of the induced Hermitian metric on the canonical line bundle $K = \wedge^{n,0} T^* X$.

Finally we quickly recall the notion of special holonomy. Recall that for any Riemannian metric g the parallel transport of Levi-Civita preserves g, so that for any $p \in X$ the holonomy group $Hol_{\nabla}(p) \subset GL(T_pX)$ is contained in the subgroup $O(g(p)) \simeq O(2n)$. For a Kähler metric, Proposition 2.33 says the parallel transport of Levi-Civita preserves the Hermitian metric h on the complex vector bundle TX. Thus, for any $p \in X$, the holonomy group $Hol_{\nabla}(p) \subset GL(T_pX)$ is contained in the smaller group $U(h(p)) \simeq U(n)$. Conversely, if $Hol_{\nabla}(p)$ is contained in some subgroup isomorphic to U(n) then it preserves some h, from which one can prove:

Proposition 2.35 (Special holonomy of Kähler manifolds). Given any Riemannian metric g on a manifold M of dimension 2n, g is a Kähler metric (for some complex structure I on M) if and only if the holonomy group at a point is contained in a subgroup isomorphic to U(n).

2.8 Hodge theory

Here we recall the basic statements of (abelian) Hodge theory. [refs]

Definition 2.36 (de Rham cohomology). Suppose *X* is any manifold. Then we define

$$H_{\mathrm{dR}}^k(X) = \frac{\mathrm{Im}\,\mathrm{d}\cap\Omega^k(X)}{\ker\mathrm{d}\cap\Omega^k(X)}.\tag{2.48}$$

$$H^k_{\mathrm{dR}}(X) \simeq H^k(X, \mathbb{R}). \tag{2.49}$$

Definition 2.38 (Formal adjoint of d**).** If *X* is a Riemannian manifold of dimension *n*, the formal adjoint of d is the operator

$$d^*: \Omega^k(X) \to \Omega^{k-1}(X) \tag{2.50}$$

given by

$$\mathbf{d}^* = (-1)^{n(k+1)+1} \star \mathbf{d} \star.$$
(2.51)

If X is a compact Riemannian manifold, we have the L^2 pairing on $\Omega^*(X)$ given by

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha(x), \beta(x) \rangle \, \mathrm{d}vol_X = \int_X \alpha \wedge \star \beta.$$
 (2.52)

Lemma 2.39 (Formal adjoint is actual adjoint on compact manifold). If *X* is a compact Riemannian manifold, d^* is the actual adjoint with respect to the L^2 pairing, i.e.

$$\langle d^* \alpha, \beta \rangle = \langle \alpha, d\beta \rangle.$$
 (2.53)

Definition 2.40 (Laplace operator on Riemannian manifold). If *X* is a Riemannian manifold, we define the form Laplacian

$$\Delta: \Omega^k(X) \to \Omega^k(X) \tag{2.54}$$

by

$$\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}.\tag{2.55}$$

Definition 2.41 (Harmonic forms). If *X* is a Riemannian manifold, we define the space of *harmonic* forms as

$$\mathcal{H}(X) = \ker \Delta. \tag{2.56}$$

It decomposes as $\mathcal{H}(X) = \bigoplus_{k=0}^{n} \mathcal{H}^{k}(X)$, where $\mathcal{H}^{k}(X) = \mathcal{H}(X) \cap \Omega^{k}(X)$.

Proposition 2.42 (Poincare duality for harmonic forms). Let *X* be a Riemannian manifold of dimension *n*. The Hodge star gives an isomorphism

$$\star: \mathcal{H}^k(X) \to \mathcal{H}^{n-k}(X). \tag{2.57}$$

If in addition X is compact, then the pairing

$$(\alpha,\beta)\mapsto \int_X \alpha\wedge\star\beta \tag{2.58}$$

is nondegenerate, and thus gives an isomorphism

$$\mathcal{H}^k(X) \simeq \mathcal{H}^k(X)^* \tag{2.59}$$

Equivalently, the pairing

$$(\alpha,\beta)\mapsto \int_X \alpha\wedge\beta$$
 (2.60)

gives an isomorphism

$$\mathcal{H}^k(X) \simeq \mathcal{H}^{n-k}(X)^*.$$
(2.61)

Theorem 2.43 (Abelian Hodge theory for compact Riemannian manifolds). Let *X* be a compact Riemannian manifold. Then:

- $\mathcal{H}(X) = \ker d \cap \ker d^*$.
- For any class $[\alpha] \in H^k_{dR}(X)$, there exists a unique harmonic representative, $\alpha' \in [\alpha] \cap \mathcal{H}^k(X)$.

Example 2.44 (Harmonic forms of degree 0 **or** *n***).** If *X* is any compact Riemannian manifold of dimension *n*, then Theorem 2.43 says $\mathcal{H}^0(X) = \ker d$, i.e. the only harmonic functions on a compact manifold are constants. Then using Proposition 2.42 we get dually that $\mathcal{H}^n(X)$ consists of all scalar multiples of the Riemannian volume form on *X*.

Definition 2.45 (Dolbeault cohomology). Let *X* be a complex manifold. Then we define

$$H^{p,q}_{\bar{\partial}}(X) = \frac{\operatorname{Im}\bar{\partial} \cap \Omega^{p,q}(X)}{\ker\bar{\partial} \cap \Omega^{p,q}(X)}.$$
(2.62)

Definition 2.46 (Formal adjoints of ∂ , $\overline{\partial}$). If *X* is a complex manifold, the formal adjoints of ∂ and $\overline{\partial}$ are

$$\partial^*: \Omega^{p,q}(X) \to \Omega^{p-1,q}(X), \qquad \bar{\partial}^*: \Omega^{p,q}(X) \to \Omega^{p,q-1}(X)$$
(2.63)

given by

$$\partial^* = -\star \bar{\partial} \star, \qquad \bar{\partial}^* = -\star \partial \star.$$
 (2.64)

Theorem 2.47 (Abelian Hodge theorem for compact Kähler manifolds). Let *X* be a compact Kähler manifold of complex dimension *n*. Then:

- $\mathcal{H}_{\mathbb{C}}(X) = \ker \overline{\partial} \cap \ker \overline{\partial}^* = \ker \partial \cap \ker \partial^*.$
- If we define

$$\mathcal{H}^{p,q}(X) = \mathcal{H}_{\mathbb{C}}(X) \cap \Omega^{p,q}(X)$$
(2.65)

then

$$\mathcal{H}_{\mathbb{C}}(X) = \bigoplus_{p,q=0}^{n} \mathcal{H}^{p,q}(X).$$
(2.66)

• For any class $[\alpha] \in H^{p,q}_{\bar{\partial}}(X)$, there exists a unique harmonic representative, $\alpha' \in [\alpha] \cap \mathcal{H}^{p,q}(X)$.

Proposition 2.48 (Poincare duality for $\bar{\partial}$ **-cohomology).** Let *X* be a Kähler manifold of complex dimension *n*. Then the Hodge star gives an isomorphism

$$\star: \mathcal{H}^{p,q}(X) \to \mathcal{H}^{n-q,n-p}(X).$$
(2.67)

If in addition X is compact, then the Hermitian pairing

$$(\alpha,\beta)\mapsto \int_X \alpha\wedge\star\bar{\beta}$$
 (2.68)

is nondegenerate, and thus gives an isomorphism

$$\mathcal{H}^{p,q}(X) \simeq \overline{\mathcal{H}^{p,q}(X)}^*.$$
(2.69)

Equivalently, the pairing

$$(\alpha,\beta)\mapsto \int_X \alpha\wedge\beta$$
 (2.70)

gives an isomorphism

$$\mathcal{H}^{p,q}(X) \simeq \mathcal{H}^{n-q,n-p}(X)^*.$$
(2.71)

Lemma 2.49 ($\partial \bar{\partial}$ -lemma). Suppose X is a compact Kähler manifold, $\alpha \in \Omega^{p,q}(X)$ and $d\alpha = 0$. Then the following are equivalent:

- $\alpha \in \operatorname{Im} d$,
- $\alpha \in \operatorname{Im} \partial$,
- $\alpha \in \operatorname{Im} \bar{\partial}$,
- $\alpha \in \operatorname{Im} \partial \overline{\partial}$.

If $\alpha \in \Omega^{p,p}_{\mathbb{R}}(X)$ then these are also equivalent to

• $\alpha \in \operatorname{Im}\left(\mathrm{i}\partial\bar{\partial}: \Omega^{p-1,p-1}_{\mathbb{R}}(X) \to \Omega^{p,p}_{\mathbb{R}}(X)\right).$

Finally we remark that all of the above statements have analogues when we consider forms valued in an auxiliary Hermitian vector bundle *E* carrying a flat connection *D*. [say a little more here?]

[...]

2.9 Kähler quotients

Definition 2.50 (Horizontal distribution). Suppose *X* is a Riemannian manifold, with a Lie group *G* acting freely on *X*. The *horizontal distribution* on *X* is

$$H = \{\rho(Z) : Z \in \mathfrak{g}\}^{\perp} \subset TX.$$

$$(2.72)$$

Definition 2.51 (Induced metric on a quotient). Suppose *X* is a Riemannian manifold, with a compact group *G* acting freely on *X* preserving *g*. Using orthogonal projection we have a canonical identification

$$T(X/G) = TX/\rho(\mathfrak{g}) \simeq H.$$
(2.73)

The *induced metric* on X/G is $g|_H$. This is a Riemannian metric on X/G. (Note that it is well defined because *g* is *G*-invariant.)

Proposition 2.52 (Symplectic quotients of Kähler manifolds are Kähler). Suppose *X* is a Kähler manifold, with a compact group *G* acting on *X* preserving both *g* and *I* (thus it also preserves ω), with a moment map μ , and such that *G* acts freely on $Y = \mu^{-1}(0)$. Then the induced metric on the symplectic quotient *X*//*G* is Kähler.

Proof. First we want to see that there is a natural almost complex structure on X//G. The tangent space T(X//G) is

$$T(X//G) = T(Y/G)$$
 (2.74)

$$= \{\rho(Z) : Z \in \mathfrak{g}\}^{\perp} \subset TY$$
(2.75)

$$= (\{\operatorname{grad} \mu_Z : Z \in \mathfrak{g}\} \oplus \{\rho(Z) : Z \in \mathfrak{g}\})^{\perp} \subset TX.$$
(2.76)

But

$$g(\operatorname{grad} \mu_Z, v) = \mathrm{d}\mu_Z \cdot v = \omega(\rho(Z), v) = g(I\rho(Z), v)$$
(2.77)

so

$$\operatorname{grad} \mu_Z = I\rho(Z). \tag{2.78}$$

Since *I* acts orthogonally, it follows that T(X//G) is preserved by *I*.

Now we need to check that the Levi-Civita connection $\nabla^{X//G}$ on T(X//G) preserves I. $\nabla^{X//G}$ can be obtained by starting with the Levi-Civita connection ∇^X , restricting to a connection in TX over $Y = \mu^{-1}(0)$, and then projecting orthogonally to T(X//G), i.e. for $v, w \in \text{Vect}(X//G) \subset \text{Vect}(X)$,

$$\nabla_w^{X/\!/G}(v) = \pi(\nabla_w^X v) \tag{2.79}$$

(To see this, one needs to check that this formula indeed gives a metric-compatible and torsion-free connection.) Then the desired statement follows from the fact that *I* is covariantly constant for ∇^X and commutes with π .

Infinitesimally what we have just done is to identify T(X//G) with the orthocomplement of the space generated by vectors $\rho(Z)$ and $I\rho(Z)$, for $Z \in \mathfrak{g}$. There is another way of thinking about this: these vector fields generate a copy of the *complexified* Lie algebra $\mathfrak{g}_{\mathbb{C}}$ inside of Vect(X). So infinitesimally it looks as if we are taking an ordinary quotient, but a quotient by some complexification $G_{\mathbb{C}}$ rather than the original G. Thus we might dream that as complex manifolds we would have

$$X//G = X/G_{\mathbb{C}}.\tag{2.80}$$

If this were literally true it would give an "easy" way of thinking about the complex structure on X//G. But if we try to realize this dream literally, we will run into problems, since there is no reason for the vector fields $I\rho(Z)$ on X to be complete: thus we cannot necessarily integrate them to a group action. Even if they are complete, we can still have problems, because $G_{\mathbb{C}}$ is not a compact group, so $X/G_{\mathbb{C}}$ is not guaranteed to be a nice space. The next example explores this a bit.

Example 2.53 (Projective space as a Kähler quotient). Take $X = \mathbb{C}^n$ with its standard Kähler metric, for which

$$\omega = \frac{\mathrm{i}}{2} \sum_{i} \mathrm{d} z_i \wedge \mathrm{d} \bar{z}_i. \tag{2.81}$$

This ω is preserved by the U(1) action simultaneously rotating all z_i ,

$$z_i \mapsto e^{i\alpha} z_i, \tag{2.82}$$

with moment map (identifying $u(1) \simeq \mathbb{R}$ as usual)

$$\mu = -\frac{1}{2} \sum_{i=1}^{n} |z_i|^2 + c, \qquad (2.83)$$

where $c \in \mathbb{R}$ is arbitrary.

Now we consider the symplectic quotient X//U(1). If c = 0 then U(1) does not act freely on $\mu^{-1}(0)$. If c < 0 then 0 is a regular value, but in a trivial way: $\mu^{-1}(0)$ is empty. The interesting case is c > 0, in which case

$$\mu^{-1}(0) = \left\{ \sum |z_i|^2 = 2c \right\} \simeq S^{2n-1}$$
(2.84)

and dividing out by U(1) gives (at least as a set) \mathbb{CP}^{n-1} .

Exercise 2.11. Check that the induced complex structure on $\mathbb{C}^n // U(1)$, promised by Proposition 2.52, is indeed the standard one on \mathbb{CP}^{n-1} . How does changing the choice of *c* change the Kähler structure?

Following the philosophy we just described, instead of taking $\mathbb{C}^n // U(1)$ we could try to take $\mathbb{C}^n / \mathbb{C}^{\times}$. This quotient is badly behaved (non-Hausdorff) as it stands, because of the point $0 \in \mathbb{C}^n$, but we certainly do have

$$(\mathbb{C}^n \setminus \{0\})/\mathbb{C}^{\times} = \mathbb{C}\mathbb{P}^{n-1} = \mathbb{C}^n//U(1).$$
(2.85)

The "explanation" of this phenomenon is that each \mathbb{C}^{\times} -orbit on X meets $\mu^{-1}(0)$ in exactly one U(1)-orbit, except for the orbit $\{0\}$ which does not meet $\mu^{-1}(0)$ at all, and thus must be thrown out if we want to compare with X//G.

Now more generally, how should we compare X//G and $X/G_{\mathbb{C}}$? They would be equal if each $G_{\mathbb{C}}$ -orbit met $\mu^{-1}(0)$ in a single *G*-orbit. At least, we can say that each *G*-orbit *O* in $\mu^{-1}(0)$ is contained in some $G_{\mathbb{C}}$ -orbit $O_{\mathbb{C}}$, and then we can ask: does $O_{\mathbb{C}}$ meet $\mu^{-1}(0)$ anywhere *else*? It turns out that this cannot happen, so at least we have $X//G \subset X/G_{\mathbb{C}}$. The remaining question is whether there might be some orbits which don't meet $\mu^{-1}(0)$ at all. Generally there are (like the orbit of $0 \in \mathbb{C}^n$ above); these orbits need to be thrown out; after restricting to their complement X^s (the set of polystable orbits), we finally get the desired

$$X//G \simeq X^s/G_{\mathbb{C}}.$$
(2.86)

2.10 Holomorphic symplectic manifolds

Definition 2.54 (Holomorphic symplectic form). If (X, I) is a complex manifold, $\Omega \in \Omega^{2,0}(X)$ is a *holomorphic symplectic form* if $d\Omega = 0$ and Ω is nondegenerate in the holomorphic sense, i.e. it induces an isomorphism $T^{1,0}X \to (T^{1,0}X)^*$. In this case we call (X, I, Ω) a *holomorphic symplectic manifold*.

Note that this definitely does *not* mean that Ω is nondegenerate on the whole $T_{\mathbb{C}}X$. Indeed, since Ω is of type (2,0) its contraction with any $v \in T^{0,1}X$ vanishes.

Morally you should think of a holomorphic symplectic form Ω as something like the analytic continuation of a real symplectic form from some real subspace of *X*.

Proposition 2.55 (Holomorphic symplectic manifolds have dimension 4n**).** If (X, I, Ω) is a holomorphic symplectic manifold, then dim_R X is a multiple of 4.

Proof. The holomorphic symplectic form Ω restricts to a nondegenerate form on each fiber of the complex vector bundle $T^{1,0}X$. Using Proposition 2.7, it follows that $T^{1,0}X$ has even complex dimension.

Actually much more is true: locally X admits holomorphic "Darboux" coordinate systems $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ such that $\Omega = \sum_{i=1}^n dp_i \wedge dq_i$.

Proposition 2.56 (Holomorphic symplectic form determines complex structure). Suppose *X* is a manifold with dim_R X = 2n, with $\Omega \in \Omega^2_{\mathbb{C}}(X)$, such that $d\Omega = 0$ and

$$T_{\mathbb{C}}X = \ker \Omega \oplus \ker \overline{\Omega}. \tag{2.87}$$

Then there is a unique complex structure *I* on *X* for which Ω is a holomorphic symplectic form.

Proof. We define a complex-linear operator $I_{\mathbb{C}}$ on $T_{\mathbb{C}}X$ to act by -i on ker Ω , and by +i on ker $\overline{\Omega}$. This $I_{\mathbb{C}}$ obeys $I_{\mathbb{C}}v = \overline{I_{\mathbb{C}}v}$, so it is the complexification of a real-linear operator I on TX, which gives an almost complex structure. The integrability of I is equivalent to requiring that $T^{0,1}X = \ker \Omega$ is an integrable distribution, i.e. that if v, w are sections of ker Ω then [v, w] is also a section of ker Ω . This follows from $d\Omega = 0$ and the covariant formula for d: for any third vector field y we have

$$d\Omega(v, w, y) = v\Omega(w, y) + w\Omega(y, v) + y\Omega(v, w) - \Omega([v, w], y) - \Omega([y, v], w) - \Omega([w, y], v)$$
(2.88)

and now all terms vanish except the one we want:

$$0 = -\Omega([v,w],y) \tag{2.89}$$

so [v, w] is a section of ker Ω as desired.

3 Hyperkähler manifolds

Useful (and inspiring) references are [10, 11, 12, 13].

3.1 **Basic definitions**

Definition 3.1 (Hyperkähler manifold). A *hyperkähler manifold* is a tuple (X, g, I_1, I_2, I_3) , where (X, g) is a Riemannian manifold equipped with three complex structures I_i obeying $I_1I_2 = I_3$, such that (X, g, I_i) is Kähler for i = 1, 2, 3.

It is crucial that we require the *single* metric *g* to be Kähler for *all* of the I_i : this is a very strong condition! We denote the three corresponding Kähler forms ω_i . Sometimes it is convenient to use instead the notation $(I_1, I_2, I_3) = (I, J, K)$ and $(\omega_1, \omega_2, \omega_3) = (\omega_I, \omega_J, \omega_K)$.

Exercise 3.1. Show that the relations $I_1I_2 = I_3$ and $I_1^2 = I_2^2 = I_3^2 = -1$ are equivalent to the full set of quaternion relations

$$I_1 I_2 = I_3, \quad I_2 I_1 = -I_3,$$
 (3.1)

$$I_2I_3 = I_1, \quad I_3I_2 = -I_1,$$
 (3.2)

$$I_3I_1 = I_2, \quad I_1I_3 = -I_2,$$
 (3.3)

$$I_1^2 = I_2^2 = I_3^2 = -1. ag{3.4}$$

Proposition 3.2 (Hyperkähler manifolds are holomorphic symplectic). If *X* is hyperkähler then $\Omega_1 = \omega_2 + i\omega_3$ is a *holomorphic symplectic form* with respect to structure I_1 (and similarly with the indices 1, 2, 3 cyclically permuted.)

Proof.

$$\Omega_1(v,w) = \omega_2(v,w) + \mathrm{i}\omega_3(v,w) \tag{3.5}$$

$$= g(I_2v, w) + ig(I_3v, w)$$
(3.6)

Thus

$$\Omega_1(I_1v, w) = g(I_2I_1v, w) + ig(I_3I_1v, w)$$
(3.7)

$$= -g(I_3v, w) + ig(I_2v, w)$$
(3.8)

$$= i\Omega_1(v, w) \tag{3.9}$$

and similarly

$$\Omega_1(v, I_1 w) = \mathrm{i}\Omega_1(v, w). \tag{3.10}$$

It follows that Ω_1 is of type (2,0) for I_1 , $\Omega_1 \in \Omega_{I_1}^{2,0}(X)$. The nondegeneracy follows from the nondegeneracy for the ω_i : namely, for any $v \in T_{I_1}^{1,0}X$,

$$\Omega_1(v,\cdot) = 0 \implies \Omega_1(v+\bar{v},\cdot) = 0 \implies \omega_2(v+\bar{v},\cdot) = 0 \implies v+\bar{v} = 0 \implies v = 0.$$
(3.11)

The remaining claims are obtained by cyclic permutations.

Corollary 3.3 (Hyperkähler manifolds have dimension 4n**).** If *X* is hyperkähler then dim_{\mathbb{R}} *X* is a multiple of 4.

Proof. This follows directly from Proposition 2.55.

Proposition 3.4 (Explicit formula for the complex structures on a hyperkähler manifold in terms of ω_i). If X is hyperkähler then

$$I_1 = \omega_3^{-1} \omega_2 \tag{3.12}$$

and cyclic permutations. (What this formula really means: view ω_2 as a map $TX \to T^*X$ namely $v \mapsto \omega_2(v, \cdot)$, and ω_3^{-1} as a map $T^*X \to TX$ namely $\omega_3(v, \cdot) \mapsto v$; then the composition $\omega_3^{-1}\omega_2 : TX \to TX$ is I_1 .)

Proof. What we have to check is that $\omega_3(I_1v, \cdot) = \omega_2(v, \cdot)$. But

$$\omega_3(I_1v, \cdot) = g(I_3I_1v, \cdot) = g(I_2v, \cdot) = \omega_2(v, \cdot)$$
(3.13)

as desired.

Corollary 3.5 (The ω_i **determine the hyperkähler metric).** If X is hyperkähler then

$$g = -\omega_1 \omega_3^{-1} \omega_2 \tag{3.14}$$

and cyclic permutations. (Here similarly we view *g* as a map $TX \rightarrow T^*X$, namely $v \mapsto g(v, \cdot)$.)

Proposition 3.6 (Condition for forms ω_i **to give a hyperkähler metric).** Suppose *X* is a manifold with symplectic forms $\omega_1, \omega_2, \omega_3$, obeying the condition that

$$-\omega_1 \omega_3^{-1} \omega_2 = -\omega_2 \omega_1^{-1} \omega_3 = -\omega_3 \omega_2^{-1} \omega_1, \tag{3.15}$$

and that this quantity, g, is *positive definite* as a symmetric bilinear form. Then g is a hyperkähler metric on X, with ω_i the associated Kähler forms.

Proof. [...]

Exercise 3.2. Suppose (X, g, I_1, I_2, I_3) is a hyperkähler manifold. Fix any $\vec{s} = (s_1, s_2, s_3) \in S^2 \subset \mathbb{R}^3$, and set

$$I_{\vec{s}} = \sum_{i=1}^{3} s_i I_i, \qquad \omega_{\vec{s}} = \sum_{i=1}^{3} s_i \omega_i.$$
(3.16)

Show that $(X, I_{\vec{s}}, g)$ is a Kähler manifold, with Kähler form $\omega_{\vec{s}}$.

In other words, a hyperkähler metric is Kähler for a whole S^2 of complex structures, not only three of them. We can think of this S^2 as the set of norm-1 imaginary quaternions. Specifying I_1 , I_2 , I_3 is equivalent to specifying the whole collection of $I_{\vec{s}}$.

Note that the antipodal map acts in a simple way: $I_{-\vec{s}} = -I_{\vec{s}}$, the opposite complex structure of $I_{\vec{s}}$ — i.e. the antipodal map exchanges holomorphic and antiholomorphic.

Exercise 3.3. Given a Riemannian manifold (X, g) and a hyperkähler structure thereon, specified by complex structures $I_{\vec{s}}$, show that we get another hyperkähler structure by choosing an element $T \in SO(3)$ and defining

$$I'_{\vec{s}} = I_{T\vec{s}}.$$
 (3.17)

Thus SO(3) naturally acts on the set of hyperkähler manifolds.

3.2 First examples

Example 3.7 (Flat quaternionic space). Being a real vector space, \mathbb{H} is a manifold of real dimension 4. If we identify $T_p\mathbb{H} \simeq \mathbb{H}$ in the obvious way, the quaternion norm $||q||^2 = q\bar{q}$

induces a metric *g* on \mathbb{H} . The operations of left-multiplication by i, j and k give complex structures I_1 , I_2 , I_3 on \mathbb{H} , obeying the quaternion algebra. Evidently these are all covariantly constant, so *g* is Kähler for all three of these complex structures, and thus \mathbb{H} is hyperkähler.

To introduce coordinates we identify \mathbb{H} with \mathbb{R}^4 via the map

$$x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \mapsto (x_0, x_1, x_2, x_3).$$
 (3.18)

Then the symplectic forms are

$$\omega_1 = \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3, \tag{3.19}$$

$$\omega_2 = \mathrm{d}x_0 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_1, \tag{3.20}$$

$$\omega_3 = \mathrm{d}x_0 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2, \tag{3.21}$$

or more uniformly

$$\omega_i = \mathrm{d}x_0 \wedge \mathrm{d}x_i + \star \mathrm{d}x_i \tag{3.22}$$

where \star denotes the Hodge star of \mathbb{R}^3 (with its standard orientation), not \mathbb{R}^4 . The holomorphic symplectic form is

$$\Omega_1 = \omega_2 + i\omega_3 = dw_1 \wedge dz_1, \qquad w_1 = x_0 + ix_1, \qquad z_1 = x_2 + ix_3.$$
(3.23)

 w_1 and z_1 are complex coordinates with respect to I_1 . Thus, in structure I_1 , \mathbb{H} is biholomorphic to \mathbb{C}^2 . Similarly we can write

$$\Omega_i = \omega_{i+1} + i\omega_{i+2} = dw_i \wedge dz_i, \qquad w_i = x_0 + ix_i, \qquad z_i = x_{i+1} + ix_{i+2}$$
(3.24)

(where we adopt the convention $x_{i+3} = x_i$.)

All this generalizes in a straightforward way to \mathbb{H}^n or better, an affine space modeled on \mathbb{H}^n , or even better, an affine space modeled on a quaternionic vector space.

Exercise 3.4. Verify the explicit formulas (3.22) for the symplectic forms ω_i on \mathbb{H} , and write a formula for $\omega_{\vec{s}}$.

It has a lot of symmetry. For example, II acts on itself by translations preserving the hyperkähler structure. Also the group O(4) acts on II by isometries, but these do *not* generally preserve the hyperkähler structure. However, we do have the following. The unit sphere in II is a Lie group, which happens to be isomorphic to SU(2).³ Thus we have an action of $SU(2) \times SU(2)$ on II by

$$(q,q') \cdot x = qxq'^{-1}.$$
 (3.26)

This gives a map $SU(2) \times SU(2) \rightarrow SO(4)$. Said otherwise, O(4) has two canonical SU(2) subgroups, which we call $SU(2)_L$ and $SU(2)_R$ (for "left" and "right.") (Incidentally, this map has kernel $\{(1,1), (-1,-1)\} \simeq \mathbb{Z}_2$, thus gives an isomorphism $SO(4) \simeq (SU(2) \times SU(2))/\mathbb{Z}_2$.)

³The isomorphism can be given explicitly by the formula

$$x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \mapsto \begin{pmatrix} x_0 + x_1 \mathbf{i} & x_2 + x_3 \mathbf{i} \\ -x_2 + x_3 \mathbf{i} & x_0 - x_1 \mathbf{i} \end{pmatrix}$$
(3.25)

but we will not need to use this anywhere.

Exercise 3.5. Show that $SU(2)_R$ acts on \mathbb{H} by *triholomorphic* isometries, i.e. isometries which are holomorphic for all of I_1 , I_2 , and I_3 .

Exercise 3.6. Show that the action of $T \in SU(2)_L$ on \mathbb{H} has

$$T^* I_{\vec{s}} = I_{T\vec{s}} \tag{3.27}$$

On the right side, by $T\vec{s}$ we mean the *conjugation* action of the unit quaternion T on the sphere of norm-1 imaginary quaternions, $\vec{s} \mapsto T^{-1}\vec{s}T$ (which gives the standard double-covering $SU(2) \rightarrow SO(3)$.)

Exercise 3.7. Show that in any complex structure $I_{\vec{s}}$, \mathbb{H} is biholomorphic to \mathbb{C}^2 .

Example 3.8 (Quotients of \mathbb{H}). It follows from Exercise 3.5 that, if we choose a subgroup $\Gamma \subset SU(2)_R$, the quotient \mathbb{H}/Γ is a hyperkähler orbifold: in particular, it carries a natural hyperkähler structure on the locus where it is a manifold. For example, if Γ is a discrete subgroup, it acts freely away from the origin, so

$$X_{\Gamma}^{\circ} = (\mathbb{H} \setminus \{0\}) / \Gamma \tag{3.28}$$

is a hyperkähler manifold. However, this hyperkähler manifold is *incomplete*, since the origin is at finite distance.



Example 3.9 ($\mathbb{R}^3 \times S^1$). Since translations preserve the hyperkähler structure, we can divide \mathbb{H} out by \mathbb{Z} acting by translations

$$x_0 \to x_0 + 2\pi n \tag{3.29}$$

to get another hyperkähler manifold,

$$X = \mathbb{H}/\mathbb{Z} \simeq \mathbb{R}^3 \times S^1. \tag{3.30}$$

In structure I_1 we have

$$\Omega_1 = -i\frac{d\mathcal{X}_1}{\mathcal{X}_1} \wedge dz_1, \qquad \mathcal{X}_1 = \exp(i(x_0 + ix_1)), \qquad z_1 = x_2 + ix_3 \tag{3.31}$$

The functions (z_1, \mathcal{X}_1) make (X, I_1) biholomorphic to $\mathbb{C} \times \mathbb{C}^{\times}$.

Exercise 3.8. Show that SO(3) acts by isometries on $X = \mathbb{H}/\mathbb{Z}$, with

$$T^* I_{\vec{s}} = I_{T\vec{s}}.$$
 (3.32)

Exercise 3.9. Show that for any \vec{s} , $X = \mathbb{H}/\mathbb{Z}$ with complex structure $I_{\vec{s}}$ is biholomorphic to $\mathbb{C} \times \mathbb{C}^{\times}$.

Crudely speaking, the different complex structures $I_{\vec{s}}$ correspond to different ways of picking which direction in \mathbb{R}^3 will get paired up with the circle direction to make a \mathbb{C}^{\times} .

Example 3.10 (Incomplete Gibbons-Hawking spaces). Now we generalize from $\mathbb{R}^3 \times S^1$ to a more general hyperkähler space with U(1) action. We begin with the case of a *free* action, i.e. a principal U(1) bundle.

Fix some open set $U \subset \mathbb{R}^3$ and let $V : U \to \mathbb{R}_{>0}$ be a positive harmonic function.⁴ Then $\Delta V = d \star dV = 0$, so if we write

$$F = -2\pi \star \mathrm{d}V \tag{3.33}$$

then we have dF = 0.

Fix a principal U(1) bundle X over U, carrying a connection Θ whose curvature is F. (Such an X exists if and only if $[F/2\pi]$ lies in the image of $H^2(U,\mathbb{Z}) \to H^2_{d\mathbb{R}}(U,\mathbb{R})$.) Globally this means a 1-form Θ on X such that $d\Theta = F$ (more precisely on the RHS we have the pullback of F to X) and $\Theta \cdot \partial_{\chi} = 1$, where ∂_{χ} denotes the vector field on X generating the U(1) action. If we choose a local trivialization of the U(1)-bundle X over a patch $U_{\alpha} \subset U$, with local fiber coordinate $\chi_{\alpha} \in \mathbb{R}/2\pi\mathbb{Z}$, then Θ is locally of the form

$$\Theta = A_{\alpha} + \mathrm{d}\chi_{\alpha},\tag{3.34}$$

with $A_{\alpha} \in \Omega^{1}(U_{\alpha})$, $dA_{\alpha} = F$. For convenience write $\tilde{\Theta} = \Theta/2\pi$.

We introduce three symplectic forms on *X*, generalizing (3.22):

$$\omega_i = \tilde{\Theta} \wedge \mathrm{d}x_i + V \star \mathrm{d}x_i. \tag{3.35}$$

To check that these are indeed closed,

$$d\omega_i = -\star dV \wedge dx_i + dV \wedge \star dx_i = 0 \tag{3.36}$$

(the last equality because all of these forms are pulled back from \mathbb{R}^3 , and on \mathbb{R}^3 we always have $\star \alpha \land \beta = \alpha \land \star \beta$ for 1-forms α, β .)

Then define

$$\Omega_1 = \omega_2 + \mathrm{i}\omega_3 \tag{3.37}$$

$$= \tilde{\Theta} \wedge dz_1 + V \star d(x_2 + ix_3) \tag{3.38}$$

$$= \tilde{\Theta} \wedge dz_1 + V(dx_3 \wedge dx_1 + idx_1 \wedge dx_2)$$
(3.39)

$$=\tilde{\Theta}\wedge dz_1 + iVdx_1\wedge dz_1 \tag{3.40}$$

$$= V\alpha_1 \wedge \mathrm{d}z_1 \tag{3.41}$$

where we introduced

$$z_1 = x_2 + ix_3, \qquad \alpha_1 = V^{-1}\tilde{\Theta} + idx_1.$$
 (3.42)

⁴Our conventions for the Laplace operator on $\Omega^0(\mathbb{R}^3)$ are: $\Delta f = d \star df = \sum_i \partial_i^2 f \text{ dvol.}$

Then ker Ω_1 is spanned by

$$\hat{\partial}_2 + i\hat{\partial}_3, \qquad 2\pi V \partial_{\chi} + i\hat{\partial}_1, \qquad (3.43)$$

where ∂_{χ} is the globally defined generator of the U(1) action (shifting χ), and $\hat{\partial}_i$ means the parallel lift of ∂_i from \mathbb{R}^3 to X, i.e. the lift obeying $\Theta \cdot \hat{\partial}_i = 0$. Thus by Proposition 2.56, Ω_1 determines a complex structure I_1 on X, which acts by +i on dz_1 and α_1 . In this structure, dz_1 is of type (1,0), so z_1 is a holomorphic map,

$$z_1: \mathbf{X} \to \mathbb{C}. \tag{3.44}$$

Morally this map makes *X* into something like a \mathbb{C}^{\times} -bundle over a patch of \mathbb{C} (generalizing the $\mathbb{C} \times \mathbb{C}^{\times}$ which we got in Example 3.9). It is not quite a \mathbb{C}^{\times} -bundle in general (since *U* was an arbitrary open subset of \mathbb{R}^3), but it does at least have a holomorphic vector field tangent to the fibers, $\partial_{\chi} - iV^{-1}\hat{\partial}_1$.

Similarly we have

$$\Omega_i = V \alpha_i \wedge \mathrm{d} z_i \tag{3.45}$$

where

$$z_i = x_{i+1} + ix_{i+2}, \qquad \alpha_i = V^{-1}\Theta + idx_i$$
 (3.46)

and we can use this to define complex structures I_2 , I_3 . Just as for $\mathbb{R}^3 \times S^1$ these obey the quaternion relation $I_1I_2 = I_3$ — we could check this directly, or just note that pointwise (3.45)-(3.46) are isomorphic to (3.24) (take $dx_i \rightarrow \sqrt{V} dx_i$ and $dx_0 \rightarrow \tilde{\Theta}/\sqrt{V}$), and the Ω_i determine the I_i by pointwise calculations, so they must obey the same relations they obeyed on \mathbb{R}^4 .

Now we compute *g* from I_1 and ω_1 :

$$\omega_1 = V \operatorname{Re} \alpha_1 \wedge \operatorname{Im} \alpha_1 + V \operatorname{Re} dz_1 \wedge \operatorname{Im} dz_1 \tag{3.47}$$

which gives

$$g = V((\operatorname{Re} \alpha_1)^2 + (\operatorname{Im} \alpha_1)^2) + V((\operatorname{Re} dz_1)^2 + (\operatorname{Im} dz_1)^2)$$
(3.48)

$$= V(V^{-2}\tilde{\Theta}^2 + dx_1^2) + V(dx_2^2 + dx_3^2)$$
(3.49)

$$= V \|dx\|^2 + V^{-1}\tilde{\Theta}^2$$
(3.50)

Thus the metric on *X* given by

$$g = V \|dx\|^2 + V^{-1}\tilde{\Theta}^2$$
(3.51)

is hyperkähler, with Kähler forms ω_i .

The principal U(1) action on X is by isometries preserving the hyperkähler structure (this is clear since nothing in ω_i depends on the fiber coordinates).

Exercise 3.10. Let *L* be a line in *U*, oriented in the direction $\vec{s} = (1, 0, 0)$. Show that $\pi^{-1}(L)$ is a complex submanifold of *X* in the complex structure I_1 . (Hint: it's sufficient to show that the holomorphic symplectic form Ω_1 vanishes along this submanifold — why?) If $U = \mathbb{R}^3$ and V = 1, describe this complex submanifold explicitly in complex coordinates.

Example 3.11 ($\mathbb{R}^4 \setminus \{0\}$ **as an incomplete Gibbons-Hawking space).** Consider Example 3.10 with $U = \mathbb{R}^3 \setminus \{0\}$. In spherical coordinates (r, θ, φ) take⁵

$$V(x) = \frac{1}{4\pi r}.\tag{3.55}$$

Then we have

$$F = \frac{1}{2}\sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\varphi \tag{3.56}$$

which has $\int_{S^2} \frac{F}{2\pi} = 1$ and thus obeys our quantization condition, so that there exists a circle bundle $X \to U$ with this curvature. The total space of such a circle bundle over S^2 is the Hopf fibration $S^3 \to S^2$.

X thus doesn't extend as a U(1) bundle over the point x = 0. Nevertheless it *does* extend as a hyperkähler manifold. Indeed, near 0 the circle fibers of X are shrinking to zero length, and it is possible to add a single point over 0, in such a way that the total space is a hyperkähler manifold with non-free U(1) action, and the quotient is \mathbb{R}^3 . To see this explicitly, let's fix a trivialization away from $\theta = \pi$, with respect to which $\Theta = A + d\chi$,

$$A = \frac{1}{2}(1 - \cos\theta) \,\mathrm{d}\varphi. \tag{3.57}$$

Now we have

$$g = \frac{1}{4\pi r} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) + \frac{r}{\pi} \left(\frac{1}{2} (1 - \cos \theta) d\varphi + d\chi\right)^2$$
(3.58)

$$= \frac{1}{4\pi r} dr^{2} + \frac{r}{\pi} \left(\frac{1}{4} d\theta^{2} + \frac{1}{2} (1 - \cos \theta) d\varphi^{2} + d\chi^{2} + (1 - \cos \theta) d\varphi d\chi \right).$$
(3.59)

Now the surprise is that the second part is just the round metric on S^3 with radius $\rho = \sqrt{r/\pi}$, and the whole *g* is isometric to the Euclidean metric on $\mathbb{R}^4 \setminus \{0\}$. To see this explicitly take

$$y_0 = \sqrt{r/\pi} \sin(\theta/2) \cos(\varphi + \chi), \qquad (3.60)$$

$$y_1 = \sqrt{r/\pi}\sin(\theta/2)\sin(\varphi + \chi), \qquad (3.61)$$

$$y_2 = -\sqrt{r/\pi}\cos(\theta/2)\sin(\chi), \qquad (3.62)$$

$$y_3 = -\sqrt{r/\pi}\cos(\theta/2)\cos(\chi), \qquad (3.63)$$

and then compute directly that

$$g = \sum_{i=0}^{3} \mathrm{d}y_i^2. \tag{3.64}$$

$$x_1 = r\cos\theta,\tag{3.52}$$

$$x_2 = r\sin\theta\cos\phi,\tag{3.53}$$

$$x_3 = r\sin\theta\sin\phi. \tag{3.54}$$

⁵Our conventions for spherical coordinates are:

Exercise 3.11. Show that the hyperkähler structure on *X* in Example 3.11 matches the standard one in $\mathbb{R}^4 \setminus \{0\}$, given in Example 3.7. This means that *X* can be extended to a complete hyperkähler manifold \bar{X} , on which U(1) acts by isometries with a single fixed point, and $\bar{X}/U(1) = \mathbb{R}^3$. In other words, starting from *X* we can "fill in the missing fiber over 0" in a smooth hyperkähler way, and the resulting hyperkähler manifold is isomorphic to \mathbb{R}^4 with its standard hyperkähler structure.

In particular, complex structure I_1 on X has holomorphic coordinates $w_1 = y_0 + iy_1$ and $z_1 = y_2 + iy_3$. Thus the loci $z_1 = 0$ and $w_1 = 0$ are complex submanifolds of I_1 , intersecting at a single point. These correspond to $\theta = 0$ and $\theta = \pi$, i.e. the preimages in X of the two half-lines $x_2 = x_3 = 0$, $\pm x_1 \ge 0$. (From Exercise 3.10 we already knew that the preimage of the half-line is a complex manifold, *except* at the point over x = 0. The discussion above shows that it actually extends to an honest complex manifold even over x = 0.)



There is a more conceptual way of arriving at Example 3.11:

Exercise 3.12. Consider again the standard hyperkähler structure on \mathbb{R}^4 (Example 3.7), which is acted on by $SU(2)_R$. Show that there is a subgroup $U(1) \subset SU(2)_R$ which acts by $(w_1, z_1) \rightarrow (e^{-i\alpha}w_1, e^{i\alpha}z_1)$. Thus this action preserves the hyperkähler structure. Moreover show that this action admits moment maps $\mu_i : \mathbb{R}^4 \rightarrow \mathfrak{u}(1)^*$ with respect to all of the ω_i . If we identify $\mathfrak{u}(1) \simeq \mathbb{R}$, we obtain a map $\vec{\mu} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by (μ_1, μ_2, μ_3) . Show that there is exactly one $x \in \mathbb{R}^3$ such that $\vec{\mu}^{-1}(x)$ is a single point; by shifting each μ_i by a constant we can arrange that this point is $0 \in \mathbb{R}^3$. Then show that the map $\vec{\mu}$ realizes $\mathbb{R}^4 \setminus \{0\}$ as a principal U(1) bundle over $\mathbb{R}^3 \setminus \{0\}$, and that the metric is of the form in Example 3.11. (Hint: at some points it is convenient to consider the complex combination $\mu_2 + i\mu_3$, which is a moment map for the holomorphic symplectic form Ω_1 .)

Example 3.12 ($(\mathbb{R}^4/\mathbb{Z}_k) \setminus \{0\}$ **as an incomplete Gibbons-Hawking space).** We can modify Example 3.11 by taking some integer k > 0 and setting

$$V = \frac{k}{4\pi r}.$$
(3.65)

Then the metric (3.51) (divided by k) matches with that of Example 3.11 except that the circumference of the circle fibers is shorter by a factor of k. In other words, this metric is obtained by dividing out the metric of $\mathbb{R}^4 / \{0\}$ by a subgroup $\mathbb{Z}_k \subset U(1)$. In turn this $U(1) \subset SU(2)_R$, so this is a special case of the quotients described in Example 3.8. Unlike the case k = 1, here we *cannot* fill in the missing point to make a manifold (if we could, we would have a point whose link is the lens space S^3 / \mathbb{Z}_k .)

Example 3.13 (Gibbons-Hawking spaces). Extending Example 3.10 and Example 3.11, let us fix $U \subset \mathbb{R}^3$, distinct points x_1, \ldots, x_k in U, and a function V on U with⁷

$$\Delta V = -\sum_{i=1}^{n} \delta(x - x_i).$$
(3.66)

Then consider a U(1)-bundle X over $U \setminus \{x_i\}$ with hyperkähler structure as in Example 3.10. Consider a small S^2 around x_i :

$$\int_{S^2} \frac{F}{2\pi} = -\int_{S^2} \star dV = -\int_{B^3} d \star dV = \int_{B^3} \delta(x - x_i) = 1$$
(3.67)

so the U(1) bundle X restricted to this S^2 has degree 1. Thus, X doesn't extend as a U(1) bundle over the point x_i .

Nevertheless it *does* extend as a hyperkähler manifold. Indeed, near x_i we have

$$V = \frac{1}{4\pi \|x - x_i\|} + \text{regular}, \qquad (3.68)$$

so the circle fibers of *X* are shrinking to zero length just as in Example 3.11. It is possible to add a single point over each x_i to get a new space \bar{X} , which is a hyperkähler manifold with non-free U(1) action, with $\bar{X}/U(1) = U$. Indeed, choosing the coordinate $x' = x - x_i$, a neighborhood *W* of x' = 0 in *X* can be identified as a principal U(1)-bundle with a neighborhood of x = 0 in Example 3.11. We have $V = V_0 + \delta V$ and $\Theta = \Theta_0 + \delta \Theta$, where V_0 and Θ_0 are as in Example 3.11, δV is a smooth (harmonic) function on *W*, and $\delta \Theta$ a smooth 1-form on *W*. We then make the same change of coordinates to (y_0, y_1, y_2, y_3) we made in Example 3.11, and compute that δV and $\delta \Theta$ do not contribute to the ω_i at y = 0: in other words, in these coordinates we have

$$\omega_i = \mathrm{d}y_0 \wedge \mathrm{d}y_i + \mathrm{d}y_{i+1} \wedge \mathrm{d}y_{i+2} + \delta\omega_i \tag{3.69}$$

where $\delta \omega_i$ vanishes at y = 0. It follows that the hyperkähler structure indeed extends over y = 0.

Example 3.14 (Eguchi-Hanson space). This is a case of Example 3.13 with two singularities. Fix distinct points $x_1, x_2 \in \mathbb{R}^3$, let $U = \mathbb{R}^3$, and

$$V(x) = \frac{1}{4\pi \|x - x_1\|} + \frac{1}{4\pi \|x - x_2\|}.$$
(3.70)

⁶This is trickier than I thought, but there is a proof at http://math.stackexchange.com/questions/561818.

⁷On the RHS δ means the "Dirac delta function" on \mathbb{R}^3 , which should be understood as a distributional 3-form; thus this equality is understood in the sense of distributions; it won't be important for our purpose to know exactly what it means, since we will deal only with some very concrete examples, but if you are interested, one source is Chapter 6 of [14].

The resulting \bar{X} is fibered over \mathbb{R}^3 with two degenerate fibers. From now on we drop the bar and just call it *X*. Let $\pi : X \to \mathbb{R}^3$ be the projection. Then $\pi^{-1}(x_1x_2)$ has the topology of S^2 .



Exercise 3.13. Show that this S^2 is a complex submanifold of X, with respect to the two complex structures $I_{\vec{s}}$ where \vec{s} is the direction from x_1 to x_2 or vice versa. (The trickiest point is to see that it is really a manifold, even when the two endpoints are included. Hint: by a rotation, we can assume without loss of generality that $\vec{s} = (1, 0, 0)$. Then away from the endpoints this S^2 is the locus $z_1 = 0$, with z_1 given in (3.42).)

Exercise 3.14. Show that the area of this S^2 , in the hyperkähler metric g, is $||x_1 - x_2||$. (Hint: complex submanifolds of Kähler manifolds are *calibrated* — the area is just $\int \omega_{\vec{s}}$.)

Exercise 3.15. Show that this S^2 has self-intersection number -2.

Exercise 3.16. Show that, in either of the complex structures $I_{\vec{s}}$ of the previous exercise, *X* is biholomorphic to $T^*\mathbb{CP}^1$. [warning, this one might be hard]

Eguchi-Hanson space is our first example where the $I_{\vec{s}}$ do not all give rise to the same complex manifold:

Exercise 3.17. Show that, in any *other* complex structure $I_{\vec{s}'}$, X has no compact 1-complexdimensional complex submanifolds. (Hint: use the fact that there is a holomorphic function $z : X \to \mathbb{C}$. The image of a compact connected 1-complex-dimensional submanifold would have to be a point. It might be convenient to assume \vec{s} is generic and $\vec{s}' = (1,0,0)$, in which case the relevant function is z_1 given in (3.42).)

Nevertheless we do have a 1-parameter group relating *some* of the $I_{\vec{s}}$:

Exercise 3.18. Show that *X* admits an action of $SO(2) \subset SO(3)$ by isometries, such that $T^*I_{\vec{s}} = I_{T\vec{s}}$. [check!]

Example 3.15 (Multi-Eguchi-Hanson spaces). [15] More generally, take $U = \mathbb{R}^3$, fix a collection of distinct points $x_1, \ldots, x_k \in \mathbb{R}^3$ and take

$$V(x) = \sum_{i=1}^{k} \frac{1}{4\pi \|x - x_i\|}.$$
(3.71)

Then a straight line segment in \mathbb{R}^3 connecting two x_i (and not meeting any others) gives
an S^2 in X. This S^2 is a complex submanifold with respect to two of the complex structures $I_{\vec{s}_i}$ just as before.

Asymptotically, the metric on this space is approximately what we would get by taking all $x_i = x_0$ for some fixed x_0 . In that case we would have simply

$$V(x) = \frac{k}{4\pi \|x - x_0\|}$$
(3.72)

which, as explained in Example 3.12, gives the hyperkähler structure of $\mathbb{R}^4/\mathbb{Z}_k$. Thus, when all the x_i are distinct, g is a kind of hyperkähler desingularization of $\mathbb{R}^4/\mathbb{Z}_k$.

A nice special case occurs when all of the x_i are collinear: then we have a single complex structure in which X contains k - 1 holomorphic spheres C_i , with intersection numbers $C_i \cdot C_{i+1} = 1$. In this complex structure X is the *minimal resolution* of the singularity $\mathbb{C}^2/\mathbb{Z}_k$ (sometimes called a "du Val singularity of type A_{k-1} ," e.g. because the intersection numbers of the C_i make up the Cartan matrix of type A_{k-1} .)



Example 3.16 (ALE spaces). One can consider the minimal resolutions X_{Γ} of the singularities at the origin in Example 3.8. Then X_{Γ} is an honest manifold, carrying a natural family of complete hyperkähler metrics [16]. These metrics asymptotically approach the metric on \mathbb{R}^4/Γ ; thus the X_{Γ} are called "ALE spaces", for "asymptotically locally Euclidean." In the case $\Gamma = \mathbb{Z}_k$ these hyperkähler metrics are the same as those of Example 3.15; for other Γ they are not Gibbons-Hawking spaces.

Example 3.17 (Taub-NUT space). This is a case of Example 3.13 with one singularity. Take $U = \mathbb{R}^3$ and

$$V(x) = 1 + \frac{1}{4\pi \|x - x_0\|}.$$
(3.73)

Now the asymptotic metric looks like that of $\mathbb{R}^3 \times S^1$ rather than \mathbb{R}^4 .

Exercise 3.19. Show that Taub-NUT space is biholomorphic to \mathbb{C}^2 , in any of its complex structures. [warning, this one might be hard]

Despite this, Taub-NUT space is definitely not the same hyperkähler manifold as II! So a hyperkähler manifold contains more information than just a family of complex manifolds.

Similarly by taking multiple singularities in *V* we could obtain the "multi-Taub-NUT" family of metrics.

3.3 The twistor family, first approach

We know that $\Omega_1 = \omega_2 + i\omega_3$ is a holomorphic symplectic form for I_1 . We can do similarly for an arbitrary complex structure $I_{\vec{s}}$, and can even arrange that the resulting holomorphic symplectic forms vary holomorphically:

Lemma 3.18 (Holomorphic symplectic forms on a hyperkähler manifold vary holomor**phically over** S^2 **).** Suppose X is a hyperkähler manifold. Let

$$V = \operatorname{Span}(\omega_1, \omega_2, \omega_3) \subset \Omega^2_{\mathbb{C}}(X).$$
(3.74)

For each $\vec{s} \in S^2$, let

$$L_{\vec{s}} = V \cap \Omega^{2,0}_{I_{\vec{s}}}(X).$$
(3.75)

Then:

- 1. dim_C $L_{\vec{s}} = 1$.
- 2. If we equip S^2 with its standard complex structure, L is a holomorphic line subbundle of the trivial rank 3 holomorphic bundle $S^2 \times V$, canonically isomorphic to $\mathcal{O}(-2) \to \mathbb{CP}^1$.

Proof. For (1) it's enough to compute for $\vec{s} = (1, 0, 0)$, and there note $\omega_2 + i\omega_3 \in \Omega_{\vec{s}}^{2,0}(X)$, $\omega_1 \in \Omega^{1,1}_{\vec{s}}(X), \omega_2 - i\omega_3 \in \Omega^{0,2}_{\vec{s}}(X)$. Thus $L_{\vec{s}}$ is 1-dimensional, spanned by $\omega_2 + i\omega_3$. For (2) here is a computational proof. Fix a complex coordinate ζ on $S^2 \setminus \{(-1,0,0)\},$

by

$$\zeta = \frac{s_3 - is_2}{1 + s_1}, \qquad (s_1, s_2, s_3) = \frac{(1 - |\zeta|^2, -2\operatorname{Im}\zeta, 2\operatorname{Re}\zeta)}{1 + |\zeta|^2}.$$
(3.76)

Thus we have

$$\zeta = 0 \leftrightarrow \vec{s} = (1, 0, 0), \quad \zeta = \infty \leftrightarrow \vec{s} = (-1, 0, 0), \tag{3.77}$$

$$\zeta = -\mathbf{i} \leftrightarrow \vec{s} = (0, 1, 0), \quad \zeta = \mathbf{i} \leftrightarrow \vec{s} = (0, -1, 0), \tag{3.78}$$

$$\zeta = 1 \leftrightarrow \vec{s} = (0, 0, 1), \quad \zeta = -1 \leftrightarrow \vec{s} = (0, 0, -1).$$
 (3.79)

Using this translation we will sometimes write I_{ζ} for $I_{\vec{s}}$ and L_{ζ} for $L_{\vec{s}}$. Now consider the holomorphic family of 2-forms

$$\Omega(\zeta) = \frac{\omega_2 + i\omega_3}{2\zeta} - i\omega_1 + \frac{\omega_2 - i\omega_3}{2}\zeta.$$
(3.80)

We want to check that $\Omega(\zeta) \in L_{\zeta}$. As a quick check note that

$$\Omega(\zeta = 1) = -i\Omega_3, \quad \Omega(\zeta = -i) = -\Omega_2, \quad \zeta\Omega(\zeta = 0) = \Omega_1.$$
 (3.81)

For general ζ , what we want to know is that that applying I_{ζ} to the first slot of $\Omega(\zeta)$ gives $i\Omega(\zeta)$, so that $\Omega(\zeta) \in L_{\zeta}$. This can be checked (a bit laboriously) using (3.76), $I_i\omega_i = g$, $I_i\omega_{i+1} = -\omega_{i+2}, I_i\omega_{i+2} = \omega_{i+1}.$

Now, $\Omega(\zeta)$ blows up as $\zeta \to 0$ or $\zeta \to \infty$. Still, for any $\zeta \in \mathbb{C}$, $\zeta \Omega(\zeta)$ is a nonvanishing element of L_{ζ} ; similarly, for any $\zeta \in \mathbb{C}^{\times} \cup \{\infty\}$, $\Omega(\zeta)/\zeta$ is a nonvanishing element of L_{ζ} . Thus we have two holomorphic trivializations of L_{ζ} over these two patches, differing by the transition function ζ^{-2} . This gives the desired holomorphic identification $L_{\zeta} \simeq \mathcal{O}(-2)$ and finishes the proof.

We will use the formula (3.80) frequently. Warning: different authors (and different papers by the same authors) have different conventions for the normalization of $\Omega(\zeta)$ and the definition of ζ .

Lemma 3.19 ((1,0)-covectors on a hyperkähler manifold vary holomorphically over S^2). Suppose X is a hyperkähler manifold. Then $(T^*)_{I_7}^{1,0}X = (1 - i\zeta I_3)(T^*)_{I_1}^{1,0}X$.

Proof. If $\beta \in (T^*)^{1,0}_{I_{\zeta}} X$ we may compute directly that

$$I_{\zeta}(1+\zeta I_3)\beta = \frac{((1-|\zeta|^2)I_1 - 2\operatorname{Im}\zeta I_2 + 2\operatorname{Re}\zeta I_3)(1+\zeta I_3)\beta}{1+|\zeta|^2}$$
(3.82)

$$= \mathbf{i}(1+\zeta I_3)\boldsymbol{\beta},\tag{3.83}$$

so at least $(1 + \zeta I_3)(T^*)_{I_1}^{1,0}X \subset (T^*)_{I_\zeta}^{1,0}X$. Moreover the map $(1 + \zeta I_3)$ is injective on $(T^*)_{I_1}^{1,0}X$, since I_3 and I_1 anticommute, so that they cannot have any simultaneous eigenvectors. So by dimension counting we are done.

Corollary 3.20 ((0, 1)**-vectors on a hyperkähler manifold vary holomorphically over** S^2 **).** Suppose *X* is a hyperkähler manifold. Then $T_{I_z}^{0,1}X = (1 + i\zeta I_3)T_{I_1}^{0,1}X$.

Proof. Take the complex conjugate of the statement in Lemma 3.19, and use the metric to identify $\overline{T}^* \simeq T$.

Exercise 3.20. Carry out the omitted computation in the proof of Lemma 3.19. Note a tricky point: here we are acting on covectors rather than vectors, so the operators I_i we are using are the *transposes* of the usual ones, so $I_2I_1 = I_3$ rather than the usual relation. (Or alternatively, carry out the analogous computation to prove Corollary 3.20 directly, in which case you would use the usual $I_1I_2 = I_3$.)

Exercise 3.21. Use Lemma 3.19 to give an alternative proof of Lemma 3.18. (You might want to look at [17].)

3.4 The twistor family, second approach

In this section we fit Lemma 3.18 and Lemma 3.19 into a slightly more abstract framework. The main aim is Proposition 3.27 below, which says that a quaternionic vector space is equivalent to a certain kind of vector bundle over \mathbb{CP}^1 .

Definition 3.21 (Quaternionic vector space). A *quaternionic vector space* is a real vector space *V* with endomorphisms I_1 , I_2 , I_3 obeying the quaternion algebra.

Exercise 3.22. Suppose *V* is a quaternionic vector space, of real dimension *n*. Show that *n* is a multiple of 4, and the group of automorphisms of *V* commuting with ρ is isomorphic to $GL(\frac{n}{4}, \mathbb{H})$. [need to use existence of basis]

Definition 3.22 (Hyperkähler vector space). A *hyperkähler vector space* is a quaternionic vector space *V*, with a positive definite symmetric bilinear form *g* obeying for all *i*

$$g(I_i v, I_i w) = g(v, w).$$
 (3.84)

Exercise 3.23. Suppose *V* is a hyperkähler vector space, of real dimension 4n. Show that the group of automorphisms of *V* preserving (I_1 , I_2 , I_3 , g) is isomorphic to Sp(n). [need to use existence of basis]

Definition 3.23 (Pseudoreal structure). Let *V* be a complex vector space. A *pseudoreal structure* on *V* is a map $\rho : V \to V$ which is conjugate-linear and has $\rho^2 = -1$.

Exercise 3.24. Suppose (V, ρ) is a pseudoreal vector space, of complex dimension *n*. Show that *n* is even, and the group of automorphisms of *V* commuting with ρ is isomorphic to $GL(\frac{n}{2}, \mathbb{H})$. [need to use standard form for pseudoreal structure?]

This result suggests a close connection between pseudoreal vector spaces and quaternionic ones. We will now make this connection more explicit. Let $H = \mathbb{C}^2$, equipped with a pseudoreal structure

$$\rho_H(z,w) = (-\bar{w},\bar{z}),$$
(3.85)

and a skew pairing,

$$\varepsilon_H((z_1, w_1), (z_2, w_2)) = z_1 w_2 - w_1 z_2.$$
 (3.86)

The two combine to give a Hermitian metric on *H*,

$$g_H((z_1, w_1), (z_2, w_2)) = \varepsilon_H((z_1, w_1), \rho_H(z_2, w_2)) = z_1 \bar{z}_2 + w_1 \bar{w}_2.$$
(3.87)

H has a \mathbb{C} -linear action of \mathbb{H} , given explicitly by (3.25).

We can also think of *H* as the vector space of global sections of $\mathcal{O}(1) \rightarrow S^2$,

$$H = H^0(\mathcal{O}(1)).$$
 (3.88)

 ρ_H then comes from a "pseudoReal" structure on the line bundle $\mathcal{O}(1)$: namely, if we let σ denote the antipodal map of \mathbb{CP}^1 , (3.85) gives

$$\rho_{\mathcal{O}(1)}: \sigma^* \mathcal{O}(1) \simeq \overline{\mathcal{O}(1)}. \tag{3.89}$$

Proposition 3.24 (Linear algebra of quaternionic vector spaces). We have:

• If *V* is a quaternionic vector space with $\dim_{\mathbb{R}} V = 4n$, then there is a canonical decomposition of complex vector spaces

$$V_{\mathbb{C}} \simeq H \otimes E, \tag{3.90}$$

where *E* is a complex vector space with dim_{$\mathbb{C}} E = 2n$, equipped with a pseudoreal structure ρ_E . Moreover:</sub>

- the real structure on $V_{\mathbb{C}}$ is $\rho_V = \rho_H \otimes \rho_E$,
- the quaternion action on $V_{\mathbb{C}}$ is induced from the action on *H*.

This construction gives an equivalence between the category of quaternionic vector spaces *V* and the category of pseudoreal vector spaces *E*.

• If *V* is a hyperkähler vector space, then in addition there is a nondegenerate skew pairing ε_E on *E*, such that the complexification of *g* is

$$g_{\mathbb{C}} = \varepsilon_H \otimes \varepsilon_E. \tag{3.91}$$

This construction gives an equivalence between the category of hyperkähler vector spaces V and the category of pseudoreal vector spaces E with compatible skew pairings.

Proof. [...]

Definition 3.25 (Twistorial vector space). A *twistorial vector space* is a holomorphic vector bundle $W \to S^2$, such that $W \otimes \mathcal{O}(-1)$ is trivial, equipped with a "Real" structure ρ_W : $\sigma^*W \xrightarrow{\sim} \overline{W}$, where σ is the antipodal map of \mathbb{CP}^1 .

Definition 3.26 (Metric twistorial vector space). A *metric twistorial vector space* is a twistorial vector space $W \rightarrow S^2$, equipped with a fiberwise nondegenerate $\Omega \in \wedge^2(W^*) \otimes \mathcal{O}(2)$, compatible with the Real structures in the sense that

$$(\rho_{\wedge^2 W^*} \otimes \rho_{\mathcal{O}(2)})^* \Omega = \overline{\Omega}, \tag{3.92}$$

and obeying a positivity condition [...]

Proposition 3.27 (Twistorial description of quaternionic vector spaces). Given a quaternionic vector space *V*, we obtain a twistorial vector space *W* by taking

$$W_{\vec{s}} = \ker(I_{\vec{s}} + \mathbf{i}) \subset V_{\mathbb{C}}.\tag{3.93}$$

Given a twistorial vector space W, we obtain a quaternionic vector space V by taking

$$V = H^0_{\mathbb{R}}(W) \tag{3.94}$$

i.e. the space of sections invariant under ρ_W . If *V* is a hyperkähler vector space, then the corresponding *W* is a metric twistorial vector space, and vice versa. All of these constructions give equivalences of categories.

Proof. [...]

3.5 The twistor space

Definition 3.28 (Twistor space). Given a hyperkähler manifold *X*, the *twistor space* of *X* is the manifold

$$\mathcal{Z} = X \times S^2 \tag{3.95}$$

equipped with the almost complex structure

$$I(x,\vec{s}) = I_{\vec{s}}(x) \oplus I_{S^2} \tag{3.96}$$

where I_{S^2} denotes the standard complex structure on S^2 .

Proposition 3.29 (Twistor space is a complex manifold). The almost complex structure I on \mathcal{Z} is integrable.

Proof. What we will use is that $\Omega(\zeta)$ determines the complex structure fiberwise, and $\Omega(\zeta)$ itself is holomorphic in ζ . (For a different proof, using instead Lemma 3.19, see [17].)

Concretely: we have

$$T^{0,1}_{(x,\zeta)}\mathcal{Z} = T^{0,1}_{x,I_{\zeta}}X \oplus T^{0,1}_{\zeta}S^2.$$
(3.97)

To see the integrability, consider a pair of vector fields in $T^{0,1}\mathcal{Z}$, of the form $v \oplus v'$ and $w \oplus w'$ with respect to this decomposition.

Part of the story is easy. Since v and w are both tangent to the fiber $X_{\zeta} = \pi^{-1}(\zeta)$ the bracket [v, w] is just the bracket on X_{ζ} , and we already know the integrability there, so $[v, w] \in T^{0,1}X$. Since v' and w' are both pulled back from the base S^2 , their bracket [v', w'] is also pulled back from the bracket on S^2 , and we already know the integrability there, so again $[v', w'] \in T^{0,1}S^2$.

All that remains is to check that $[v, w'] \in T^{0,1} \mathcal{Z}$. Choose local coordinates x^i for X and write

$$v = v^i(x,\zeta)\partial_i, \qquad w' = f(x,\zeta)\partial_{\bar{\zeta}}, \qquad \Omega = \Omega_{ij}(\zeta)dx^i \wedge dx^j.$$
 (3.98)

Here $v \in T^{0,1}X$ means it obeys the constraint

$$\Omega_{ij}(\zeta)v^i = 0. \tag{3.99}$$

Then

$$[v, f\partial_{\bar{\zeta}}] = (v^i \partial_i f) \partial_{\bar{\zeta}} - (\partial_{\bar{\zeta}} v^i) \partial_i.$$
(3.100)

The first term is evidently in $T^{0,1}S^2$; for the second term note that applying $\partial_{\bar{\zeta}}$ to (3.99) gives $\Omega_{ij}(\zeta)\partial_{\bar{\zeta}}v^i = 0$, so the second term is in $T^{0,1}X$; thus the whole RHS is in $T^{0,1}\mathcal{Z}$, which is what we want.

Proposition 3.30 (Properties of twistor spaces). Suppose *X* is a hyperkähler manifold with twistor space \mathcal{Z} . Then:

- 1. \mathcal{Z} is a complex manifold, with a holomorphic projection $\pi : \mathcal{Z} \to S^2$.
- 2. \mathcal{Z} carries a twisted fiberwise holomorphic symplectic form,

$$\Omega \in \Omega^{2,0}_{\text{fiber}}(\mathcal{Z}) \otimes \pi^* \mathcal{O}(2)$$
(3.101)

where $\Omega_{\text{fiber}}^{2,0}(\mathcal{Z}) = \wedge^2 (T_{\text{vert}}^{1,0}\mathcal{Z})^*$.

3. \mathcal{Z} carries a *real structure* (antiholomorphic involution) $\rho : \mathcal{Z} \to \mathcal{Z}$, such that:

(a) ρ covers the antipodal involution σ on S^2 :



(b) $\rho^*\Omega = \overline{\Omega}$. (In formulating this condition we use the standard conjugate-linear lift of σ to act on $\mathcal{O}(2)$.)

Proof. We already proved (1). For (2) use the Ω of (3.80). Finally for (3) take

$$\rho(x, \vec{s}) = (x, -\vec{s}).$$
(3.102)

The statement that $\rho^* \Omega = \overline{\Omega}$ then amounts to the fact that if we substitute $-1/\overline{\zeta}$ in (3.80) we get $-\overline{\Omega}$.

The fiber $\pi^{-1}(\vec{s})$ is isomorphic to $(X, I_{\vec{s}})$ as a complex manifold.

Proposition 3.31 (Real sections of twistor spaces). Suppose *X* is a hyperkähler manifold with twistor space \mathcal{Z} . Then \mathcal{Z} , as a bundle over S^2 , has holomorphic sections s_x corresponding to the points $x \in X$. These sections are *real*, i.e. $s_x(-\vec{s}) = \rho(s_x(\vec{s}))$. The normal bundle to $s_x(S^2) \subset \mathcal{Z}$ is isomorphic to $\mathcal{O}(1)^{\oplus 2n}$.

Proof. Set

$$s_x(\zeta) = (x, \zeta). \tag{3.103}$$

Evidently this is a holomorphic and real section. The desired statement about the normal bundle follows from Lemma 3.19: indeed choosing a basis $\{\alpha_1, \ldots, \alpha_{2n}\}$ for the 2*n*dimensional vector space $(T_{x,I_1}^{1,0}X)^*$ we get two trivializations of the conormal bundle to $s_x(S^2)$, by the sections $\{(1 + \zeta I_3)\alpha_i \oplus 0\}$ over $\{\zeta \neq \infty\}$ and $\{(1/\zeta + I_3)\alpha_i\} \oplus 0$ over $\{\zeta \neq 0\}$. The transition function relating these two trivializations is $1/\zeta \mathbf{1}$, showing the conormal bundle is $\mathcal{O}(-1)^{\oplus 2n}$, thus the normal bundle is $\mathcal{O}(1)^{\oplus 2n}$ as desired.

The real sections can be thought of as horizontal for a sort of nonlinear flat connection which gives the identification between the fibers of \mathcal{Z} .



Definition 3.32 (Pseudo-hyperkähler structure). Suppose *X* is a manifold. A *pseudo-hyperkähler structure* on *X* is all the data (X, g, I_1, I_2, I_3) of a hyperkähler structure except that we do not impose the condition that *g* be positive definite.

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Theorem 3.33 ((Re)construction of hyperkähler manifolds from twistor spaces). [17] Suppose given a manifold \mathcal{Z} carrying all the structures of Proposition 3.30. Then let X be the space of all real holomorphic sections of \mathcal{Z} having normal bundle isomorphic to $\mathcal{O}(1)^{\oplus 2n}$. X carries a canonical structure of manifold and a canonical pseudo-hyperkähler structure. If \mathcal{Z} is the twistor space of a pseudo-hyperkähler manifold X', then $X' \subset X$, and the pseudo-hyperkähler structures agree.

Proof. We need some results from deformation theory of complex manifolds. A useful reference for this stuff is [6]. Here I can only give a sketch of how it works.

The deformations of a compact complex submanifold $Y \subset Z$ are "controlled by" the sheaf cohomology of the holomorphic normal bundle N = TZ/TY. The simplest situation is the situation where $H^1(Y, N) = 0$. In this case the set parameterizing the deformations is actually a complex manifold, and its tangent space at a given Y is $H^0(Y, N)$.

We are going to apply this to the situation where *Y* is the image of one of the sections of \mathcal{Z} parameterized by *X*. Then $Y \simeq \mathbb{CP}^1$ and $N \simeq \mathcal{O}(1)^{\oplus 2n}$. The first bit of good news is that in this situation we indeed have $H^1(Y, N) = 0$, because $H^1(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. Moreover, *N* carries the structure of metric twistorial vector space as in Definition 3.26, and thus by Proposition 3.27, $H^0_{\mathbb{R}}(Y, N)$ is a hyperkähler vector space. As *Y* varies this gives (I_1, I_2, I_3, g) on *X'*.

Finally we just have to check the integrability $d\omega_i = 0$. This follows directly from the fact that $d\Omega = 0$ on the fibers of \mathcal{Z} [...]

Two remarks about Theorem 3.33:

- I have never heard of an example for which X' ≠ X, but it seems hard in general to rule out the possibility that Z could have some other real sections having nothing to do with the points of the original X.
- The space X_C of *all* holomorphic sections of Z has an antiholomorphic involution induced by *ρ*; X is the fixed locus. It thus provides a natural *complexification* of X. For example, when X = H, Z is the total space of O(1) ⊕ O(1) → S² (see Example 3.34). Then X_C is a complex 4-dimensional vector space, equipped with a real structure; we recover the original X = H by restricting to the real points.

3.6 First examples of twistor spaces

Example 3.34 (Twistor space of \mathbb{R}^4 **).** For the standard hyperkähler structure on \mathbb{H} all this becomes very concrete. If we define

$$w(\zeta) = w_1 - \bar{z}_1 \zeta, \tag{3.104}$$

$$z(\zeta) = z_1 + \bar{w}_1 \zeta, \tag{3.105}$$

then the holomorphic 2-form (3.80) can be written as

$$\Omega(\zeta) = \frac{1}{2\zeta} \mathrm{d}w(\zeta) \wedge \mathrm{d}z(\zeta) \tag{3.106}$$

Indeed,

$$\frac{1}{2\zeta} dw(\zeta) \wedge dz(\zeta) = \frac{1}{2\zeta} (dw_1 \wedge dz_1 + \zeta (dw_1 \wedge d\bar{w}_1 + dz_1 \wedge d\bar{z}_1) - \zeta^2 d\bar{z}_1 \wedge d\bar{w}_1) \quad (3.107)$$

$$= \frac{\omega_2 + i\omega_3}{2\zeta} - i\omega_1 + \frac{\omega_2 - i\omega_3}{2}\zeta \qquad (3.108)$$

matching (3.80).

It follows (using the result of Exercise 3.26 below) that $w(\zeta)$ and $z(\zeta)$ are fiberwise holomorphic. But they are evidently also holomorphic in ζ . So they are holomorphic functions on \mathcal{Z} . Thus we have two coordinate systems on \mathcal{Z} : $(\zeta, z(\zeta), w(\zeta))$ covering the patch $\zeta \neq \infty$ and $(1/\zeta, z(\zeta)/\zeta, w(\zeta)/\zeta)$ covering $\zeta \neq 0$. The transition map in the fibers is thus multiplication by $1/\zeta$. This identifies \mathcal{Z} as the total space of the rank 2 holomorphic bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to S^2. \tag{3.109}$$

The antiholomorphic involution ρ is

$$\rho(\zeta, z, w) = (-1/\bar{\zeta}, -\bar{w}/\bar{\zeta}, \bar{z}/\bar{\zeta}).$$
(3.110)

Exercise 3.25. Verify the formula (3.110) for the antiholomorphic involution.

Exercise 3.26. If *X* is a complex manifold, with dim_C *X* = 2, with holomorphic symplectic form Ω , and $\Omega = \alpha \land \beta$ for $\alpha, \beta \in \Omega^1(X)$, show that in fact $\alpha, \beta \in \Omega^{1,0}(X)$.

Example 3.35 (Twistor space of $\mathbb{R}^3 \times S^1$). For $\mathbb{R}^3 \times S^1$ the picture is more interesting. The shift by \mathbb{Z} acts as $(w_1, z_1) \rightarrow (w_1 + 2\pi n, z_1)$. We can build coordinates by taking shift-invariant combinations of the coordinates we used in Example 3.34: on the patch $\zeta \neq \infty$ we take

$$\eta = z - \zeta w, \qquad \mathcal{X} = \exp(\mathrm{i}w) \tag{3.111}$$

while on $\zeta \neq 0$ we take

$$\eta' = z/\zeta^2 - w/\zeta, \qquad \mathcal{X}' = \exp(iz/\zeta) \tag{3.112}$$

Thus the transition map is

$$(\zeta',\eta',\mathcal{X}') = (1/\zeta,\eta/\zeta^2,\mathcal{X}\exp(i\eta/\zeta))$$
(3.113)

So η is a local coordinate on the fiber of $\mathcal{O}(2) \to S^2$; let *L* denote the line bundle over $\mathcal{O}(2) \to S^2$ with transition function $e^{i\eta/2\zeta}$; what we just showed is that \mathcal{Z} is the total space of L^2 , with the zero section deleted (because \mathcal{X} never takes the value 0.)

Note that in particular \mathcal{Z} is *not* algebraic, because of the appearance of the exponential function here. (Were it algebraic, it would follow on general grounds that *L* is pulled back from the base \mathbb{CP}^1 , but that's not the case here.)

Exercise 3.27. Write down the (fiberwise, twisted) holomorphic symplectic form Ω and the antiholomorphic involution in these complex coordinates for \mathcal{Z} .

The next example is a key one: it has many of the features which appear for moduli spaces of Higgs bundles (in fact this example literally occurs as the moduli space of Higgs bundles on the curve $C = T^2$ with gauge group G = U(1).)

Example 3.36 ($\mathbb{R}^2 \times T^2$). Now fix some $\tau \in \mathbb{C}$, with Im $\tau > 0$, and let *X* be \mathbb{R}^4 modulo shifts of w_1 by the lattice

$$\Lambda_{\tau} = \frac{2\pi}{\sqrt{\mathrm{Im}\,\tau}} (\mathbb{Z} \oplus \mathbb{Z}\tau) \subset \mathbb{C}. \tag{3.114}$$

(The funny prefactor is engineered so that the area of $\mathbb{C}/\Lambda_{\tau}$ is $(2\pi)^2$, independent of τ .) Note that in choosing to shift w_1 , as opposed to some other complex coordinate, we have privileged the complex structure I_1 . In structure I_1 , moreover, we see immediately that

$$X \simeq \mathbb{C} \times T_{\tau}^2, \tag{3.115}$$

where by T_{τ}^2 we mean the complex torus $\mathbb{C}/\Lambda_{\tau}$. Similarly in structure $-I_1$ we have

$$X \simeq \mathbb{C} \times T^2_{-\bar{\tau}}.\tag{3.116}$$

Thus, as a complex manifold, (X, I_1) or $(X, -I_1)$ really *depend* on τ — different choices (not related by $SL(2, \mathbb{Z})$) give inequivalent complex manifolds.

What about the other structures, for $\zeta \in \mathbb{C}^{\times}$? Let's start with the slightly easier case $\tau = i$. In this case the \mathbb{Z}^2 action just shifts x_0 and x_1 independently by multiples of 2π . Away from $\zeta = 0$ and $\zeta = \infty$, we can write holomorphic coordinates as

$$\mathcal{X}_A = \exp\left(rac{z_1}{2\zeta} - \mathrm{i}x_1 + rac{ar{z}_1}{2}\zeta
ight), \ \mathcal{X}_B = \exp\left(rac{\mathrm{i}z_1}{2\zeta} + \mathrm{i}x_0 - rac{\mathrm{i}ar{z}_1}{2}\zeta
ight).$$

As a check, note that at $\zeta = 1$ we have $\mathcal{X}_A = \exp(x_2 - ix_1)$, $\mathcal{X}_B = \exp(-x_3 + ix_0)$, holomorphic for I_3 , and at $\zeta = -i$ we have $\mathcal{X}_A = \exp(-x_3 - ix_1)$, $\mathcal{X}_B = \exp(-x_2 + ix_0)$, holomorphic for I_2 . For general $\zeta \in \mathbb{C}^{\times}$ we compute directly

$$\frac{\mathrm{d}\mathcal{X}_A \wedge \mathrm{d}\mathcal{X}_B}{\mathcal{X}_A \mathcal{X}_B} = -\mathrm{i}\Omega(\zeta),\tag{3.117}$$

which shows in particular (again using Exercise 3.26) that \mathcal{X}_A and \mathcal{X}_B are holomorphic on (X, I_{ζ}) . Thus, for all $\zeta \in \mathbb{C}^{\times}$, we get in structure I_{ζ}

$$X \simeq \mathbb{C}^{\times} \times \mathbb{C}^{\times}. \tag{3.118}$$

For $\tau \neq i$ the picture is very similar, just with slightly more complicated formulas:

$$\mathcal{X}_{A,B} = \exp\left(\zeta^{-1}Z_{A,B} + \mathrm{i}\theta_{A,B} + \zeta\bar{Z}_{A,B}\right)$$
(3.119)

where we introduced

$$\theta_A = -\frac{1}{\sqrt{\mathrm{Im}\,\tau}} x_1, \qquad \theta_B = \sqrt{\mathrm{Im}\,\tau} x_0 - \frac{\mathrm{Re}\,\tau}{\sqrt{\mathrm{Im}\,\tau}} x_1, \tag{3.120}$$

$$Z_A = \frac{z_1}{2\sqrt{\operatorname{Im}\tau}}, \qquad Z_B = \tau Z_A. \tag{3.121}$$

In these coordinates the \mathbb{Z}^2 action just shifts θ_A and θ_B independently by multiples of 2π . The I_1 -holomorphic combination of these is $\theta_B - \tau \theta_A$.

We have again

$$\frac{\mathrm{d}\mathcal{X}_A \wedge \mathrm{d}\mathcal{X}_B}{\mathcal{X}_A \mathcal{X}_B} = -\mathrm{i}\Omega(\zeta) \tag{3.122}$$

and so again \mathcal{X}_A and \mathcal{X}_B are holomorphic, giving $X \simeq \mathbb{C}^{\times} \times \mathbb{C}^{\times}$. In particular, for $\zeta \in \mathbb{C}^{\times}$, the space (X, I_{ζ}) as a complex manifold *does not* depend on τ .

Exercise 3.28. Check (3.117), or more generally (3.122). As we have explained, this immediately implies that \mathcal{X}_A and \mathcal{X}_B are indeed holomorphic functions on \mathcal{Z} . You could also try checking this directly from the definition of the complex structure on \mathcal{Z} .

Exercise 3.29. Check that $(\mathcal{X}_A, \mathcal{X}_B)$ indeed give a *biholomorphism* $(X, I_{\zeta}) \simeq \mathbb{C}^{\times} \times \mathbb{C}^{\times}$, and thus $(\mathcal{X}_A, \mathcal{X}_B, \zeta)$ give a biholomorphism between an open subset of \mathcal{Z} and $\mathbb{C}^{\times} \times \mathbb{C}^{\times} \times \mathbb{C}^{\times}$.

Exercise 3.30. Write the antiholomorphic involution ρ in terms of $(\mathcal{X}_A, \mathcal{X}_B, \zeta)$.

Example 3.37 (Twistor spaces of multi-Eguchi-Hanson metrics). Now we consider the twistor spaces of the metrics of Example 3.15.

We have commented before that in complex structure I_1 there is a holomorphic function $z_1 = x_2 + ix_3$. Of course we can get a similar function in any complex structure I_{ζ} . Now we want to choose the normalizations of these functions in such a way that they vary holomorphically with ζ , and thus give a holomorphic function η on Z. One way of thinking about this is: for each ζ we get a complex coordinate by considering the *holomorphic moment map* of the U(1) action with respect to $\Omega(\zeta)$. Concretely this gives

$$\eta = \frac{x_2 + ix_3}{2} - ix_1\zeta + \frac{x_2 - ix_3}{2}\zeta^2.$$
(3.123)

This function blows up at $\zeta = \infty$, but near there we can switch to the function $\eta' = \eta/\zeta^2$. Globally the pair (η, η') gives a *holomorphic map* from \mathcal{Z} to the total space of the bundle $\mathcal{O}(2) \to \mathbb{CP}^1$. or more prosaically, we have an $\mathcal{O}(2)$ -valued coordinate η . (The same situation occurred in Example 3.35.)

To complete our description of \mathcal{Z} we need (in each patch) one more local holomorphic coordinate. The answer turns out as follows. If the singularities of *V* are at points x_1, \ldots, x_n define corresponding sections η_1, \ldots, η_n of $\mathcal{O}(2)$ by (3.123). Then \mathcal{Z} carries an $\mathcal{O}(2)$ -valued coordinate η and two $\mathcal{O}(n)$ -valued coordinates u, v, obeying the equation

$$uv = \prod_{i=1}^{n} (\eta - \eta_i).$$
(3.124)

More invariantly we could say that we have a map

$$\pi: \mathcal{Z} \to \left\{ uv = \prod_{i=1}^{n} (\eta - \eta_i) \right\} \subset [(\mathcal{O}(n) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)) \to \mathbb{CP}^1].$$
(3.125)

This map is an isomorphism on all but finitely many fibers: the exception is the fibers over ζ for which some $\eta_i(\zeta)$ become equal. In these fibers we have some compact complex submanifolds, the $S^{2's}$ discussed in Example 3.15; these get contracted by the map π . Thus for most values of ζ , (\mathcal{M}, I_{ζ}) is an affine variety, while for finitely many special values (\mathcal{M}, I_{ζ}) is a resolution of a singular affine variety. The simple example n = 2, Example 3.14, is an affine variety $\{uv = (\eta - \eta_1)(\eta - \eta_2)\}$ in all structures I_{ζ} except for two; in those special structures we get $T^*\mathbb{CP}^1$, realized as a resolution of the singular variety $uv = (\eta - \eta_1)^2$.

The holomorphic symplectic form is

$$\Omega = i \frac{du \wedge dv}{n \frac{\partial}{\partial \eta} \prod(\eta - \eta_i)}.$$
(3.126)

Note that when n = 1 the equation (3.124) can be eliminated by solving for η , leaving the coordinates u, v parameterizing $\mathcal{O}(1) \oplus \mathcal{O}(1)$; this matches the twistor space of \mathbb{R}^4 (Example 3.34), as it should.

Exercise 3.31. Show that the description of the twistor space in Example 3.37 is indeed correct. (Hint: construct the functions u, v, concretely as solutions of the Cauchy-Riemann equations. These equations are determined by the (0, 1) vector fields we gave in (3.43). The idea is to consider what these equations become in the special case where we look at a function which is locally of the form $u = f(x)e^{i\chi}$ — or more invariantly, a function which obeys $\partial_{\chi} u = iu$. Such functions are equivalently sections of a line bundle over the base $\mathbb{R}^3 \setminus \{x_i\}$, carrying a canonical connection, and the Cauchy-Riemann equations take a nice form in terms of this line bundle. Similarly v will be a section of the dual line bundle, locally of the form $v = g(x)e^{-i\chi}$).

3.7 Semiflat metrics

Now we start considering more interesting torus fibrations. The hyperkähler metrics we describe below appeared first in [18]; see also [19] for a more mathematically oriented exposition, and closely related discussion in [20].

Example 3.38 (Semiflat metric in one dimension, trivially fibered). This is a generalization of Example 3.36 to allow a bundle of tori, with varying modulus.

Fix a 1-dimensional complex manifold \mathcal{B} and holomorphic functions

$$Z_{A,B}: \mathcal{B} \to \mathbb{C} \tag{3.127}$$

such that the quantity

$$\tau = \frac{\mathrm{d}Z_B}{\mathrm{d}Z_A} \tag{3.128}$$

is valued in the upper half-plane. Then let *X* be $\mathcal{B} \times T^2$, with T^2 coordinatized by $\theta_{A,B} \in \mathbb{R}/2\pi\mathbb{Z}$. Finally, much like Example 3.36, define functions $\mathcal{X}_{A,B} : X \to \mathbb{C}^{\times}$ by

$$\mathcal{X}_{A,B} = \exp\left(\zeta^{-1}Z_{A,B} + \mathrm{i}\theta_{A,B} + \zeta\bar{Z}_{A,B}\right),\tag{3.129}$$

and for any $\zeta \in \mathbb{C}^{\times}$ define $\Omega(\zeta) \in \Omega^2_{\mathbb{C}}(X)$ by

$$\frac{\mathrm{d}\mathcal{X}_A \wedge \mathrm{d}\mathcal{X}_B}{\mathcal{X}_A \mathcal{X}_B} = -\mathrm{i}\Omega(\zeta). \tag{3.130}$$

To recover Example 3.36, take $\mathcal{B} = \mathbb{C}$, $Z_A = z$, $Z_B = \tau z$, $\theta_A = -x_1$, $\theta_B = x_0$.

Exercise 3.32. Show that there is a hyperkähler structure on the space *X* in Example 3.38, with holomorphic symplectic form $\Omega(\zeta)$. (Hint: since we know $\Omega(\zeta)$ concretely, we can compute directly what the ω_i and *g* must be. What is not immediately obvious is that they obey the necessary algebraic relations to give a hyperkähler metric. You could verify this directly, but you can also obtain it as a consequence of the twistor reconstruction, Theorem 3.33: then all you have to do is construct \mathcal{Z} with the required structures. The tricky point is the condition on normal bundles: what does it mean concretely about the functions $\mathcal{X}_{A,B}$?)

Exercise 3.33. This is a continuation of Exercise 3.32, but actually it can be done independently, *assuming* the result of that exercise. Compute the symplectic forms ω_i and the metric g. What can you say about the complex manifold (X, I_1) ? Show that the group $U(1) \times U(1)$ acts by triholomorphic isometries on X (translations on the torus fibers.)

Exercise 3.34. Consider again the hyperkähler manifold from Exercise 3.32. Choose a contractible patch $W \subset \mathcal{B}$ on which the function Z_A is injective. Using Z_A to embed $W \hookrightarrow \mathbb{C}$, we can view $U = W \times \mathbb{R}$ as a subset of $\mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$. Then the function $V = \text{Im } \tau$ is a positive harmonic function on U. We can thus consider the Gibbons-Hawking space (Example 3.10) associated to this data. It has translation invariance in the \mathbb{R} direction. Show that the hyperkähler manifold from Exercise 3.32 is isomorphic to the quotient of this Gibbons-Hawking space by $\mathbb{Z} \subset \mathbb{R}$.

Example 3.39 (Semiflat metric in one dimension, nontrivially fibered). Here is a further generalization of Example 3.38, to allow a torus *bundle* with nontrivial monodromy. The notation needed to take care of this monodromy makes things look considerably more complicated, but the local geometry is exactly the same as Example 3.38.

We again fix a 1-dimensional complex manifold \mathcal{B} . Now we also fix a local system Γ of rank 2 lattices over \mathcal{B} , equipped with an antisymmetric integer-valued pairing

$$\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \to \mathbb{Z} \tag{3.131}$$

and a map

$$Z: \Gamma \to \mathbb{C} \tag{3.132}$$

such that:

- *Z* is a homomorphism on each fiber i.e. for each point $u \in \mathcal{B}$, we get a homomorphism $Z(u) : \Gamma_u \to \mathbb{C}$,
- *Z* is holomorphic, if we equip Γ with the complex structure it acquires as a covering space of *B*. (Said otherwise: if we choose a section *γ* of Γ over some patch *U* ⊂ *B*,

then *Z* induces a function $Z_{\gamma} : U \to \mathbb{C}$, and the condition is that this function should always be holomorphic.)

• Choosing a local basis $\{\gamma_1, \gamma_2\}$ with $\langle \gamma_1, \gamma_2 \rangle = 1$, the 2-form

$$\omega = dZ_{\gamma_2} \wedge d\bar{Z}_{\gamma_1} - dZ_{\gamma_1} \wedge d\bar{Z}_{\gamma_2}$$
(3.133)

is positive.

(To recover Example 3.38, take Γ to be the trivial local system with fiber \mathbb{Z}^2 , and then for $\gamma = (a, b)$ take $Z_{\gamma} = aZ_A + bZ_B$.)

Now define a torus bundle over \mathcal{B} by

$$X = \text{Hom}(\Gamma, U(1)), \qquad (3.134)$$

i.e. the fiber of X over $u \in \mathcal{B}$ is Hom $(\Gamma_u, U(1))$, which is indeed a torus $(S^1)^2$. Given a local section γ of Γ over a patch $U \subset \mathcal{B}$, we have the evaluation map

$$\varphi_{\gamma}: X|_U \to U(1). \tag{3.135}$$

Then finally we define a map

$$\mathcal{X}: \Gamma \times \mathbb{C}^{\times} \to \mathbb{C}^{\times} \tag{3.136}$$

by

$$\mathcal{X}_{\gamma} = \exp\left(\zeta^{-1}Z_{\gamma} + \zeta\bar{Z}_{\gamma}\right)\varphi_{\gamma}.$$
(3.137)

Fixing a local section γ of Γ over $U \subset \mathcal{B}$, this gives a function

$$\mathcal{X}_{\gamma}: U \times \mathbb{C}^{\times} \to \mathbb{C}^{\times} \tag{3.138}$$

Now fix a local basis $\{\gamma_1, \gamma_2\}$ of Γ over $U \subset \mathcal{B}$, with $\langle \gamma_1, \gamma_2 \rangle = 1$, and finally define $\Omega(\zeta) \in \Omega^2_{\mathbb{C}}(X|_U)$ by

$$\Omega(\zeta) = i \frac{d\mathcal{X}_{\gamma_1}(\zeta) \wedge d\mathcal{X}_{\gamma_2}(\zeta)}{\mathcal{X}_{\gamma_1}(\zeta)\mathcal{X}_{\gamma_2}(\zeta)}.$$
(3.139)

Although we defined it over a patch $U \subset \mathcal{B}$, the Ω obtained is independent of the choice of basis $\{\gamma_1, \gamma_2\}$, and thus it extends to a form $\Omega(\zeta) \in \Omega^2_{\mathbb{C}}(X)$. We can write it more compactly as

$$\Omega(\zeta) = i \langle\!\langle d \log \mathcal{X}(\zeta), d \log \mathcal{X}(\zeta) \rangle\!\rangle$$
(3.140)

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the inverse pairing, $\Gamma^* \times \Gamma^* \to \mathbb{Z}$, and we think of $d \log \mathcal{X}(\zeta)$ as a 1-form valued in Γ^* .

Exercise 3.35. Show that the 2-form $\Omega(\zeta)$ in Example 3.39 is the holomorphic symplectic form for a hyperkähler structure on *X*. Compute the symplectic forms ω_i and the hyperkähler metric *g*. (The computation needed here should be almost identical to the one in Exercise 3.32). Show that each fiber of the projection to \mathcal{B} is a flat torus. (This is the reason why this metric is called "semiflat.")

This hyperkähler structure does not have a global $U(1) \times U(1)$ action, but it does have it *locally*, i.e. on patches where Γ can be trivialized. More invariantly we could say that it has an action of a *bundle* of groups.

3.8 The Ooguri-Vafa space

So far we have considered honest torus fibrations. Now we move on to the first example where we have *singular* fibers.

Example 3.40 (Ooguri-Vafa space). [21] We return to the Gibbons-Hawking ansatz, Example 3.13. The idea is to make a circle bundle with a \mathbb{Z} shift symmetry; then on dividing out by the \mathbb{Z} action we will get a T^2 bundle. The simplest thing to try would be to take the 1-dimensional lattice of points

$$x_n = (2\pi n, 0, 0) \in \mathbb{R}^3$$
 (3.141)

and then take V to be

$$\sum_{n \in \mathbb{Z}} \frac{1}{4\pi \|x - x_n\|}.$$
(3.142)

However there is a difficulty: for large enough *n* we have

$$\frac{1}{4\pi \|x - x_n\|} \approx \frac{1}{8\pi^2 |n|} \tag{3.143}$$

and thus this sum is logarithmically divergent. We can remove this divergence by "sub-tracting an infinite constant", i.e. we define

$$V = \frac{1}{4\pi \|x\|} + \sum_{n \neq 0 \in \mathbb{Z}} \left(\frac{1}{4\pi \|x - x_n\|} - \frac{1}{8\pi^2 |n|} \right) + C$$
(3.144)

where *C* is any constant. Now the divergence problem is cured, but we get a new problem: because we have introduced some minus signs it is not clear that *V* is positive, and indeed *V* is *not* positive when *x* is far from the lattice of singularities. To get an idea of the difficulty, recall the coordinate $z = z_1 = x_2 + ix_3$, and consider the limit of large |z|: here we may replace the sum of point sources $\sum \delta(x - x_n)$ by a continuous string source $\frac{1}{2\pi}\delta(z)dx_1$, where now δ denotes the *two*-dimensional delta function.

Such a source would lead to a two-dimensional Laplace equation,

$$\Delta V = -\frac{1}{2\pi}\delta(z). \tag{3.145}$$

Thus for $|z| \gg 1$ we expect

$$V \approx -\frac{1}{4\pi^2} \log|z| + C' + C$$
 (3.146)

for some constant C'. This expectation can be made rigorous using the Poisson summation formula, which gives in fact

$$V = -\frac{1}{4\pi^2} \log|z/\Lambda| + \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \cos(nx_1) K_0(n|z|)$$
(3.147)

where $\Lambda = 4\pi e^{-\gamma + 4\pi^2 C}$, γ is Euler's constant, and K_0 is the modified Bessel function. This function has asymptotics $K_0(M) \sim \sqrt{\frac{\pi}{2M}} e^{-M}$, so each term in the sum is exponentially small for large |z|. Dropping these exponentially suppressed terms we get

$$V \approx V^{\rm sf} = -\frac{1}{4\pi^2} \log|z/\Lambda|. \tag{3.148}$$

Let us briefly consider the Gibbons-Hawking metric where *V* is *exactly* given by V^{sf} of (3.148) (i.e. we consider the "approximation" where we drop the sum in (3.147).) V^{sf} is singular at z = 0, and for $|z| \ge \Lambda$ we will not have V > 0, so the biggest base we can take is

$$U = \{0 < |z| < \Lambda\} \times \mathbb{R}.\tag{3.149}$$

The resulting Gibbons-Hawking metric g^{sf} is invariant under continuous translations in the x_1 direction, $x_1 \rightarrow x_1 + \alpha$ for any $\alpha \in \mathbb{R}$. In particular, it is invariant under the discrete subgroup $\mathbb{Z} \subset \mathbb{R}$ acting by $x_1 \rightarrow x_1 + 2\pi n$. After dividing out by this action g^{sf} descends to one of the semiflat metrics of Example 3.39, torus fibered over the punctured disc $0 < |z| < \Lambda$: see Exercise 3.37 below. However, this quotient metric seems very unlikely to admit a reasonable extension over z = 0.

The Gibbons-Hawking metric g obtained from the full V of (3.144) rather than V^{sf} shares the bad large-|z| behavior of the approximate one. Indeed, when $1 \ll |z| < \Lambda$, g is very well approximated by g^{sf} . At small |z|, though, the behavior of g is much better than that of g^{sf} : it is smooth. g is not invariant under continuous translations in the x_1 direction, but does still have the discrete \mathbb{Z} symmetry $x_1 \rightarrow x_1 + 2\pi n$, preserving the hyperkähler structure.⁸ After dividing out by this \mathbb{Z} action we obtain a hyperkähler space X, the Ooguri-Vafa manifold. Crucially, the Ooguri-Vafa manifold and its hyperkähler structure do extend smoothly over z = 0, since near the point singularities of V we have the usual behavior $V \sim \frac{1}{4\pi r}$ as in Example 3.13.

As usual for Gibbons-Hawking spaces, in complex structure I_1 , X has the holomorphic function z. The fiber of X over a generic z is a compact complex torus (circle fibration over a circle), while over z = 0 we get a singular fiber (torus with a node, aka "ordinary double point," a place which looks locally like the locus $\{xy = 0\} \subset \mathbb{C}^2$.)



⁸There is a subtle choice here: the \mathbb{Z} action on the base has to be lifted to the total space of the U(1) bundle X, and there are a circle's worth of ways to do this. Thus the Ooguri-Vafa metric depends on two parameters: the real parameter C in (3.144) — which we have absorbed into Λ — and also a circle-valued parameter controlling the choice of lift of the \mathbb{Z} action. It is actually natural to combine these into a single *complex* parameter Λ ; this is the point of view taken in [22].

Exercise 3.36. Let *X* be the Gibbons-Hawking hyperkähler manifold with $V = V^{sf}$ given by (3.148). Write a formula for the connection form Θ relative to some trivialization of *X* as a U(1) bundle away from the line z = 0.

Exercise 3.37. Let *X* be the Gibbons-Hawking hyperkähler manifold with $V = V^{sf}$ given by (3.148). Verify that X/\mathbb{Z} is an example of a semiflat metric as in Example 3.39, where the base \mathcal{B} is the punctured disc $0 < |z| < \Lambda$, and Γ is a rank 2 local system of lattices, whose monodromy around z = 0 is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$. What is the map *Z*?

Exercise 3.38. [23] Let *X* be the Ooguri-Vafa space. Show that there is an I_{ζ} -holomorphic disc in *X* with boundary on the torus fiber over *z*, if and only if $\zeta/z \in i\mathbb{R}$. (Hint: to see that the disc *does* exist, you can construct it directly, along the lines of our earlier constructions of holomorphic spheres in Gibbons-Hawking spaces; to see that it *doesn't* exist, use the fact that the integral of $\Omega(\zeta)$ over such a disc would have to vanish.)

Exercise 3.39. Use the Poisson summation formula to verify (3.147). [warning, this is hard]

Example 3.41 (Twistor description of Ooguri-Vafa space). Let *X* be the Ooguri-Vafa space as in Example 3.40. As with our previous examples, we may ask: what are the holomorphic functions on *X* in complex structure I_{ζ} ? One global holomorphic function is easy to construct: it is

$$\mathcal{X}_A = \exp\left(\zeta^{-1}\frac{z}{2} - \mathrm{i}x_1 + \zeta\frac{\bar{z}}{2}\right). \tag{3.150}$$

This is the same formula (3.123) we have encountered earlier, except that we have exponentiated it, to make it well defined on the quotient by the \mathbb{Z} action $x_1 \rightarrow x_1 + 2\pi n$.

To get a second coordinate is considerably more difficult. If we wanted to get a holomorphic function on the *approximate* Ooguri-Vafa space with $V = V^{sf}$, we could use the function written in Example 3.39,

$$\mathcal{X}_{B}^{\mathrm{sf}} = \exp\left(\zeta^{-1}Z_{B} + \mathrm{i}\chi + \zeta\bar{Z}_{B}\right),\tag{3.151}$$

where

$$Z_B = \frac{1}{4\pi i} (z \log(z/\Lambda) - z). \tag{3.152}$$

The formula (3.151) does not define a global function on *X*, for two reasons: first, χ is only a *local* fiber coordinate; second, Z_B is multivalued because of the logarithm. We choose the principal branch of the logarithm, so that there is a cut at $z/\Lambda \in \mathbb{R}_-$, and also take a trivialization of the circle bundle defining *X* on the complement of this cut. It is possible to choose this trivialization in such a way that when we cross the cut we get $\chi \rightarrow \chi + x_1$. With these choices made, (3.151) defines an honest function on the complement of the cut, and crossing the cut leads to the jumps $Z_B \rightarrow Z_B + \frac{z}{2}$ and $\chi \rightarrow \chi - x_1$, which combine to give $\mathcal{X}_B^{\text{sf}} \rightarrow \mathcal{X}_B^{\text{sf}} \mathcal{X}_A$. The fact that the jump of $\mathcal{X}_B^{\text{sf}}$ is by multiplication with a *holomorphic* function is to be expected, since $\mathcal{X}_B^{\text{sf}}$ is itself holomorphic on both sides of the

cut. (Compare the result of Exercise 3.37, where the same situation is described in a more global way, by introducing a local system with nontrivial monodromy instead of working with an explicit branch cut.)

Changing from V^{sf} to the full *V* changes the Cauchy-Riemann equations, in such a way that the function $\mathcal{X}_B^{\text{sf}}$ is *not* holomorphic anymore (though \mathcal{X}_A still is.) However it turns out that we can find a true holomorphic function \mathcal{X}_B which has the same *asymptotic* behavior as $\mathcal{X}_B^{\text{sf}}$ as we take $\zeta \to 0$ or $\zeta \to \infty$.

The formula will look slightly bizarre at first exposure: we write

$$\mathcal{X}_B = \mathcal{X}_B^{\mathrm{sf}} \mathcal{X}_B^{\mathrm{inst}} \tag{3.153}$$

where the "correction" is

$$\mathcal{X}_{B}^{\text{inst}}(\zeta) = \exp\left[\frac{\mathrm{i}}{4\pi} \int_{\ell_{+}} \frac{\mathrm{d}\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_{A}(\zeta'))\right]$$
(3.154)

$$-\frac{\mathrm{i}}{4\pi}\int_{\ell_{-}}\frac{\mathrm{d}\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_{A}(\zeta')^{-1})\bigg],\qquad(3.155)$$

with

$$\ell_{\pm} = \{ \pm z / \zeta' \in \mathbb{R}_{-} \}.$$
(3.156)

Note that the integrals indeed converge, because of our careful choice of the integration contours: along these contours $\mathcal{X}_A(\zeta')$ is exponentially decaying, so that $\log(1 - \mathcal{X}_A(\zeta'))$ is also exponentially decaying.

What is surprising here is that X_B is actually an I_{ζ} -holomorphic function on X. To see that this is indeed true, one can compute directly (Exercise 3.40) that the holomorphic symplectic form (3.80) is given by

$$\Omega(\zeta) = i \frac{d\mathcal{X}_A(\zeta) \wedge d\mathcal{X}_B(\zeta)}{\mathcal{X}_A(\zeta)\mathcal{X}_B(\zeta)}.$$
(3.157)

A tricky aspect of this story is that the function \mathcal{X}_B is actually only *piecewise* holomorphic. Indeed, the integrals in (3.154) are not defined when $\zeta \in \ell_{\pm}$. One can however see that the *limits* from the two sides exist,

$$\mathcal{X}_{B}^{\pm}(\zeta) = \lim_{\epsilon \to 0^{\pm}} \mathcal{X}_{B}(\zeta e^{i\epsilon}), \qquad (3.158)$$

by deforming the contour of integration so that it lies along ℓ_{\pm} except for a little semicircular detour around the pole of the integrand at $\zeta' = \zeta$.



Thus the two limits differ by the integral around the pole at $\zeta' = \zeta$, which can be evaluated by the residue theorem, giving the jump formulas

$$\mathcal{X}_{B}^{+}(\zeta) = \mathcal{X}_{B}^{-}(\zeta)(1 - \mathcal{X}_{A}(\zeta))^{-1} \text{ for } \zeta \in \ell_{+},$$
(3.159)

$$\mathcal{X}_B^+(\zeta) = \mathcal{X}_B^-(\zeta)(1 - \mathcal{X}_A(\zeta)^{-1}) \text{ for } \zeta \in \ell_-.$$
(3.160)

In light of this discontinuity the formula (3.157) might seem strange: how can it be that $\Omega(\zeta)$ given by this formula is still well defined and smooth, when $\mathcal{X}_B(\zeta)$ is not? The point is that the jumps (3.159), (3.160) are actually *symplectomorphisms*. Indeed, writing $x = \log \mathcal{X}_A$ and $y = \log \mathcal{X}_B$ to simplify notation a bit, these jumps are both of the form

$$(x,y) \mapsto (x,y+f(x)) \tag{3.161}$$

and of course the form

$$\Omega = \mathrm{d}x \wedge \mathrm{d}y \tag{3.162}$$

is invariant under this transformation. Thus, if we have a pair of functions (x, y) which are smooth except for a discontinuity of this sort at some codimension-1 wall, Ω is perfectly smooth (or more precisely it admits a smooth extension over the wall).

 $\mathcal{X}_{B}^{\text{inst}}$ is bounded as $\zeta \to 0$ and $\zeta \to \infty$. In this sense the asymptotics of \mathcal{X}_{B} are essentially the same as those of $\mathcal{X}_{B}^{\text{sf}}$, as we claimed.

Exercise 3.40. Verify the formula (3.157). This computation is a slight adaptation of one in [22], but it may be more interesting to try it yourself; the idea is that even though the integrals (3.154) cannot be evaluated in closed form, once we plug into (3.157) and Fourier expand in e^{ix_1} under the integral sign we obtain integrals which *can* be done, producing Bessel functions which eventually get matched to those appearing in (3.147). The key integral identity needed is

$$\int_0^\infty \frac{\mathrm{d}t}{t} t^n \exp\left(-M(t^{-1}+t)\right) = 2K_n(2M).$$
(3.163)

Exercise 3.41. Verify that $\mathcal{X}_{B}^{\text{inst}}$ is bounded as $\zeta \to 0$ and $\zeta \to \infty$.

Let us briefly comment on the relation of this example to mirror symmetry. For any fixed ζ , the space (X, I_{ζ}) has two open domains $\text{Im}(z/\zeta) > 0$ and $\text{Im}(z/\zeta) < 0$. In each of these domains we have a pair of holomorphic coordinates $(\mathcal{X}_A, \mathcal{X}_B)$, which moreover are Darboux coordinates in the sense of (3.157). These coordinates can be analytically continued beyond their respective domains, and on the overlaps the analytic continuations differ by the jumps (3.159), (3.160). In other words, our picture of X is that it is glued together from two patches, each patch canonically mapping to $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ (probably embedded in $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ though I do not think this has quite been proven), overlapping in a domain with two connected components, with the transition functions (3.159), (3.160) on the two components. (Strictly speaking this description omits the fiber over z = 0; the function \mathcal{X}_A and the analytic continuation of \mathcal{X}_B from one side actually do extend over this fiber [check], but \mathcal{X}_B can vanish there, so they no longer map to $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$.)

This kind of description of the space X is reminiscent of a general approach for constructing Calabi-Yau manifolds by gluing together tori $(\mathbb{C}^{\times})^{2n}$, employed (in some form) by Kontsevich-Soibelman, Gross-Siebert, Auroux, Gross-Hacking-Keel, following ideas of Strominger-Yau-Zaslow. In that approach one focuses on a single complex structure on X, for which the torus fibers are special Lagrangian: that means we need some I_{ζ} with $|\zeta| = 1$. **Exercise 3.42.** Show that the torus fibers of *X* are special Lagrangian with respect to the pair $(\omega_{\zeta}, \Omega_{\zeta})$ if and only if $|\zeta| = 1$.

Let us take structure I_3 i.e. $\zeta = 1$. The gluing maps are supposed to be related to holomorphic discs on a mirror manifold X^{\vee} , which is thought of just as a symplectic manifold, but which is also realized as a torus fibration over the same base as X. In some sense the locus where the gluing takes place is to be identified with the locus of torus fibers in X^{\vee} which contain boundaries of holomorphic discs in X^{\vee} .

In the case of the Ooguri-Vafa space the mirror manifold X^{\vee} is X again with its symplectic structure ω_2 , i.e. $\zeta = i$. Then indeed the two gluing maps we just discussed correspond to two holomorphic discs in X^{\vee} , namely the ones in Exercise 3.38: note that these have boundaries on the fibers with $z \in \mathbb{R}$, precisely the locus where the discontinuities of $\mathcal{X}_B(\zeta = 1)$ are.

One of the important features of the Ooguri-Vafa space is that it describes the local geometry near singular fibers of more interesting elliptically fibered hyperkähler spaces; but more precisely we need a slight generalization for this purpose:

Example 3.42 (Generalized Ooguri-Vafa space). We reconsider Example 3.40 with a slight modification: instead of adding just a *constant* to *V* we allow something more general, replacing (3.144) by

$$V = \frac{1}{4\pi \|x\|} + \sum_{n \neq 0 \in \mathbb{Z}} \left(\frac{1}{4\pi \|x - x_n\|} - \frac{1}{8\pi^2 |n|} \right) + f$$
(3.164)

where *f* is any smooth harmonic function which is invariant under the shift $x_3 \rightarrow x_3 + 2\pi n$. All of the constructions in Example 3.40 then go through as before and give a hyperkähler metric which is torus fibered over some disc in the complex plane, with one nodal fiber.

3.9 Differential geometry of hyperkähler manifolds

Proposition 3.43 (Hyperkähler manifolds are Ricci-flat). If *X* is a hyperkähler manifold, then the hyperkähler metric *g* is Ricci-flat.

Proof. View *X* as a Kähler manifold in structure I_1 , with dim_C X = n. The canonical bundle $K_X = \wedge^n (T^*)^{1,0} X$ with its induced Levi-Civita connection admits the global covariantly constant section Ω_1 , so its curvature is zero; thus, by Proposition 2.34, Ric = 0.

Proposition 3.43 suggests one possible strategy for constructing examples of hyperkähler structures: first construct a Ricci-flat Kähler metric and then try to prove it is actually part of an hyperkähler structure. To implement this strategy the next theorem is the key tool.

Theorem 3.44 (Yau's theorem: existence of Ricci-flat Kähler metrics). [24] Suppose that:

- *X* is a compact complex manifold,
- the canonical bundle K_X is trivial (as a complex line bundle),
- $\alpha \in H^2_{dR}(X)$ is a Kähler class, i.e. there exists some Kähler metric on X with $[\omega] = \alpha$.

Then there exists a unique Ricci-flat Kähler metric on *X* with $[\omega] = \alpha$.

Note that Theorem 3.44 is not explicit: it guarantees the existence of *some* Ricci-flat metric but tells us relatively little about what that metric actually is. A nice sketch of the proof can be found in [25].

Theorem 3.45 (Compact, Kähler, uniquely holomorphic symplectic manifolds are uniquely hyperkähler). Suppose that:

- *X* is a compact complex manifold,
- the space of holomorphic symplectic forms on X is 1-complex-dimensional, spanned by Ω ∈ Ω^{2,0}(X),
- $\alpha \in H^2_{dR}(X)$ is a Kähler class, i.e. there exists some Kähler metric on X with $[\omega] = \alpha$.

Then there exists a unique hyperkähler structure (*X*, *I*₁, *I*₂, *I*₃, *g*) such that $[\omega_1] = \alpha$, *I*₁ is the given complex structure on *X*, and $\omega_2 + i\omega_3 = c\Omega$ for some $c \in \mathbb{R}$.

Proof. Since Ω^n is a nowhere vanishing holomorphic section of the canonical bundle K_X , we have $c_1(X) = 0$; thus we can use Theorem 3.44 to conclude that there exists a unique Ricci-flat Kähler metric g on X with $[\omega] = \alpha$. Now let I_1 be the given complex structure on X, and $\omega_1 = \omega$. What we need to show is that g is actually a hyperkähler metric.

First we use the "Bochner principle" which says that holomorphic objects on Ricci-flat compact manifolds are covariantly constant. More precisely: let $\langle \cdot, \cdot \rangle$ denote the L^2 inner product on $\Omega^*(X)$ or $\Omega^*(X) \otimes TX$. Recall the $\bar{\partial}$ -Laplacian

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \qquad (3.165)$$

which has in particular $\Delta_{\bar{\partial}}\Omega = 0$ (trivially, since $\bar{\partial}^*\Omega = 0$ for degree reasons.) A Weitzenbock formula (see [26] for a proof), plus the fact that Ric = 0, gives a simple relation between this and the covariant Laplacian:

$$\Delta_{\bar{\partial}} = \nabla^* \nabla. \tag{3.166}$$

Thus we have

$$0 = \langle \Omega, \Delta_{\bar{\partial}} \Omega \rangle = \langle \nabla \Omega, \nabla \Omega \rangle, \qquad (3.167)$$

so $\nabla \Omega = 0$.

Now define ω_2 and ω_3 by $\omega_2 + i\omega_3 = \Omega$, and define $I_2 = g^{-1}\omega_2$, $I_3 = g^{-1}\omega_3$. It is straightforward to show $I_2I_1 = -I_3$, $I_3I_1 = I_2$; also $\Omega g^{-1}\Omega = 0$, so $(I_2 + iI_3)^2 = 0$, so $I_2^2 = I_3^2$ and $I_2I_3 = -I_3I_2$. Moreover, by construction I_2 is skew-symmetric with respect to an orthogonal basis, since its matrix in such a basis is just the matrix of ω_2 . Using the structure theorem for skew-symmetric matrices, it follows that in some orthogonal basis I_2 is a direct sum of blocks $\begin{pmatrix} 0 & c_i \\ -c_i & 0 \end{pmatrix}$ with all $c_i > 0$. All that remains is to show that all of the c_i appearing in this decomposition are equal; then by an overall rescaling of ω_2 and ω_3 we can arrange they are all 1, which completes the proof.

So, suppose that the c_i are not all equal. Then decompose *TX* into blocks E_i labeled by the distinct c_i . Evidently the E_i are preserved by I_2 , and E_i can be characterized as

the $-c_i^2$ eigenspace of I_2^2 . We have a similar block decomposition for the action of I_3 , and since $I_3^2 = I_2^2$, the blocks must coincide. So the E_i are also preserved by I_3 . Then using $I_2^2I_1 = -I_2I_3$ they are also preserved by I_1 . Moreover, since I_2 is covariantly constant, the E_i are also preserved by parallel transport. But in this case we can construct a covariantly constant endomorphism of TX which acts by a different constant on each E_i , and thus commutes with I_1 . Then applying this endomorphism to Ω we would get other holomorphic symplectic forms on X, violating our assumption.

Example 3.46 (Elliptically fibered K3 surface). Now let's construct a hyperkähler space with torus fibers over the *compact* base \mathbb{CP}^1 . Loosely speaking, for each $u \in \mathbb{CP}^1$ we will consider an elliptic curve

$$X_u = \{y^2 z = x^3 + A(u)xz^2 + B(u)z^3\} \subset \mathbb{CP}^2,$$
(3.168)

and let *X* be the union of the X_u .

We want *X* to carry a global holomorphic 2-form. How can we make it? On each fiber X_u we have the holomorphic 1-form dx/y. We want to wedge this with a holomorphic 1-form coming from the base. Such a 1-form doesn't literally exist, because $K_{\mathbb{CP}^1} \simeq \mathcal{O}(-2)$; but there is a nowhere vanishing *twisted* 1-form,

$$\eta \in H^0(K_{\mathbb{CP}^1} \otimes \mathcal{O}(2)). \tag{3.169}$$

Our desired holomorphic symplectic form will be

$$\Omega = \eta \wedge \frac{\mathrm{d}x}{y}.\tag{3.170}$$

For this to make global sense, dx/y needs to be valued in $\mathcal{O}(-2)$. Also, for the equation (3.168) to make sense, y^2z and x^3 must be valued in the same place. To make all this work, we make *x* valued in $\mathcal{O}(4)$, *y* in $\mathcal{O}(6)$, and *z* in $\mathcal{O}(0)$. Then A(u) must be a (fixed) section of $\mathcal{O}(8)$ and B(u) a section of $\mathcal{O}(12)$. Now we are ready to make a proper definition of *X*:⁹

$$X = \{y^2 z = x^3 + A(u)xz^2 + B(u)z^3\} \subset \mathbb{P}[\mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O}(0)] \to \mathbb{CP}^1$$
(3.171)

The fiber X_u is smooth as long as the polynomial $x^3 + A(u)x + B(u)$ doesn't have any double zeroes in x, i.e. as long as the discriminant

$$\Delta(u) = 27B(u)^2 + 4A(u)^3 \tag{3.172}$$

does not vanish. Since Δ is a section of $\mathcal{O}(24)$, if it has no multiple zeroes we get 24 singular fibers. To look more closely at the singular fibers, note that at each of them we have $A(u) \neq 0$ and $B(u) \neq 0$ (otherwise Δ would have a higher-order zero at u). Thus the singular fiber is of the form

$$\{y^2 z = x^3 + Axz^2 + Bz^3\} \subset \mathbb{CP}^2$$
(3.173)

⁹What the notation $\mathbb{P}[\mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O}(0)]$ means: we take the locus where *x*, *y*, *z* don't simultaneously vanish, then quotient by simultaneous rescalings $(x, y, z) \to (\lambda x, \lambda y, \lambda z)$ with $\lambda \in \mathbb{C}^{\times}$; this defines a bundle over \mathbb{CP}^1 with each fiber isomorphic to \mathbb{CP}^2 .

with $A, B \neq 0$ and $27B^2 - 4A^3 = 0$. This gives a nodal torus (like the one in the Ooguri-Vafa space.) It is straightforward moreover to check that if Δ has no multiple zeroes then the total space of X is smooth. It carries the holomorphic symplectic form Ω . Finally, X is Kähler. [simple proof? add Kähler form coming from \mathbb{CP}^2 fibers to ones coming from base?] Thus by Theorem 3.45, X admits hyperkähler metrics.

We have obtained a family of hyperkähler manifolds: one for each $(A(u), B(u), \alpha)$, where $A(u) \in H^0(\mathcal{O}(8))$, $B(u) \in H^0(\mathcal{O}(12))$ are such that $\Delta(u)$ has only simple zeroes, and α is a Kähler class.

As we vary A(u), B(u) and α the diffeomorphism type of the space does not change, so we could think of this as a family of hyperkähler structures on a single manifold. Most of the complex structures on this manifold are *not* of the type we constructed here: indeed most of them are not even algebraic.

Exercise 3.43. Check that the 1-form dx/y is holomorphic on the curve (3.168).

Exercise 3.44. Check that if Δ has no multiple zeroes then *X* is smooth.

In general, Yau's Theorem gives very little explicit information about the Ricci-flat metrics it provides. In the case of the K3 surface we have at least some *asymptotic* information, thanks to the work of Gross-Wilson [27]. *Very roughly* this says the following. Fix an elliptically fibered K3 surface *X* as in Example 3.46. The base \mathbb{CP}^1 has a natural Kähler metric. Moreover there exists a family of Kähler classes $\alpha(R) \in H^{1,1}_{\mathbb{R}}(X)$, parameterized by $R \in \mathbb{R}_+$, such that the corresponding hyperkähler metrics g(R) can be described concretely:

$$g(R) \approx g_{avv}(R) + O(e^{-MR}) \tag{3.174}$$

where *M* is controlled by the distance between the singular fibers, and $g_{app}(R)$ are obtained by taking generalized Ooguri-Vafa metrics (Example 3.42) in small discs around the 24 singular fibers and gluing them to semiflat metrics (Example 3.39) on the rest of X — where the holomorphic data Z_{γ} entering the semiflat metric are certain periods of the holomorphic symplectic form $R\Omega$ — and then dividing by R^2 . Then the torus fibers of g(R) have area $4\pi^2/R^2$. As $R \to \infty$ these metrics have a Gromov-Hausdorff limit, in which the torus fibers collapse onto the Kähler base \mathbb{CP}^1 .

3.10 Hyperkähler quotients

Definition 3.47 (Hyperkähler moment map). Suppose *X* is hyperkähler and a compact group *G* acts on *X* by isometries preserving the hyperkähler structure. Then a *hyperkähler moment map* for this action is a map

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3) : X \to \mathfrak{g}^* \otimes \mathbb{R}^3 \tag{3.175}$$

where μ_i is a moment map for the action with respect to ω_i .

Exercise 3.45. Suppose $\vec{\mu}$ is a hyperkähler moment map as above. Then for $\vec{s} \in S^2$, defining

$$\mu_{\vec{s}}: X \to \mathfrak{g}^* \tag{3.176}$$

by

$$\mu_{\vec{s}} = \vec{\mu} \cdot \vec{s} \tag{3.177}$$

show that $\mu_{\vec{s}}$ is a moment map for the action with respect to $\omega_{\vec{s}}$.

Exercise 3.46. Suppose *X* is a Gibbons-Hawking space as in Example 3.13. Then *X* has a U(1) action by isometries preserving the hyperkähler structure and also has a projection $X \to \mathbb{R}^3$. Show that this projection is a hyperkähler moment map for the U(1) action (when we identify $\mathfrak{u}(1) \simeq \mathbb{R}$ as usual).

Definition 3.48 (Hyperkähler quotient). Suppose *X* is hyperkähler and a compact group *G* acts on *X*, with hyperkähler moment map $\vec{\mu}$. Then the *hyperkähler quotient* of *X* by *G* is

$$X////G = \vec{\mu}^{-1}(0)/G. \tag{3.178}$$

Theorem 3.49 (Hyperkähler quotients are hyperkähler). Suppose *X* is hyperkähler and a compact group *G* acts on *X*, with hyperkähler moment map $\vec{\mu}$. Also suppose *G* acts freely on $\vec{\mu}^{-1}(0)$. Then *X*////*G* is a manifold, with

$$\dim_{\mathbb{R}}(X////G) = \dim_{\mathbb{R}} X - 4 \dim_{\mathbb{R}} G, \qquad (3.179)$$

and the quotient metric on *X*////*G* is hyperkähler.

Proof. First we show that if *G* acts freely on $\vec{\mu}^{-1}(0)$ then 0 is a regular value for $\vec{\mu}$.

To show $d\vec{\mu}$ surjective it is sufficient to show that the equation

$$d\mu_1 \cdot Z_1 + d\mu_2 \cdot Z_2 + d\mu_3 \cdot Z_3 = 0 \tag{3.180}$$

implies $Z_1 = Z_2 = Z_3 = 0$. We may rewrite this as

$$\iota_{\rho(Z_1)}\omega_1 + \iota_{\rho(Z_2)}\omega_2 + \iota_{\rho(Z_3)}\omega_3 = 0$$
(3.181)

which in turn is equivalent to

$$I_1\rho(Z_1) + I_2\rho(Z_2) + I_3\rho(Z_3) = 0.$$
(3.182)

To show this implies $Z_1 = Z_2 = Z_3 = 0$ is the content of Exercise 3.47 below. Using this result, we get that 0 is indeed a regular value for $\vec{\mu}$, so $\vec{\mu}^{-1}(0)$ is a manifold.

Now, consider the function

$$M_1 = \mu_2 + \mathrm{i}\mu_3 : X \to \mathfrak{g}_{\mathbb{C}}.\tag{3.183}$$

For any $Z \in \mathfrak{g}$ we have

$$dM_1 \cdot Z = \iota_{\rho(Z)}(\omega_2 + i\omega_3) = \iota_{\rho(Z)}\Omega_1 \in \Omega^{1,0}(X),$$
(3.184)

so M_1 is holomorphic. Thus $Y = M_1^{-1}(0)$ is an I_1 -complex submanifold, hence in particular Kähler for ω_1 . Moreover *G* acts on *Y* preserving ω_1 , with moment map μ_1 , and we have

$$X////G = \vec{\mu}^{-1}(0)/G = (M_1^{-1}(0) \cap \mu_1^{-1}(0))/G = Y//G.$$
(3.185)

Thus by Proposition 2.52, I_1 descends to the quotient X //// G, and the quotient metric on X //// G is Kähler for I_1 . Similarly it is Kähler for I_2 , I_3 .

Exercise 3.47. Suppose *V* is a quaternionic vector space, with a compatible metric and a real subspace $W \subset V$, such that I_iW is orthogonal to *W* for i = 1, 2, 3. Suppose also

$$w_0 + I_1 w_1 + I_2 w_2 + I_3 w_3 = 0, \qquad w_i \in W.$$
(3.186)

Then show that $w_0 = w_1 = w_2 = w_3 = 0$.

The idea of Theorem 3.49 is that the single hyperkähler quotient space can be realized as a Kähler quotient in many different ways. There is also yet another way to think of it, using the fact that the Kähler quotient is (ignoring issues of stability) a quotient by a complexification $G_{\mathbb{C}}$. Thus modulo stability we have

$$X////G = M_1^{-1}(0)/G_{\mathbb{C}} = X//G_{\mathbb{C}}$$
(3.187)

i.e. it looks like *X* //// *G* is a *holomorphic symplectic* version of the symplectic quotient. This point of view is useful for constructing the twistor space of *X* //// *G*: it arises as a *fiberwise* version of the holomorphic symplectic quotient, i.e. (again ignoring issues of stability)

$$\mathcal{Z}(X////G) = \mathcal{Z}(X)//G_{\mathbb{C}}.$$
(3.188)

Example 3.50 (Hyperkahler quotient of \mathbb{H} by U(1)). Suppose $X = \mathbb{H}$ with its standard hyperkähler structure, and the standard $U(1) \subset SU(2)_R$ action from Exercise 3.12. It follows from the result of Exercise 3.12 that for any choice of $\vec{\mu}$, $\vec{\mu}^{-1}(0)$ is a circle, and thus $\mathbb{H}////U(1)$ is a point.

Let us look in more detail at how this goes. We have the moment maps

$$\mu_1 = \frac{1}{2}(|w|^2 - |z|^2) + c, \qquad (3.189)$$

$$M_1 = \mu_2 + i\mu_3 = -iwz + \alpha, \tag{3.190}$$

for constants $c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$. We want to determine $\vec{\mu}^{-1}(0)$. Setting $M_1 = 0$ means

$$wz = -i\alpha \tag{3.191}$$

Assuming $\alpha \neq 0$ for a moment, this gives a \mathbb{C}^{\times} parameterized by say w, with $z = -i\alpha/w$. Now we consider the condition $\mu_1 = 0$: this says

$$0 = \frac{1}{2} \left(|w|^2 - \frac{|\alpha|^2}{|w|^2} \right) + c$$
(3.192)

which has a single positive solution for $|w|^2$, irrespective of the value of *c*. Thus $\vec{\mu}^{-1}(0)$ is a circle, parameterized by the phase of *w*. Finally dividing out by U(1) gives a single point, so in this case we got $\mathbb{H}////U(1) = \{pt\}$.

If $\alpha = 0$ the story looks a little different: the condition (3.191) says

$$wz = 0, \tag{3.193}$$

and the condition $\mu_1 = 0$ becomes

$$|w|^2 - |z|^2 = -2c. ag{3.194}$$

So if c > 0 we get w = 0, $|z| = \sqrt{2c}$, and dividing out by U(1) gives a point as before. Similarly if c < 0 we get z = 0, $|w| = \sqrt{2c}$, and dividing out by U(1) gives a point. If c = 0 then we are in the singular situation w = z = 0, so U(1) does not act freely.

Thus altogether we have found that if $(\alpha, c) \neq (0, 0)$ we have $\mathbb{H}////U(1) = \{pt\}$, and if $(\alpha, c) = (0, 0)$, U(1) does not act freely on $\vec{\mu}^{-1}(0)$. (It happens that we get a point anyway, but morally speaking we should consider this as a singular situation, where we would not expect to get a manifold.)

Note that although the analysis looks different for $\alpha \neq 0$ and $\alpha = 0, c \neq 0$ the eventual conclusion is the same.



This is as we should expect: (α, c) are just different components of a single hyperkähler moment map, and could be rotated into one another by action of $SU(2)_R$. Said otherwise, the analysis looks unsymmetric only because we chose to work in the complex coordinates given by the arbitrarily-chosen complex structure I_1 .

Finally note that the "hard" part in this example was solving the real equation (3.192) which came from $\mu_1 = 0$. If we are only interested in getting the quotient as a complex manifold in structure I_1 , we could try to sidestep this problem by looking instead at $M_1^{-1}(0)/\mathbb{C}^{\times}$, which modulo problems of stability is supposed to be equal to the desired $\mathbb{H}////U(1)$. For $\alpha \neq 0$, $M_1^{-1}(0)$ is $\{zw = -i\alpha\}$, a single \mathbb{C}^{\times} -orbit, which indeed is stable (it intersects the locus $\mu_1 = 0$). For $\alpha = 0$ it's trickier: the locus zw = 0 contains two \mathbb{C}^{\times} -orbits — one open and one just the point z = w = 0. For $c \neq 0$ the open orbit is stable; for c = 0 the point orbit is stable, but U(1) doesn't act freely there.

Example 3.51 (Modifying hyperkähler manifolds with U(1) **symmetry).** [28] Suppose *X* is any hyperkähler manifold with a U(1) action by triholomorphic isometries, with a moment map $\vec{\mu}_X$. Then we can consider

$$X' = X \times \mathbb{H}.\tag{3.195}$$

Putting our standard $U(1) \subset SU(2)_R$ action on \mathbb{H} , with its standard moment map $\vec{\mu}_{\mathbb{H}}$, we get a diagonal U(1) action on X', with moment map

$$\vec{\mu} = \vec{\mu}_X + \vec{\mu}_{\mathbb{H}}.\tag{3.196}$$

Note the U(1) action is free except at points (x, 0) where $x \in X$ is a fixed point of the U(1) action there. Thus provided we choose $\vec{\mu}_X$ so that it does not vanish at these points x, the hyperkähler quotient

$$Y = X' //// U(1)$$
(3.197)

is a hyperkähler manifold, of the same dimension as *X*. We think of it as a *modification* of *X*.

Example 3.52 (Multi-Eguchi-Hanson spaces as hyperkähler quotients). Take $X = \mathbb{H}^n$. In structure I_1 , X is naturally identified with \mathbb{C}^{2n} with coordinates $(z_1, w_1, \ldots, z_n, w_n)$. It has a triholomorphic $U(1)^n$ action (coming from $SU(2)_R$ action on each factor), acting by $z_i \to e^{i\alpha_i}z_i, w_i \to e^{-i\alpha_i}w_i$, which has a moment map $X \to \mathbb{R}^3 \otimes_{\mathbb{R}} \mathfrak{u}(1)^n \simeq \mathbb{R}^{3n}$, just given by n copies of the one in Example 3.50. Now consider the "center of mass" subgroup $G \subset U(1)^n$ given by the condition $\prod_i g_i = 1$. We have $G \simeq U(1)^{n-1}$. Choose a tuple $x = (x_1, \ldots, x_n) \in \mathbb{R}^{3n}$. Now we consider the hyperkähler quotient

$$Y = X / / / _{x} G. (3.198)$$

We claim this quotient is a multi-Eguchi-Hanson space as in Example 3.15, with *n* singularities at the positions x_i .

To describe this we study the quotient at the level of the twistor space. So we begin with the twistor space of \mathbb{H}^n , which is $\mathcal{O}(1)^{\oplus 2n}$, with $\mathcal{O}(1)$ -valued coordinates z_i , w_i . The action of *G* here complexifies nicely to an action of $G_{\mathbb{C}} = (\mathbb{C}^{\times})^n$. The moment map for the *i*-th \mathbb{C}^{\times} action is $-iw_iz_i$. Thus in taking the symplectic quotient by $G_{\mathbb{C}}$ we first impose the moment map conditions

$$-i(w_1 z_1 - w_i z_i) = \eta_1 - \eta_i \tag{3.199}$$

for all *i*, where η_i is the section of $\mathcal{O}(2)$ corresponding to the point $x_i \in \mathbb{R}^3$.

We still need to divide out by the $G_{\mathbb{C}}$ -action. To write the result in a compact way, we define an $\mathcal{O}(2)$ -valued function by

$$\eta = iw_1 z_1 + \eta_1. \tag{3.200}$$

This is a natural object to consider: it is a moment map for the remaining \mathbb{C}^{\times} action, for which a representative generator is $w_1 \to e^{i\alpha}w_1$, $z_1 \to e^{-i\alpha}z_1$. Now combining (3.199) and (3.200) we get

$$\mathbf{i}w_i z_i = \eta - \eta_i. \tag{3.201}$$

Finally defining the $G_{\mathbb{C}}$ -invariant coordinates $u = \prod_{i=1}^{n} w_i$ and $v = \prod_{i=1}^{n} (iz_i)$, and taking the product over all *i* in (3.201), we get

$$uv = \prod_{i=1}^{n} (\eta - \eta_i).$$
(3.202)

We have seen this equation before: it is obeyed by the functions u, v, η on the twistor space for a multi-Eguchi-Hanson space (Example 3.37). [still need to show the space is literally equal]

4 Transition to infinite dimensions

Although our main eventual interest is in finite-dimensional hyperkähler spaces, the way we construct them will involve a detour into infinite dimensions: namely we will

divide infinite-dimensional vector spaces by infinite-dimensional groups, in such a way that the quotient is finite-dimensional.

In this section, to get prepared, we briefly describe the necessary modifications of some of our constructions to the infinite-dimensional setting. It might be a good idea to skip this section until you feel the need to look at it.

Let us motivate a bit what we have to do. The most obvious infinite-dimensional spaces in our context are spaces of C^{∞} sections of vector bundles. In order to develop the symplectic quotient directly in this context we would need (at least) a version of the inverse function theorem.

For this purpose we will pass to *completions* of the spaces of C^{∞} sections with respect to certain "Sobolev" norms. This has the advantage that these completions are Banach spaces, for which more off-the-shelf technology exists, including a reasonable inverse function theorem. In the end we will have to pass from the completions back to the C^{∞} spaces of our real interest. The theory of elliptic regularity ensures that this works without difficulty.

You may wonder why we do not look for some version of the inverse function theorem which works for the spaces of C^{∞} sections directly. Actually a candidate theorem does exist: it is called the Nash-Moser theorem. It applies to a rather specific sort of infinite-dimensional spaces, namely *tame Frechet spaces*, and to *tame maps* between them. The spaces of C^{∞} sections are indeed tame Frechet spaces, and the maps we need to consider are tame maps. My impression is that this tameness is nearly equivalent to the existence of Sobolev norms and the statement that the maps are well behaved with respect to these; so in the end perhaps this method would not be so different from what we do here.

4.1 Banach spaces

Here I write only the bare minimum, in part because I am not really an expert; to really learn this material well, you'd better look elsewhere. [provide some refs]

Definition 4.1 (Banach space). A *Banach space* is a normed vector space V over \mathbb{R} or \mathbb{C} , which is complete as a metric space.

Definition 4.2 (Norm of a linear map). Say *V* and *W* are Banach spaces and $T : V \rightarrow W$ is a linear map. Then we define

$$||T|| = \sup_{v \in V: ||v|| = 1} ||T(v)||.$$
(4.1)

We say *T* is bounded if $||T|| < \infty$.

Unlike the finite-dimensional situation, linear maps of Banach spaces need not be continuous:

Proposition 4.3 (Continuous = bounded). If *V* and *W* are Banach spaces, a linear map $T: V \rightarrow W$ is continuous if and only if it is bounded.

For our purposes we are mostly interested in Banach spaces as topological vector spaces — we don't care about the specific norm, only about the topology induced by

that norm. (Indeed, our main examples later will be Sobolev spaces, where in general we don't have a particularly canonical norm, just a class of norms which all induce the same topology.) Thus we take bounded linear maps as our notion of morphism, even though they don't preserve the norm. We take bounded linear isomorphisms as our notion of isomorphism; this is partially justified by the next theorem and exercise:

Theorem 4.4 (Open mapping theorem for Banach spaces). Suppose $T : V \rightarrow W$ is a surjective map of Banach spaces. Then *T* is an open map.

Exercise 4.1. Suppose *V* and *W* are Banach spaces, and $T : V \to W$ is a bounded linear isomorphism. Show that T^{-1} is also bounded. (Hint: use the open mapping theorem to show T^{-1} is continuous.)

Example 4.5 (Spaces of bounded linear maps). If *V* and *W* are Banach spaces, let L(V, W) denote the space of all bounded linear maps $V \to W$. L(V, W) is a Banach space. The *dual* of *V* is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$.

 V^* is not too small — at least it contains the dual space to any finite-dimensional $W \subset V$, as follows from:

Theorem 4.6 (Hahn-Banach theorem). If *V* is a Banach space, $W \subset V$ and $\psi \in W^*$, then ψ extends to an element $\tilde{\psi} \in V^*$, with $\|\tilde{\psi}\| = \|\psi\|$.

The Hahn-Banach theorem implies that the canonical map $V \rightarrow V^{**}$ is injective and norm-preserving. If it is an isomorphism, then *V* is called *reflexive*.

When considering Banach spaces we are particularly interested in their *closed* subspaces (these are the ones which are themselves Banach spaces).

Exercise 4.2. Suppose *V*, *W* are Banach spaces and $T : V \rightarrow W$ is a bounded linear map. Then show that ker *T* is closed.

Exercise 4.3. Suppose *V*, *W* are Banach spaces and $T : V \to W$ is a bounded linear map. Suppose there exists c > 0 such that $||T(x)|| \ge c||x||$ for all $x \in V$. Then show that Im *T* is closed.

Among these we will be especially interested in the complemented ones:

Definition 4.7 (Complemented subspace). If *V* is a Banach space and $W \subset V$ a closed subspace, we say *W* is *complemented* if there exists a closed subspace $W' \subset V$ such that the natural map $W \oplus W' \to V$ is an isomorphism.

The notion of "isomorphism" in this definition can be taken to be just linear isomorphism, since this map is evidently bounded, and so if it is a linear isomorphism it is also an isomorphism of topological vector spaces. We should remark that every $W \subset V$ admits a complement W' in the algebraic sense (using the existence of a Hamel basis), so the real issue is whether W' can be taken to be *closed* or not.

Exercise 4.4. Suppose *V* is a Hilbert space and $W \subset V$ is a closed subspace. Show that *W* is complemented.

Exercise 4.5. Suppose *V* is a Banach space and $W \subset V$ is a closed subspace of finite dimension or finite codimension. Show that *W* is complemented.

It doesn't seem so easy to come up with examples of closed subspaces which are not complemented, but it is known that they do exist in any Banach space which is not a Hilbert space. One concrete example: take *V* to be the space of bounded sequences in \mathbb{R} with the L^{∞} (supremum) norm, and $W \subset V$ the subspace of sequences which tend to zero; then *W* is closed but not complemented.

Definition 4.8 (Nondegenerate pairing of Banach spaces). If V, W are Banach spaces over \mathbb{R} , a *nondegenerate pairing* between V and W is a bilinear map

$$V \times W \to \mathbb{R} \tag{4.2}$$

continuous in each factor, such that the induced maps

$$V \to W^*, \qquad W \to V^*$$
 (4.3)

are injections. (Some authors would call this a "weakly nondegenerate" pairing.)

The topology induced by the norm on a Banach space is sometimes called the *strong topology*. Another one which is useful (e.g. in the formulation of Theorem 5.34 below) is the weak topology:

Definition 4.9 (Weak topology). If V is a Banach space, the *weak topology* on V is the weakest topology for which all elements of V^* are continuous.

4.2 Calculus on Banach spaces

Definition 4.10 (Derivatives of maps of Banach spaces). Suppose $f : V \to W$ is a map of Banach spaces. We say f is *differentiable* at v, and $Df_v : V \to W$ is its *derivative* at v, if

$$\lim_{h \to 0} \frac{\|f(v+h) - f(v) - Df_v(h)\|}{\|h\|} = 0.$$
(4.4)

If $U \subset V$ is open and f is differentiable at all $v \in U$, then we get a map $Df : U \to L(V, W)$ defined by $Df(v) = Df_v$. If Df is continuous, we say f is C^1 on U. Similarly, if $D^r f$ exists for all r, we say f is C^{∞} or *smooth* on U.

Theorem 4.11 (Inverse function theorem for Banach spaces). Suppose $f : V \to W$ is a smooth map of Banach spaces, and for some $v \in V$ the map $Df_v : V \to W$ is a bounded isomorphism. Then there exists a neighborhood U of v such that $f|_U$ admits a smooth two-sided inverse.

4.3 Sobolev spaces

Example 4.12 (Sobolev spaces). Suppose *E* is a complex vector bundle with Hermitian metric *h* over a compact Riemannian manifold *X*. Then for $k \in \mathbb{Z}_{\geq 0}$ and 1 ,

we define the *Sobolev norm* on smooth sections of *E* as follows. First fix an *h*-unitary connection ∇ on *E*. Then write

$$\|s\|_{p,k}^{p} = \int_{X} \left(\|s\|^{p} + \|\nabla s\|^{p} + \|\nabla \nabla s\|^{p} + \dots + \|\nabla^{k}s\|^{p} \right) dvol_{X}.$$
(4.5)

(Here the various terms are sections of tensor bundles $E \otimes (TX)^m$, considered with the norms induced by *h* and the metric on *X*.) Then $L_k^p(E)$ is the *completion* of the space of smooth sections of *E* with respect to $\|\cdot\|_{p,k}$ (modulo identification of functions which agree almost everywhere). While the precise norm depends on the choice of ∇ , the different Banach spaces $L_k^p(E)$ we obtain from the different norms are all isomorphic.

Crudely speaking, $L_k^p(E)$ consists of functions which are L^p and have k derivatives that are also in L^p . There is actually an extension of this definition to arbitrary $k \in \mathbb{R}$ (fractional number of derivatives!) which is a bit more technical to describe: to get the required norm $\|\cdot\|_{p,k}$ one has to use a partition of unity to reduce to a patch of \mathbb{R}^n , then use Fourier transform. There is an efficient description of this in [29], which I like. When k < 0 these should be thought of as distributions, and indeed $L_{-k}^p(E)$ is the dual of $L_k^p(E)$. Alternatively, morally $L_{-k}^p(E)$ means sections of E which are locally a bit singular, but become L^p after they are integrated k times.

We will only need the case p = 2, i.e. we will work always with L^2 objects admitting various numbers of derivatives.

4.4 Banach manifolds

A good review of this material is in [30].

Definition 4.13 (Banach manifold). A (smooth) *Banach manifold* is a Hausdorff topological space *X*, equipped with a maximal atlas of open charts U_{α} with smooth injective maps

$$\phi_{\alpha}: U_{\alpha} \to V_{\alpha} \tag{4.6}$$

such that each V_{α} is a Banach space, $\phi_{\alpha}(U_{\alpha})$ is an open subset, and the transition maps $\phi_{\alpha} \circ \phi_{\beta}^{-1}$ are smooth.

From now on in this section let *X* be a Banach manifold.

Definition 4.14 (Vector bundle). Vector bundles *V* over *X* are defined in the usual way: they are Banach manifolds with projection $\pi : V \to X$, admitting local trivializations with smooth transition functions.

Definition 4.15 (Tangent bundle and differentials). For $x \in X$ the *tangent space* T_xX is defined in the "classical" way: we take $\sqcup_{\alpha:x\in U_{\alpha}}V_{\alpha}$, modulo the relation $v \in V_{\alpha} \sim v' \in V_{\alpha'}$ if v' = Jv, where $J : V_{\alpha} \to V_{\alpha'}$ is the differential of the transition map $\phi_{\alpha'}^{-1} \circ \phi_{\alpha}$. This is again a Banach space. The T_xX fit together into a vector bundle over X. For a smooth map $f : X \to Y$ one then gets in the usual way a smooth map $df : TX \to TY$.

This definition of tangent vectors is equivalent to taking equivalence classes of curves with the same 1-jet, but generally *not* equivalent to looking at derivations acting on smooth

functions (in general there are more derivations than tangent vectors, unless the model Banach space is reflexive; on the other hand our main application will be to Sobolev spaces, which *are* reflexive, so maybe we needn't worry too much about this.)

Definition 4.16 (Cotangent bundle and differential forms). The cotangent bundle, differential forms, and the d operator can be defined by the same constructions as usual.

Definition 4.17 (Submanifold). A subset $Y \subset X$ is a *submanifold* if for every $y \in Y$ there exists a chart U_{α} of X, such that $\phi_{\alpha}(Y \cap U_{\alpha}) = W_{\alpha} \cap \phi_{\alpha}(U_{\alpha})$ for some *closed* subspace $W_{\alpha} \subset V_{\alpha}$.

This is just like the usual definition of (embedded) submanifold; the key extra word in the infinite-dimensional setting is "closed."

4.5 Symplectic forms and symplectic quotients

Definition 4.18 (Symplectic Banach manifold). The definition of symplectic Banach manifold is the same as Definition 2.8, except that the notion of nondegeneracy is in the weak sense: we require only that the map

$$T_x X \to T_x^* X \tag{4.7}$$

$$T_x X \mapsto \iota_v \omega = \omega(v, \cdot) \tag{4.8}$$

is an *injection* for each $x \in X$, not an isomorphism.

Definition 4.19 (Banach Lie group). A Banach Lie group *G* is a Banach manifold equipped with a group structure, such that the multiplication map $G \times G \rightarrow G$ and the inversion map $G \rightarrow G$ are smooth. The Lie algebra \mathfrak{g} is the Banach space T_eG and acquires a Lie bracket in the usual way.

Proposition 4.20 (Free proper quotients are Banach manifolds). Suppose *X* is a Banach manifold with a Banach Lie group *G* acting properly and freely, and the tangent spaces to orbits, $T_x(Gx) \subset T_xX$, are closed and complemented. Then X/G is a manifold and $TX/\rho(\mathfrak{g}) \simeq T(X/G)$, as in Proposition 2.5. [ref Bourbaki]

In the situation where we will use Proposition 4.20, the tangent spaces to *G*-orbits have finite codimension; in that case they are automatically complemented if they are closed.

Definition 4.21 (Moment maps and symplectic quotients). The definition of moment maps and symplectic quotients are the same as Definition 2.11 and Definition 2.12.

Theorem 4.22 (Symplectic quotient is symplectic). Suppose *X* is a symplectic Banach manifold, with a Banach Lie group *G* acting properly, with moment map μ . Suppose that *G* acts freely on $\mu^{-1}(0)$, and we impose some additional conditions:

- $\mu^{-1}(0)$ is a submanifold of *X*.
- The tangent spaces $T = T_x(Gx)$ to *G*-orbits are closed and complemented.
- Each tangent space obeys $T = (T^{\perp_{\omega}})^{\perp_{\omega}}$.

Then *X*//*G* is a Banach manifold, and there is a symplectic form $\omega_{X//G}$ on *X*//*G*, with the property

$$\pi^* \omega_{X//G} = \iota^* \omega. \tag{4.9}$$

Proof. (From [31].) Follow the pattern of the proof of Proposition 2.13. The infinitedimensional setting leads to the following hiccups:

- We just *assume* that μ⁻¹(0) is a submanifold rather than proving it; it would be nice to see in a general way why this follows "formally" from the *G*-action being free, as in the finite-dimensional case, but this seems to be subtle, or at least, I had trouble making it work. In the case we need, we will prove it by hand.
- To see that $\mu^{-1}(0)/G$ is a manifold we use Proposition 4.20.
- The "dimension counting" argument which showed ω nondegenerate in the finitedimensional case gets replaced by a use of the assumption T = (T^{⊥ω})^{⊥ω}.

Definition 4.23 (Hyperkähler quotients). [again same as in finite dimensions]

Theorem 4.24 (Hyperkähler quotient is hyperkähler). [...]

4.6 Ellipticity

We give here a very brief and sketchy account of ellipticity. Something much nicer can be found in [29].

Suppose *X* is a manifold carrying vector bundles *E*, *F*. We consider a smooth differential operator $D : E \to F$ and a smooth section *f* of *F*. We would like to know when we can solve the equation

$$De = f \tag{4.10}$$

for *e*, and what kind of regularity properties *e* should have.

To get oriented let us consider a very special case, the Laplace operator in a Euclidean vector space *V*,

$$D = \sum_{i=1}^{n} \partial_i^2. \tag{4.11}$$

Here, we could try to solve (4.10) by Fourier transformation: at least formally the differentiations become multiplications, and we get

$$-\|p\|^{2}\hat{e}(p) = \hat{f}(p)$$
(4.12)

for $p \in V^*$, which would be solved by

$$\hat{e}(p) = -\frac{\hat{f}(p)}{\|p\|^2}$$
(4.13)

or, returning to position space,

$$e(x) = -\int_{V^*} (\mathrm{d}vol_{V^*}) e^{\mathrm{i}p \cdot x} \frac{\hat{f}(p)}{\|p\|^2}$$
(4.14)

This looks like it has some chance of working. The large-*p* behavior of the integrand is very good when *f* is smooth: indeed, in that case $\hat{f}(p)$ decays faster than any polynomial in *p*. The only problem is to control the behavior of $\hat{f}(p)$ near p = 0. That problem has to do with the *low frequency* modes of *f*. Its analogue is different on a compact manifold: it becomes the condition that *f* is orthogonal to the kernel of D^* . Apart from this difficulty, this strategy really works and can be applied to the Laplacian acting on functions on a compact Riemannian manifold *X* (with appropriate patching, partitions of unity and such).

Moreover, the solutions obtained in this way have very good regularity: if f itself was smooth then they are also smooth, essentially because we can differentiate under the integral sign in (4.14) — this just introduces more powers of p, but $\hat{f}(p)$ decays fast enough that the integral still converges.

In contrast, the same strategy would fail drastically for the Laplacian on a manifold with indefinite signature metric. The problem is that in this case the factor $1/||p||^2$ in (4.14) has singularities propagating all the way out to $p = \infty$ rather than just at p = 0.

The property that the highest-order part of the Fourier transformed operator (*symbol*) is invertible except at p = 0 is what we call *ellipticity*. It is associated with many good analytical properties, some of which we will use later on.

Let us specialize to the operator

$$d + d^* : \Omega^1(C) \to \Omega^2(C) \oplus \Omega^0(C)$$
(4.15)

on a Riemannian 2-manifold *C* (indeed this is essentially the operator we will need below.) In local coordinates around x = 0, orthonormal at 0, this operator could be written as

$$f_1 dx_1 + f_2 dx_2 \mapsto (\partial_2 f_1 - \partial_1 f_2) dx_1 \wedge dx_2 + (\partial_1 f_1 + \partial_2 f_2)$$

$$(4.16)$$

or relative to the bases $\{dx_1, dx_2\}$ and $\{dx_1 \land dx_2, 1\}$,

$$\begin{pmatrix} \partial_2 & -\partial_1 \\ \partial_1 & \partial_2 \end{pmatrix} \tag{4.17}$$

Thus the symbol is

$$\mathbf{i} \begin{pmatrix} p_2 & -p_1 \\ p_1 & p_2 \end{pmatrix} \tag{4.18}$$

which is indeed invertible except at $p_1 = p_2 = 0$. Thus this operator is elliptic.

For the record, we can formulate things a little more precisely:

Definition 4.25 (Symbol of a differential operator). The (principal) *symbol* of an order ℓ differential operator D mapping between smooth vector bundles E and F over X is a section $\sigma \in \pi^* \operatorname{Hom}(E, F)$, where $\pi : T^*X \to X$ is the projection. It has a fancy invariant

$$\sum_{i_1,\dots,i_\ell} F_{i_1\dots i_\ell} \partial_{i_1} \cdots \partial_{i_\ell}$$
(4.19)

where each $P_{i_1 \cdots i_k} \in \text{Hom}(E, F)$, and then replace $\partial_i \to ip_i$, where p_i are the corresponding local coordinates on T^*X , to get

$$\sigma = \mathbf{i}^{\ell} \sum_{i_1, \dots, i_{\ell}} F_{i_1 \cdots i_{\ell}} p_{i_1} \cdots p_{i_{\ell}}$$
(4.20)

Exercise 4.6. Verify that Definition 4.25 is well formulated, i.e. that the symbol σ so obtained does not depend on the coordinate system we choose.

Definition 4.26 (Elliptic differential operator). A differential operator *D* is elliptic if its symbol $\sigma \in \pi^* \operatorname{Hom}(E, F)$ is invertible at all points off the zero section of T^*X .

Theorem 4.27 (Elliptic regularity). Suppose *D* is an elliptic differential operator of order ℓ . Then for any *k*, *D* extends to an operator acting on Sobolev spaces

$$D_k: L^2_k(E) \to L^2_{k-\ell}(E),$$

and we have:

- *D_k* is *Fredholm*, i.e. it has closed range, and finite-dimensional kernel and cokernel.
- ker D_k and coker D_k contain only smooth sections (so in particular they are independent of k.)
- The formal adjoint D^* is also elliptic, and Im D is the L^2 orthocomplement of ker D^* .

This should be thought of as a broad generalization of the fact that harmonic functions on a compact Riemannian manifold are smooth (in fact constant).

5 Moduli of bundles

In this section fix a compact Riemann surface (1-dimensional complex manifold) *C*, with a Kähler metric, of total area 1:

$$\int_C \omega_C = 1. \tag{5.1}$$

We are going to define a moduli space $\mathcal{N} = \mathcal{N}_{K,d}(C)$ which can be thought of in two ways:

• \mathcal{N} is a moduli space of unitary connections on complex vector bundles over *C*, of rank *K* and degree *d* over *C*, obeying a "harmonicity" condition on their curvature (Einstein connections). From this point of view we see naturally that \mathcal{N} is *symplectic*.

Moreover from this point of view there is a natural sense in which \mathcal{N} is independent of the complex structure of C: indeed we can identify it with a moduli space of representations of $\pi_1(C)$ (or an extension of $\pi_1(C)$) which depends only on the *topology* of C.

• *N* is a moduli space of holomorphic vector bundles over *C*, of rank *K* and degree *d*. From this point of view we see naturally that *N* is *complex*. Its complex structure does depend on the complex structure on *C*.

In fact the two structures are compatible and so N is naturally Kähler (where it is smooth).

Although we will not use this fact, $\mathcal{N}_{K,d}(C)$ is actually algebraic. For those who are algebraically inclined, a very nice introductory reference to this material is [32]. The notes [33] also appear very good.

5.1 Degree of vector bundles

Definition 5.1 (Degree of a vector bundle). The *degree* of a complex vector bundle *E* over *C* is

$$\deg E = c_1(E) \cdot [C] \in \mathbb{Z},\tag{5.2}$$

where $c_1(E) \in H^2(C, \mathbb{Z})$ denotes the first Chern class, and $[C] \in H_0(C, \mathbb{Z})$ the fundamental class.

Proposition 5.2 (Chern-Weil formula for degree).

$$\deg E = \frac{\mathrm{i}}{2\pi} \int_C \mathrm{Tr} \, F_D \tag{5.3}$$

where $F_D \in \Omega^2(\text{End } E)$ is the curvature of any connection *D* in *E*.

In practice Proposition 5.2 is what we will use. The only reason we do not adopt this as the definition is that it does not make immediately manifest the fact that deg E is an *integer*.

Exercise 5.1. Show that the vector bundle *E* and the line bundle det *E* have the same degree.

Another interpretation of the degree which is useful for developing intuition is:

Proposition 5.3 (Degree counts zeroes). Suppose *E* is a complex vector bundle over *C*, of rank *K*. Fix sections e_1, \ldots, e_K of *E* which are transverse in the sense that they give a trivialization of *E* away from finitely many points $z_i \in C$. Let $a_i \in \mathbb{Z}$ denote the winding number of the determinant det (e_1, \ldots, e_K) evaluated on a small circle around z_i (measured relative to a local trivialization of *E* which extends over z_i). Then

$$\deg E = \sum_{i} a_i. \tag{5.4}$$

In particular, if *E* is a holomorphic line bundle, choose a meromorphic section e of *E*, and then deg *E* is the number of zeroes of e minus the number of poles (counted with multiplicity.)
Exercise 5.2. Show that the line bundle $\mathcal{O}(n)$ over \mathbb{CP}^1 has degree *n*.

5.2 Moduli spaces by symplectic quotient

Proposition 5.4 (Classification of complex vector bundles). All rank *K* complex vector bundles over *C* of degree *d* are equivalent.

Proof. (From [32].) Rank *K* complex vector bundles over *C* up to equivalence correspond to homotopy classes of maps $C \to BU(K)$. Since *C* has dimension 2, by cellular approximation we can replace BU(K) by its 3-skeleton, which is \mathbb{CP}^1 ; and maps $C \to \mathbb{CP}^1$ are classified by their degree.

Thus, suppose we fix a single rank *K* complex vector bundle *E*, of degree *d*. Then every holomorphic vector bundle of rank *K* and degree *d* is equivalent to $(E, \bar{\partial})$ for some $\bar{\partial}$ -operator $\bar{\partial}$ on *E*. Let $\mathcal{A}^{\bar{\partial}}$ denote the space of all such $\bar{\partial}$. Then $\mathcal{A}^{\bar{\partial}}$ is an affine space modeled on the infinite-dimensional complex linear space $\Omega^{0,1}(\text{End } E)$.

The moduli space $\mathcal{N}_{K,d}(C)$, which we want to construct, could be thought of as \mathcal{A}^{∂} modulo equivalence — where now "equivalence" means the (infinite-dimensional, complex) group $\mathfrak{G}_{\mathbb{C}}$ of sections of GL(E), which acts on $\mathcal{A}^{\overline{\partial}}$ by conjugation:

$$D \mapsto D^g = (g^{-1})^* D = g D g^{-1}.$$
 (5.5)

So formally we want to say

$$\mathcal{N}_{K,d}(C) = \mathcal{A}^{\partial} / \mathfrak{G}_{\mathbb{C}}.$$
(5.6)

It is not so easy to make a "nice" space in this way. The difficulty comes from the fact that $\mathfrak{G}_{\mathbb{C}}$ is a complex group. (Of course $\mathfrak{G}_{\mathbb{C}}$ is also infinite-dimensional, but that's not really the problem!)

One neat trick for getting around this problem, explained in [34], is to think of $\mathfrak{G}_{\mathbb{C}}$ as the complexification of a group \mathfrak{G} whose action is better controlled, and then interpret (5.6) as a Kähler quotient (still infinite-dimensional.) For this purpose, let us fix a Hermitian metric *h* on *E*. After so doing, we have an infinite-dimensional real group \mathfrak{G} of *unitary* equivalences, i.e. sections of U(E). Let \mathcal{A}^h denote the space of all *h*-unitary connections on *E*. \mathcal{A}^h is an affine space, modeled on the infinite-dimensional *real* linear space $\Omega^1(\mathfrak{u}(E))$.

Proposition 5.5 (Unitary connections = holomorphic structures). $\mathcal{A}^{\partial} \simeq \mathcal{A}^{h}$.

Proof. Each $\bar{\partial} \in \mathcal{A}^{\partial}$ can be extended in a canonical way to a unitary connection $D \in \mathcal{A}^{h}$ (Chern connection), and conversely, each unitary connection $D \in \mathcal{A}^{h}$ can be restricted to an operator $\bar{\partial} \in \mathcal{A}^{\bar{\partial}}$.

The compatibility of these two pictures of the space is summarized in the next exercise:

Exercise 5.3. Show that the real linear space $\Omega^1(\mathfrak{u}(E))$ carries a natural complex structure *I*, and with this complex structure it is canonically identified with the complex linear space $\Omega^{0,1}(\operatorname{End} E)$. Moreover show that $I = \star$.

$$\omega(\dot{A}_1, \dot{A}_2) = -\int_C \operatorname{Tr}(\dot{A}_1 \wedge \dot{A}_2).$$
(5.7)

We would like to think of this as a symplectic form.

Strictly speaking, though, we have only defined the notion of "symplectic form" for Banach manifolds. So now is a good time to switch to that setting. We fix some k > 1(you can imagine that it is very big, say k = 100), and replace \mathcal{A}^h by the larger space \mathcal{A}^h_k of L^2_k connections.¹⁰ \mathcal{A}^h_k is a Banach manifold, in fact an affine space modeled on the Banach space $\Omega^1_k(\text{End } E) = L^2_k(T^* \otimes \text{End } E)$. Then (5.7) gives a symplectic structure on \mathcal{A}^h_k . From now on we will throw in the subscript k to pass to L^2_k versions of the various spaces of sections we need to consider. On a first reading, you might profit by ignoring the k's and pretending that we work always with C^{∞} objects.

Altogether we now have a single space $\mathcal{A}_k (\simeq \mathcal{A}_k^{\bar{\partial}} \simeq \mathcal{A}_k^h)$ with both symplectic and complex structures.

Exercise 5.4. Verify that the complex and symplectic structures on A_k are compatible, i.e. $\omega(I\dot{A}_1, I\dot{A}_2) = \omega(\dot{A}_1, \dot{A}_2)$.

Exercise 5.5. Verify that A_k is Kähler, and that the metric is

$$g(\dot{A}_1, \dot{A}_2) = -\int_C \operatorname{Tr}(\dot{A}_1 \wedge \star \dot{A}_2).$$
(5.8)

Moreover, A_k carries a natural action of \mathfrak{G}_{k+1} :

Proposition 5.6. For k > 1, the action of \mathfrak{G} on \mathcal{A}^h by conjugation,

$$D \mapsto D^g = g D g^{-1}, \qquad D \in \mathcal{A}_k^h.$$
 (5.9)

extends to an action of \mathfrak{G}_{k+1} on \mathcal{A}_k .

Proof. This follows from the general fact that any differential operator of order 1 mapping smooth sections $E \to F$ extends to a map on Sobolev spaces $L^2_{k+1}(E) \to L^2_k(F)$. [plus Sobolev multiplication!]

Proposition 5.7. The Lie algebra of \mathfrak{G}_k is $\Omega_k^0(\text{End } E)$.

The action of \mathfrak{G}_{k+1} on \mathcal{A}_k preserves the Kähler structure. We could check this directly, but it also follows from:

Proposition 5.8 (Moment map for \mathfrak{G}_{k+1} **action on** \mathcal{A}_k **).** The action of \mathfrak{G}_{k+1} on \mathcal{A}_k admits a moment map, given by

$$\mu(D) = -F_D - 2\pi i \frac{d}{K} \mathbf{1} \omega_C \tag{5.10}$$

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¹⁰We have not formally defined " L_k^2 connection," but hope to get by just with the remark that it means a connection of the form $D + \alpha$, where D is a smooth connection and $\alpha \in L_k^2(\mathfrak{u}(E))$.

where ω_C denotes the Kähler form on *C*. More precisely, this means: suppose given an element $Z \in \text{Lie } \mathfrak{G}_{k+1} = \Omega^0_{k+1}(\mathfrak{u}(E))$, then we have

$$\mu_Z(D) = -\int_C \operatorname{Tr}(ZF_D) - 2\pi i \frac{d}{K} \int_C \omega_C \operatorname{Tr}(Z).$$
(5.11)

Proof. The key facts are the action of $Z \in \text{Lie } \mathfrak{G}_{k+1} = \Omega^0_{k+1}(\mathfrak{u}(E))$ by infinitesimal gauge transformation,

$$\rho(Z) = -DZ \tag{5.12}$$

and the differential of μ_Z , acting on a tangent vector $\dot{A} \in \Omega^1_k(\mathfrak{u}(E)) = T\mathcal{A}^h_k$:

$$d\mu_Z \cdot \dot{A} = -\int_C \operatorname{Tr}(Z \, D \dot{A}). \tag{5.13}$$

We compute

$$\omega(\rho(Z), \dot{A}) = \int_C \operatorname{Tr}(DZ \wedge \dot{A})$$
(5.14)

$$= -\int_{C} \operatorname{Tr}(Z \, D\dot{A}) \tag{5.15}$$

$$= \mathrm{d}\mu_Z \cdot \dot{A}. \tag{5.16}$$

Exercise 5.6. Verify (5.12) and (5.13).

Exercise 5.7. Check that μ given by (5.10) is \mathfrak{G} -equivariant.

Proposition 5.8 says in particular that $\mu^{-1}(0)$ is the subset of \mathcal{A}_k^h consisting of connections with

$$F_D = -2\pi i \frac{d}{K} \mathbf{1} \omega_C. \tag{5.17}$$

Let us put this funny-looking formula in a more general context:

Definition 5.9 (Einstein connection). If *X* is a Riemannian manifold and *E* a complex vector bundle over *X* with connection *D*, we say *D* is *Einstein* if the curvature $F_D \in \Omega^2(\text{End } E)$ is

$$F_D = i\mathbf{1}\alpha \tag{5.18}$$

where $\alpha \in \Omega^2(X)$ is harmonic.

Exercise 5.8. Show that (5.17) is equivalent to the condition that *D* is an Einstein connection over the Kähler Riemann surface *C*. (Hint: see Example 2.44.)

In the important special case d = 0, (5.17) just says that *D* is a *flat* connection. For $d \neq 0$, (5.17) says at least that *D* is *projectively* flat, i.e. it would descend to a flat connection on the bundle $\mathbb{P}(E)$ of projective spaces, and moreover the remaining central curvature is completely fixed.

Following our plan, we want to take the symplectic quotient $A_k //\mathfrak{G}_{k+1} = \mu^{-1}(0)/\mathfrak{G}_{k+1}$. As a technical preliminary we need to understand the extent to which \mathfrak{G}_{k+1} acts freely and properly on $\mu^{-1}(0)$.

Definition 5.10 (Irreducible connections). We call a unitary connection *D* in *E irreducible* if there does not exist any subbundle $E' \subset E$ such that *D* preserves E'.

Proposition 5.11 (Gauge group acts almost freely on irreducible connections). If $g \in \mathfrak{G}_{k+1}$ and *D* is an irreducible connection, then $D^g = D$ if and only if *g* acts on *E* by multiplication by a constant scalar.

Proof. If *g* is multiplication by a constant scalar then we certainly have $D^g = D$. Conversely, suppose $D^g = D$. Then *g* is covariantly constant for the connection in End *E* induced by *D*, which implies that *g* is determined by its value g(p) for any $p \in C$, and that g(p) commutes with the holonomy of *D* around any loop based at *p*. Thus the eigenspaces of g(p) are preserved by parallel transport. This contradicts the irreducibility of *D* unless g(p) is a scalar. Finally the covariant constancy of *g* shows that *g* is a constant scalar.

Letting $\mathcal{A}_k^s \subset \mathcal{A}_k$ be the set of irreducible connections, and defining the "effective gauge group"

$$\mathfrak{G}_{k+1}^{\text{eff}} = \mathfrak{G}_{k+1}/U(1) \tag{5.19}$$

we could rephrase Proposition 5.11 as saying that $\mathfrak{G}_{k+1}^{\text{eff}}$ acts freely on \mathcal{A}_k^s .

Exercise 5.9. Show that the moment map μ given in (5.10) descends to give a moment map for $\mathfrak{G}_{k+1}^{\text{eff}}$.

Proposition 5.12 (Unitary gauge group acts properly). The action of \mathfrak{G}_{k+1} on \mathcal{A}_k is proper.

Proof. ([35] Proposition 7.1.14) What we need to show is that if we have a sequence (g_n, D_n) in $\mathfrak{G}_{k+1} \times \mathcal{A}_k$ such that the D_n converge to some D_∞ , and the gauge-transformed connections $D_n^{g_n}$ converge to some D^* , then after passing to a subsequence, the g_n also converge to some g_∞ with $D_\infty^{g_\infty} = D^*$. The idea is to take a subsequence of the g_n which converges in a single fiber E_x (this exists because the unitary group $U(E_x)$ is compact). Then the condition $D_\infty^{g_\infty} = D^*$ is a differential equation on g_∞ which determines what it must be in all other fibers, and the fact that $D_n^{g_n} \to D^*$ forces $g_n \to g_\infty$.

Encouraged by Proposition 5.11 and Proposition 5.12, let $A^s \subset A$ be the subset of irreducible unitary connections, and define:

Definition 5.13 (Moduli spaces of Einstein connections).

$$\mathcal{N}_{K,d}^{s}(C) = \mathcal{A}_{k}^{s} / / \mathfrak{G}_{k+1}^{\text{eff}}, \qquad \mathcal{N}_{K,d}(C) = \mathcal{A}_{k} / / \mathfrak{G}_{k+1}^{\text{eff}}, \tag{5.20}$$

where we use (5.10) to define μ .

These are both infinite-dimensional quotients. Nevertheless they still make sense, at least as topological spaces. Our experience with finite-dimensional quotients would suggest that $\mathcal{N}_{K,d}^s(C)$ should be a Kähler manifold, and this is indeed true, as we will show momentarily.

First we look a bit more at the *formal* picture. The tangent space to $\mathcal{N}_{K,d}^{s}(C)$ at a given Einstein connection D should be the kernel of $d\mu$ modulo $\rho(\text{Lie}(\mathfrak{G}^{\text{eff}}))$. A nice way to say this is that the tangent space "should be" the cohomology H^{1} of the complex

$$0 \to \Omega^0(\mathfrak{u}(E)) \xrightarrow{D} \Omega^1(\mathfrak{u}(E)) \xrightarrow{D} \Omega^2(\mathfrak{u}(E)) \to 0$$
(5.21)

(It's indeed a complex: $D^2 = 0$ acting on sections of u(E), since $F_D \sim 1$ and hence is central. Of course this is as it should be since infinitesimal gauge transformations should be tangent to $\mu^{-1}(0)$.)

To get some control over this infinite-dimensional problem, we first think of it in a slightly different way: instead of dividing out by the image of $D : \Omega^0 \to \Omega^1$ we impose the gauge-fixing equation $D^* = 0$, where D^* is the formal adjoint of D (the idea is that the kernel of D^* should be an L^2 orthocomplement to the image of D.) This leads us to consider the operator

$$\hat{D} = D \oplus D^* : \Omega^1(\mathfrak{u}(E)) \to \Omega^2(\mathfrak{u}(E)) \oplus \Omega^0(\mathfrak{u}(E)).$$
(5.22)

The good news then is that this operator is *elliptic* in the sense of subsection 4.6, just as for the example of $d + d^*$ discussed there (indeed the symbol of \hat{D} is just K^2 copies of the symbol of $d + d^*$.) Thus we will be able to use the elliptic regularity theorem, Theorem 4.27.

We will need the following preliminary:

Lemma 5.14 (Almost-vanishing for the gauge complex). If $D \in \mathcal{A}_{k'}^s$ then coker $D \subset \Omega^2(\mathfrak{u}(E))$ and coker $D^* \subset \Omega^0(\mathfrak{u}(E))$ are both 1-dimensional.

(i.e. both H^0 and H^2 of the complex (5.21) are 1-dimensional.)

Proof. By elliptic regularity we can identify coker $\hat{D} = \ker \hat{D}^*$, where

$$\hat{D}^* = D \oplus D^* : \Omega^0(\mathfrak{u}(E)) \oplus \Omega^2(\mathfrak{u}(E)) \to \Omega^1(\mathfrak{u}(E)).$$
(5.23)

Since Im *D* and Im D^* are L^2 -orthogonal in $\Omega^1(\mathfrak{u}(E))$ we will have ker $\hat{D}^* = \ker D \oplus \ker D^*$. Any $\alpha \in \ker D \subset \Omega^0(\mathfrak{u}(E))$ must be a constant multiple of **1**, since *D* is an irreducible connection: this gives the desired statement for $\Omega^0(\mathfrak{u}(E))$. For $\Omega^2(\mathfrak{u}(E))$ we essentially use Poincare duality: we have

$$D^* = -\star D\star \tag{5.24}$$

so any $\alpha \in \ker D^* \subset \Omega^2(\mathfrak{u}(E))$ has $D \star \alpha = 0$; said otherwise, $\ker D^* \subset \Omega^2(\mathfrak{u}(E))$ is just spanned by $\star(\mathbf{1})$.

Now we are ready to prove that \mathcal{N} is well behaved:

Theorem 5.15 ($\mathcal{N}_{K,d}^s(C)$ is Kähler). $\mathcal{N}_{K,d}^s(C)$ is a Kähler manifold.

Proof. We want to apply Theorem 4.22 to see that $\mathcal{N}_{K,d}(C)$ is symplectic. For this we need to verify the hypotheses. First, we want to show that $\mu^{-1}(0) \subset \mathcal{A}_k^s$ is a submanifold. This follows from the Banach space inverse function theorem, applied to the map

$$\mu: \mathcal{A}_k^s \to \Omega^2(\mathfrak{u}(E))/(\star \mathbf{1}), \tag{5.25}$$

which is a submersion according to Lemma 5.14. Second, we want to show that the tangent spaces to \mathfrak{G} -orbits are closed and complemented. $D\Omega^0 \oplus D^*\Omega^2$ is the image of a Fredholm operator and thus closed, and the two summands are L^2 -orthogonal, so that they are both separately closed; thus $D\Omega^0$ is closed, and complemented by $D^*\Omega^2 \oplus \operatorname{coker} \hat{D}$, as desired. Third, we want to show that the tangent spaces to \mathfrak{G} -orbits have $(T^{\perp_{\omega}})^{\perp_{\omega}} = T$; for this see an argument in [35].

The proof that $\mathcal{N}_{K,d}(C)$ is not only symplectic but actually Kähler follows exactly the lines of the finite-dimensional case. See [31] for an account.

Theorem 5.16 (Dimension of $\mathcal{N}_{K,d}^{s}(C)$ **).** dim_C $\mathcal{N}_{K,d}^{s}(C) = (g-1)K^{2} + 1$ where *g* is the genus of *C*.

Proof. We will compute the index

$$\operatorname{ind} \hat{D} = \dim \ker \hat{D} - \dim \operatorname{coker} \hat{D}.$$
(5.26)

Very generally, for any elliptic operator, the index can be computed using the Atiyah-Singer index theorem. When the degree d = 0, though, we actually don't need to know the index theorem: all we need to know is one of the basic preliminaries in some proofs of that theorem, namely that ind \hat{D} only depends on the symbol of \hat{D} . Indeed, for the operator \hat{D} we have, the symbol is the same as that for the trivial connection on the trivial rank *K* bundle (in local coordinates D = d + A, and the *A* part is lower-order.) But for the trivial connection we just get K^2 copies of the de Rham complex, whose index is known by de Rham's theorem (Theorem 2.37): it is $-\chi(C) = 2g - 2$. Thus we get

ind
$$\hat{D} = (2g - 2)K^2$$
. (5.27)

On the other hand Lemma 5.14 says

$$\dim \operatorname{coker} \hat{D} = 2. \tag{5.28}$$

Combining these gives the desired result, when d = 0. For general d we need to know a bit more about the structure of the index formula. [maybe better to defer to later, identify tangent space with $H^1(\text{End } E)$ and use Riemann-Roch]

The formula in Theorem 5.16 looks a bit suspicious for g < 2. But in these cases the only way to have an irreducible Einstein connection is to have K = 1. In that case we get dimension 0 for g = 0 and dimension 1 for g = 1, which is indeed correct — see Example 5.17 below.

5.3 Connections versus representations

Exercise 5.10. Show that $\mathcal{N}_{K,0}^{s}(C)$, considered as a set, has a canonical bijection to the set of all irreducible representations $\pi_1(C) \to U(K)$ modulo equivalence. (Hint: this is actually the shadow of a stronger statement, namely that the category of flat connections is equivalent to the category of representations. So the real guts of the question is to construct functors in both directions. To go from a flat connection, use the universal cover

of *C*.) Similarly show that $\mathcal{N}_{K,0}(C)$ has a canonical bijection to the set of all representations $\pi_1(C) \to U(K)$ modulo equivalence. Finally show (or at least sketch) that these bijections are actually homeomorphisms.

Exercise 5.11. Suppose *C* is a torus and K > 1. Show that $\mathcal{N}_{K,0}^{s}(C)$ is empty, and describe $\mathcal{N}_{K,0}(C)$. (Hint: use the result of Exercise 5.10.)

Now how about the case of general d, where we consider connections obeying (5.17) instead of just flat connections? We remark that ω_C is an arbitrary positive 2-form on C with total integral 1. At least formally we can imagine taking ω_C to be more and more concentrated at a single point z_0 , until in the limit we get $\omega_C = \delta(z_0)$. In this limit the connections D obeying (5.17) are becoming flat away from z_0 , but developing a singularity at z_0 , with the holonomy around z_0 of finite order, given by multiplication by the root of unity $e^{-2\pi i d/K}$.



This motivates the following:

Exercise 5.12. Present *C* as a polygon with edges identified in the standard way.



This gives a description of $\pi_1(C)$ by generators $(A_1, B_1, A_2, B_2, ..., A_g, B_g)$ subject to the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = \mathbf{1}.$$
 (5.29)

Consider an *extension*

$$1 \to \mathbb{Z}/K\mathbb{Z} \to \tilde{\pi}_1(C) \to \pi_1(C) \to 1$$
(5.30)

obtained by deforming this relation to

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = Z$$
(5.31)

where *Z* is a new generator, and adding the relation

$$Z^K = \mathbf{1}.\tag{5.32}$$

Show that $\mathcal{N}_{K,d}(C)$ is the set of equivalence classes of representations $\tilde{\pi}_1(C) \to U(K)$ for which *Z* acts by $e^{-2\pi i d/K}$ **1**. (Hint: what can you say about the holonomy around

the boundary of the polygon?) Similarly show that $\mathcal{N}_{K,d}^{s}(C)$ is the subset of equivalence classes of *irreducible* representations.

Exercise 5.13. Use the result of Exercise 5.12 to describe $\mathcal{N}_{2,1}(C)$ when *C* has genus 1. (This amounts to classifying pairs of matrices *A*, *B* in *U*(2) such that $ABA^{-1}B^{-1} = -1$, up to simultaneous conjugation. It might be useful to start with the case where *A*, *B* \in *SU*(2); using the 2 : 1 projection *SU*(2) \rightarrow *SO*(3) this amounts to looking for pairs of matrices $\overline{A}, \overline{B} \in SO(3)$ which commute, and yet do not belong to a common maximal torus $SO(2) \subset SO(3)$.)

5.4 The case of line bundles

Example 5.17 (Jacobians). The special case K = 1 is very concrete. In light of Proposition 5.23 below, all degrees are essentially the same, so let us consider d = 0. Then we introduce the notation

$$\operatorname{Jac}(C) = \mathcal{N}_{1,0}^{s}(C), \tag{5.33}$$

and call Jac(C) the *Jacobian* of *C*. It is the space of flat connections on the trivial line bundle over *C*, modulo gauge equivalence. Taking holonomies around loops gives a map

$$Jac(C) \to Hom(\pi_1(C), U(1)) = H^1(C, U(1)).$$
 (5.34)

In fact this map is an isomorphism, as follows from Exercise 5.10 above.

Thus Jac(C) is the compact torus $H^1(C, U(1))$, of real dimension 2*g*. In particular, using this description we see that as a smooth manifold Jac(C) is *independent* of the chosen metric on *C*. To describe it even more concretely, choose a *marking* of *C*, i.e. a basis

$$\{A_1, \dots, A_g, B_1, \dots, B_g\}, \qquad A_i, B_i \in H_1(C, \mathbb{Z}),$$
(5.35)

such that

$$A_i \cap A_j = 0, \quad B_i \cap B_j = 0, \quad A_i \cap B_j = \delta_{ij}.$$

$$(5.36)$$

Then defining

$$\mathcal{X}_{A_i} = \operatorname{Hol}_{A_i} D, \qquad \mathcal{X}_{B_i} = \operatorname{Hol}_{B_i} D,$$
(5.37)

the $(\mathcal{X}_{A_i}, \mathcal{X}_{B_i})$ give an explicit diffeomorphism Jac $C \simeq U(1)^{2g}$.

Now, from $Jac(C) \simeq H^1(C, U(1))$ we get an identification

$$T_x \operatorname{Jac}(C) \simeq \mathrm{i} H^1(C, \mathbb{R}). \tag{5.38}$$

On the other hand, our realization of $T_x \operatorname{Jac}(C)$ via the symplectic quotient identifies it as $\operatorname{ker}(d \oplus d^*)$, i.e. the space of harmonic forms,

$$T_x \operatorname{Jac}(C) \simeq \mathrm{i}\mathcal{H}^1(C). \tag{5.39}$$

The relation between (5.38) and (5.39) is the standard one given by the Hodge theorem, Theorem 2.43: in each cohomology class we choose the unique harmonic representative.

Let us now describe the symplectic and Kähler structures on Jac(C). As usual for symplectic quotients, the symplectic form ω is just obtained by restriction of the original symplectic form (5.7) from A to $i\mathcal{H}^1(C)$,

$$\omega(\dot{A}_1, \dot{A}_2) = -\int_C \dot{A}_1 \wedge \dot{A}_2. \tag{5.40}$$

This pairing in fact depends only on the cohomology classes of α and β , not on the actual harmonic forms chosen to represent them! Said otherwise, this formula represents an intrinsically defined symplectic form on $H^1(C, U(1))$. We could write it even more concretely: in the coordinates \mathcal{X}_{A_i} , \mathcal{X}_{B_i} we introduced above,

$$\omega = -\sum_{i} \frac{\mathrm{d}\mathcal{X}_{A_{i}}}{\mathcal{X}_{A_{i}}} \wedge \frac{\mathrm{d}\mathcal{X}_{B_{i}}}{\mathcal{X}_{B_{i}}}.$$
(5.41)

In particular this form does not depend on the complex structure we chose on *C*. Thus, as a symplectic manifold, Jac(C) does not depend on the complex structure of *C*.



On the other hand the *complex* structure of Jac(C) definitely does depend on the complex structure of C: it is the one inducing the Hodge decomposition

$$H^{1}(C,\mathbb{C}) = H^{0,1}(C) \oplus H^{1,0}(C).$$
 (5.42)

Thus the Kähler metric on Jac(C) also depends on the complex structure of *C*. Explicitly, we can write the Kähler metric as

$$g(\dot{A}_1, \dot{A}_2) = -\int \dot{A}_1 \wedge \star \dot{A}_2$$
 (5.43)

where \dot{A}_1 , $\dot{A}_2 \in i\mathcal{H}^1(C)$. Loosely speaking, as we deform the complex curve *C*, the "angles" of the flat complex torus Jac(*C*) change.

Exercise 5.14. Prove the formula (5.41) for the symplectic form on Jac(*C*).

Exercise 5.15. Prove that the complex structure on Jac(C) induced by the Kähler quotient construction is the one claimed above.

Example 5.18 (Jacobians of genus 1 **curves).** Suppose that *C* has genus 1 and complex modulus τ . In this case we can describe all this in a completely explicit way. We fix coordinates (x, y) on *C* with $x \in \mathbb{R}/\mathbb{Z}$, $y \in \mathbb{R}/\mathbb{Z}$. Then the complex coordinate on *C* is

$$z = x + \tau y \in \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}).$$
(5.44)

A general flat U(1)-connection is gauge equivalent to one of the form D = d + A with

$$A = \mathrm{i}\theta_x \,\mathrm{d}x + \mathrm{i}\theta_y \,\mathrm{d}y \tag{5.45}$$

$$= (2\operatorname{Im}\tau)^{-1}(\bar{\alpha}\,\mathrm{d}z - \alpha\,\mathrm{d}\bar{z}) \tag{5.46}$$

where

$$\alpha = \theta_y - \tau \theta_x. \tag{5.47}$$

Note that

$$\bar{\partial}_D = \bar{\partial} + (-2(\operatorname{Im}\tau)^{-1}\alpha)d\bar{z}$$
(5.48)

so α is a holomorphic coordinate on Jac *C*. Thus we have two coordinate systems on Jac *C*: one real coordinate system

$$(\theta_x, \theta_y) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \tag{5.49}$$

the other a holomorphic coordinate

$$\alpha \in \mathbb{C} / \left(2\pi \mathbb{Z} \oplus 2\pi \tau \mathbb{Z} \right). \tag{5.50}$$

Using the holomorphic coordinate α we see that Jac *C* is biholomorphic to *C* itself. The Kähler form is

$$\omega = \mathrm{d}\theta_x \wedge \mathrm{d}\theta_y = \frac{\mathrm{i}}{2\,\mathrm{Im}\,\tau} \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}. \tag{5.51}$$

(Comparing with our notation for the general case, we have $e^{i\theta_x} = \mathcal{X}_A$, $e^{i\theta_y} = \mathcal{X}_B$.)

Exercise 5.16. Write an explicit formula for the Kähler metric on Jac(C) when *C* has genus $g \ge 1$. (You will need — or rediscover — the notion of *period matrix* of *C*, generalizing the τ which appeared in Example 5.18.)

Lemma 5.19 (Existence of Einstein connections on line bundles). Suppose \mathcal{L} is a holomorphic line bundle over C. Then there exists a Hermitian metric h on \mathcal{L} such that the Chern connection D_h is Einstein. The metric h with this property is unique up to scalar multiple.

Proof. First choose some arbitrary *h* on \mathcal{L} . The curvature of the Chern connection is then $F_{D_h} \in \Omega^2(C)$, which generally is not harmonic. We want to improve *h* to some *h'* so that $iF_{D_{h'}} \in \mathcal{H}^2(C)$.

We write $h' = e^f h$ for some (yet unknown) $f : C \to \mathbb{R}$. Then we have, using (2.39),

$$F_{D_{h'}} = F_{D_h} + \bar{\partial}\partial f. \tag{5.52}$$

Now we can apply the abelian Hodge theorem (Theorem 2.43) to the cohomology class $[iF_{D_h}] \in H^2_{dR}(C)$. It says that there exists a unique $\beta \in \Omega^1(C)$ such that

$$\mathbf{i}F_{D_h} + \mathbf{d}\beta \in \mathcal{H}^2(C). \tag{5.53}$$

But then the $\partial \partial$ -lemma (Lemma 2.49) says that $d\beta$ can also be written as $i\partial \partial f$ for some real function f. This gives the desired f. It is unique up to shifts by a solution of $\partial \bar{\partial} f = 0$, but those are just harmonic functions on C, i.e. constants.

There is another way of thinking about this result. We return to our original context where (E, h) is a *fixed* Hermitian vector bundle of rank *K*, and consider the case K = 1.

Corollary 5.20 (Gauge-theoretic stability of line bundles). Suppose $\bar{\partial} \in \mathcal{A}^{\partial}$. Then the $\mathfrak{G}_{\mathbb{C}}$ -orbit of $\bar{\partial}$ intersects $\mu^{-1}(0)$ precisely in a \mathfrak{G} -orbit.

Proof. By Lemma 5.19 the holomorphic line bundle $(E, \bar{\partial})$ admits a Hermitian metric h', unique up to scalar multiple, such that the Chern connection for $(E, \bar{\partial}, h')$ is Einstein. But since all Hermitian metrics on E are equivalent, there exists $g \in \mathfrak{G}_{\mathbb{C}}$ such that $g^*h' = h$; concretely we can take g defined by $gv = \sqrt{h/h'}v$. Then the Chern connection for $(E, \bar{\partial}^{g^{-1}}, h)$ is Einstein, as desired.

This theorem says that, for rank K = 1, our original strategy for constructing a moduli space of holomorphic vector bundles works perfectly: we really do have

$$\mathcal{A}^{\partial}/\mathfrak{G}_{\mathbb{C}} = \mathcal{A}^{h}/\!/\mathfrak{G} = \operatorname{Jac} C.$$
(5.54)

5.5 Good properties of $\mathcal{N}_{K,d}(C)$

We have already seen that when K = 1, $\mathcal{N}_{K,0}(C) = \text{Jac } C$ is a compact Kähler torus. The behavior is almost as good for general K and d:

Corollary 5.21 (Compactness of $\mathcal{N}_{K,d}(C)$ **).** $\mathcal{N}_{K,d}(C)$ is compact.

Proof. This follows directly from the result of Exercise 5.12 since a representation of a finite extension of $\pi_1(C)$ into U(K) is determined by a finite collection of matrices in U(K), and U(K) is compact.

Exercise 5.17. Show that if (d, K) = 1 then $\mathcal{N}_{K,d}(C) = \mathcal{N}_{K,d}^{s}(C)$.

Corollary 5.22 ($\mathcal{N}_{K,d}(C)$ is nice when (d, K) = 1). When (d, K) = 1, $\mathcal{N}_{K,d}(C)$ is a compact Kähler manifold.

Proof. This is just the combination of Theorem 5.15, Corollary 5.21, and Exercise 5.17.

What does $\mathcal{N}_{K,d}(C)$ look like concretely, when K > 1? Here is a preliminary observation:

Proposition 5.23 (Tensorization with line bundle gives an isometry). Suppose \mathcal{L} is a holomorphic line bundle on *C*, of degree *d*'. Then the map $E \to E \otimes \mathcal{L}$ gives a holomorphic isometry of Kähler manifolds

$$\mathcal{N}^{s}_{K,d}(C) \xrightarrow{\sim} \mathcal{N}^{s}_{K,d+Kd'}(C).$$
 (5.55)

Proof. By Lemma 5.19 we can choose a metric inducing an Einstein connection $D_{\mathcal{L}}$ on \mathcal{L} . Now suppose given an Einstein connection D in E. Then the connection $D' = D \otimes 1 + 1 \otimes D_{\mathcal{L}}$ in $E' = E \otimes \mathcal{L}$ has curvature

$$F_{D'} = F_D + F_{D_c} \mathbf{1}$$
(5.56)

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(note in writing this equation we used the fact that $\text{End}(E') \simeq \text{End}(E)$). Thus D' is also Einstein. This gives the desired map.

As a consequence of Proposition 5.23, one can say loosely that $\mathcal{N}_{K,d}^s(C)$ "depends on d only modulo K" (although one does not quite get a *canonical* isomorphism.) Moreover, tensorization by degree *zero* bundles keeps us within the same moduli space, i.e. $\mathcal{N}_{K,d}^s(C)$ has an action of Jac(C) by isometries. So crudely speaking, $\mathcal{N}_{K,d}^s(C)$ has g complex directions in which it looks like the compact complex torus Jac(C), while the other $(g-1)(K^2-1)$ directions are more interesting.

Exercise 5.18. Verify (5.56).

Exercise 5.19. Verify that the map in Proposition 5.23 is a holomorphic isometry of Kähler manifolds.

5.6 Stable vector bundles

It still remains to explain what $\mathcal{N}_{K,d}(C)$ and $\mathcal{N}_{K,d}^s(C)$ precisely mean in terms of our original aim of studying holomorphic vector bundles. For K > 1 the situation is a bit different than for K = 1: not every holomorphic vector bundle admits an Einstein connection, so not every holomorphic vector bundle will correspond to a point of $\mathcal{N}_{K,d}(C)$. In this section we introduce the class of bundles which do.

Definition 5.24 (Slope of a vector bundle). The *slope* of a holomorphic vector bundle *E* over *C* is

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E}.$$
(5.57)

Definition 5.25 (Stable holomorphic vector bundle). A holomorphic vector bundle *E* over *C* is called:

- *stable* if, for every holomorphic subbundle $E' \subset E$, we have $\mu(E') < \mu(E)$,
- *polystable* if *E* is a direct sum of stable bundles of the same slope,
- *semistable* if, for every holomorphic subbundle $E' \subset E$, we have $\mu(E') \leq \mu(E)$.

We have inclusions

$$\{\text{stable}\} \subset \{\text{polystable}\} \subset \{\text{semistable}\}. \tag{5.58}$$

In favorable cases this hierarchy collapses:

Exercise 5.20. Show that if (K, d) = 1 then every semistable bundle of rank *K* and degree *d* is stable.

Example 5.26 (Stability for bundles over \mathbb{CP}^1 **).** Grothendieck's lemma says that all holomorphic vector bundles over \mathbb{CP}^1 are direct sums of line bundles, $E = \bigoplus_{i=1}^{K} \mathcal{O}(n_i)$, and

moreover that two such sums are isomorphic only if they include the same n_i . The slope of such a bundle is the average of the n_i ,

$$\mu(E) = \frac{\sum n_i}{K}.$$
(5.59)

Thus *E* is unstable unless all n_i are equal; if all n_i are equal it is polystable; it is stable only if K = 1.

Example 5.27 (Instability of $\mathcal{O}(-1) \oplus \mathcal{O}(1)$). What does it mean to say that the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ is unstable? One way to think of it is to consider a certain 1-parameter family of vector bundles E_t , obtained as *extensions*

$$0 \to \mathcal{O}(-1) \to E_t \to \mathcal{O}(1) \to 0, \tag{5.60}$$

parameterized by classes $t \in H^1(\mathcal{O}(-2)) \simeq \mathbb{C}$. Concretely what this means is that we take the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ and modify its transition function on the overlap between north-pole and south-pole patches to

$$\begin{pmatrix} z^{-1} & t \\ 0 & z \end{pmatrix}.$$
 (5.61)

For $t \neq 0$ we have $E_t \simeq \mathcal{O}(0) \oplus \mathcal{O}(0)$ while for t = 0 instead $E_t \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$. Thus we have the unstable bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ precisely at t = 0, which under the slightest perturbation "decays" to the polystable bundle $\mathcal{O}(0) \oplus \mathcal{O}(0)$.

The existence of this family means that if we naively try to construct a moduli space of *all* holomorphic bundles over \mathbb{CP}^1 up to equivalence, without imposing any stability condition, then any open set around the unstable bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ necessarily contains $\mathcal{O}(0) \oplus \mathcal{O}(0)$, i.e. the space would have to be non-Hausdorff.

Exercise 5.21. Check that indeed for $t \neq 0$ we have $E_t \simeq \mathcal{O}(0) \oplus \mathcal{O}(0)$. Hint: use the fact that the transition matrix can be factorized into one piece polynomial in z and one piece polynomial in z^{-1} ("Birkhoff factorization"),

$$\begin{pmatrix} z^{-1} & t \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & t^{-1}z \end{pmatrix} \begin{pmatrix} -t^{-1} & 0 \\ z^{-1} & t \end{pmatrix}.$$
 (5.62)

[need to briefly introduce language of coherent sheaves here, locally free sheaves, their degrees and ranks, additivity in exact sequences, characterization of subbundles by fact that quotient is torsion-free, fact that sheaf kernel of bundle map is a bundle]

Lemma 5.28 (Saturation of subsheaves of bundles over curves). If *E* is a holomorphic vector bundle and $S \subset E$ a subsheaf, then there exists a subbundle $F \subset E$ with rank F = rank *S* and deg $F \ge \deg S$.

Proof. Let *F* be the kernel of the projection $E \to (E/S)/T$, where *T* is the torsion subsheaf of *E*/*S*; then *F* is a subsheaf of *E* and *E*/*F* is torsion-free, which means *F* is a subbundle of *E*, as desired.

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We will use Lemma 5.28 below for subsheaves which arise as the images of holomorphic bundle maps. (The image sheaf of a map $E \to E'$ is a subsheaf of the locally free sheaf of sections of E', so it is itself a locally free sheaf, and thus it is the space of sections of some bundle, but nevertheless it is not generally a subbundle of E'; for instance, over \mathbb{CP}^1 , the image of a nontrivial map $\mathcal{O} \to \mathcal{O}(1)$ is the subsheaf of $\mathcal{O}(1)$ consisting of all sections vanishing at a point. This subsheaf is isomorphic to the sheaf of sections of \mathcal{O} , but \mathcal{O} is not a subbundle of $\mathcal{O}(1)$.)

Corollary 5.29 (Alternative characterization of stability). In Definition 5.25 we may replace "subbundle of *E*" by "subsheaf $S \subset E$ with rank S < rank E."

Proposition 5.30 (Stable bundles are simple). If *E* and *E'* are both stable holomorphic vector bundles with the same rank and degree, then Hom(E, E') is 1-dimensional if $E \simeq E'$, and trivial otherwise.

Proof. [32] If $E \not\simeq E'$ and Hom(E, E') is nontrivial, *or* dim Hom(E, E') > 1, then there exists some $\varphi : E \to E'$ such that $\varphi \neq 0$ and φ is not an isomorphism. Since $\varphi \neq 0$ we must have rank ker $\varphi < \text{rank } E$. Similarly, since φ is not an isomorphism, we must have rank Im $\varphi < \text{rank } E'$ (else det φ would be a nonzero map det $E \to \text{det } E'$, which would have to be an isomorphism since deg E = deg E'.) But we have deg ker $\varphi + \text{deg Im } \varphi = \text{deg } E$ and rank ker $\varphi + \text{rank Im } \varphi = \text{rank } E$ from which it follows that either $\mu(\text{ker } \varphi) \geq \mu(E)$ or $\mu(\text{Im } \varphi) \geq \mu(E) = \mu(E')$. This contradicts the stability of E and E'.

Finally we remark on one fine point. As we will see below, the objects appearing in $\mathcal{A}^h / \mathcal{O}$ are stable or at least polystable bundles, not general semistable ones. On the other hand, the moduli space we are constructing is often described as the moduli space of semistable vector bundles. To explain the reason for this apparent difference we first state:

Proposition 5.31 (Existence of Jordan-Hölder filtration). If $(E, \bar{\partial})$ is any holomorphic vector bundle, there exists a filtration by holomorphic subbundles,

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E, \tag{5.63}$$

such that all $\mu(E_i) = \mu(E)$ and each quotient E_i/E_{i-1} is stable. The filtration need not be unique, but the induced bundle

$$\operatorname{gr} E = \bigoplus_{i} E_i / E_{i-1} \tag{5.64}$$

is uniquely determined up to equivalence.

If *E* is stable then the Jordan-Hölder filtration has just one step, and gr E = E. More generally we have:

Proposition 5.32 (Associated graded to a semistable bundle is polystable). If *E* is semistable, then gr *E* is polystable.

Definition 5.33 (*S***-equivalence).** If *E* and *E*' are semistable holomorphic vector bundles over *C*, we say *E* and *E*' are *S*-equivalent if and only if gr *E* and gr *E*' are equivalent.

Thus in each *S*-equivalence class there is a unique polystable representative up to ordinary equivalence. It follows that the set of semistable bundles up to *S*-equivalence is the same as the set of polystable bundles up to ordinary equivalence.

5.7 Gauge-theoretic meaning of stability

Now we are ready to deal with the question of whether indeed

$$\mathcal{A}^{\bar{\partial}}/\mathfrak{G}_{\mathbb{C}} = \mathcal{A}^{h}/\!/\mathfrak{G}.$$
(5.65)

Theorem 5.35 below says that this will be true after deleting unstable bundles from \mathcal{A}^{∂} . We need a preliminary:

Theorem 5.34 (Weak Uhlenbeck compactness). Suppose that $(D_n)_{n=1}^{\infty}$ is a sequence in \mathcal{A}_1^h , with $||F(D_n)||$ bounded. Then after passing to a subsequence, there exist $g_n \in \mathfrak{G}_2$ such that $D_n^{g_n}$ converge in the weak topology on \mathcal{A}_1^h .

This should be understood as analogous to the compactness Corollary 5.21 for flat connections: Theorem 5.34 says that even if we allow some bounded amount of curvature, we still get a kind of weak compactness when we divide out by gauge transformations. [explain why strong compactness is too much to ask?]

Now we are ready for the main theorem about moduli of holomorphic vector bundles:

Theorem 5.35 (Narasimhan-Seshadri theorem). [36] We have:

- For any $[D] \in \mathcal{N}_{K,d}^s(C)$ the (0,1) part $\bar{\partial}_D$ induces the structure of *stable* holomorphic vector bundle on E. Conversely, any stable holomorphic structure on E is equivalent to $\bar{\partial}_D$ for a unique $[D] \in \mathcal{N}_{K,d}^s(C)$.
- For any $[D] \in \mathcal{N}_{K,d}(C)$ the (0,1) part $\bar{\partial}_D$ induces the structure of *polystable* holomorphic vector bundle on *E*. Conversely, any polystable holomorphic structure on *E* is equivalent to $\bar{\partial}_D$ for a unique $[D] \in \mathcal{N}_{K,d}(C)$.

Proof. The proof we follow here is given in [37]. We only give a sketch. Moreover, we just discuss the case d = 0 — the other cases are very similar but involve more ugly notation.

First suppose $[D] \in \mathcal{N}_{K,d}^s(C)$. Thus *D* is an irreducible flat connection. Suppose that $E' \subset E$ is some subbundle preserved by $\bar{\partial}_D$; we want to show that deg E' < 0. This is an instance of the general principle that "curvature decreases in holomorphic subbundles." Let E'' be the orthocomplement of E'; then *D* splits as

$$D = \begin{pmatrix} D_{E'} & -\beta^{\dagger} \\ \beta & D_{E''} \end{pmatrix}$$
(5.66)

where $\beta \in \Omega^{1,0} \otimes \text{Hom}(E', E'')$ (the (0, 1) component of β vanishes because E' is preserved by $\bar{\partial}_D$). Then we compute

$$F_D = \begin{pmatrix} F_{D_{E'}} - \beta^{\dagger} \wedge \beta & -D_{\operatorname{Hom}(E'',E')}\beta^{\dagger} \\ D_{\operatorname{Hom}(E',E'')}\beta & F_{D_{E''}} - \beta \wedge \beta^{\dagger} \end{pmatrix} = 0.$$
(5.67)

$$F_{D_{F'}} = \beta^{\dagger} \wedge \beta, \tag{5.68}$$

and thus

$$\deg E' = \frac{\mathrm{i}}{2\pi} \int_C \operatorname{Tr} F_{D_{E'}} = \frac{\mathrm{i}}{2\pi} \int_C \operatorname{Tr} \beta^{\dagger} \wedge \beta.$$
(5.69)

Now, i Tr($\beta^{\dagger} \wedge \beta$) is a seminegative form, so we get deg $E' \leq 0$, with equality only if $\beta = 0$.

If *D* is irreducible, then we must have $\beta \neq 0$; in that case we get deg E' < 0, so $(E, \overline{\partial}_D)$ is stable, as desired. If *D* is not irreducible, then we may have $\beta = 0$; but in this case $(E, \overline{\partial}_D)$ decomposes as direct sum of two holomorphic bundles, each of degree 0, each carrying a flat unitary connection. By induction on the rank we may assume that these two are polystable; thus $(E, \overline{\partial}_D)$ is polystable, as desired.

Conversely suppose we have some D such that $\bar{\partial}_D$ induces a stable holomorphic structure. Now we want to find a flat connection which induces the same holomorphic structure, i.e. is in the same $\mathfrak{G}_{\mathbb{C}}$ -orbit. Call this orbit \mathcal{O} . The moral idea is to look for a minimum of the L^2 norm of F_D (Yang-Mills functional)

$$||F_D||^2 = \int_C \operatorname{Tr}(F_D \wedge \star F_D).$$
(5.70)

For technical convenience, instead of the L^2 norm, we use a functional *J* which has the same minima as ||F||:

$$J(D)^{2} = \int_{C} \left[\nu \left(\frac{\star F_{D}}{2\pi i} \right) \right]^{2}, \qquad \nu(M) = \operatorname{Tr} \sqrt{M^{*}M} = \sum |\lambda_{i}|.$$
(5.71)

Let *m* be the infimum of *J* on \mathcal{O} , and suppose that \mathcal{O} contains a connection *D'* with J(D') = m. We will show that $F_{D'} = 0$.

For this, we consider an infinitesimal gauge transformation by a self-adjoint element $Z \in \text{Lie} \mathfrak{G}_{\mathbb{C}} = \Omega^{0}(\text{End} E)$ — i.e. we transform by e^{tZ} and work to first order in t. The leading change of $\star F_{D'}$ is $itD'^{*}D'Z$ (because D' changes by $t\bar{\partial}_{D'}Z - t\partial_{D'}Z$, so $F_{D'}$ changes by $t(\partial_{D'}\bar{\partial}_{D'} - \bar{\partial}_{D'}\partial_{D'})Z = \star itD'^{*}D'Z$). [check sign] Now we consider Z obeying

$$iD'^*D'Z = -\star F_{D'}.$$
 (5.72)

Elliptic theory shows that we can indeed find a *Z* obeying (5.72). Given such a *Z*, gauge transformation by e^{tZ} for small *t* reduces J(D'), unless already $F_{D'} = 0$. Thus, since we assume J(D') = m, we must have $F_{D'} = 0$. [interpret as gradient flow?]

Thus the problem remaining is to show that \mathcal{O} indeed contains a minimum of *J*. For this consider a sequence $(D_n)_{n=1}^{\infty}$ in \mathcal{O} with

$$J(D_n) \to m. \tag{5.73}$$

By the Uhlenbeck compactness theorem, Theorem 5.34, we can make gauge transformations such that some subsequence converges in the weak topology on A_1^h to a limiting L_1^2 connection D_{∞} . This weak convergence is sufficient to imply that

$$J(D_{\infty}) \le m. \tag{5.74}$$

[explain why] Thus, if $D_{\infty} \in \mathcal{O}$, then $J(D_{\infty}) = m$ and D_{∞} is the desired minimum of J on \mathcal{O} .

So what if $D_{\infty} \notin \mathcal{O}$? Then we want to derive a contradiction. [remark D_{∞} still defines a holomorphic structure even though it's only L_1^2 : from Atiyah-Bott Lemma 14.8] In this case there is no holomorphic isomorphism $(E, \bar{\partial}_D) \to (E, \bar{\partial}_{D_{\infty}})$. Still, we can show that at least there is a nonzero holomorphic map

$$\gamma: (E, \bar{\partial}_D) \to (E, \bar{\partial}_{D_{\infty}}). \tag{5.75}$$

For this we use the fact that each $D_n \in \mathcal{O}$, and so there does exist a nonzero holomorphic map $\gamma_n : (E, \bar{\partial}_D) \to (E, \bar{\partial}_{D_n})$, so the operator $\bar{\partial}_{D_n^*} \otimes \bar{\partial}_D$ on Hom(E, E) has a nontrivial kernel. Some elliptic estimates show that this kernel cannot disappear in the limit (roughly: if the lowest eigenvalue of the associated Laplacian is zero for all $\bar{\partial}_{D_n}$ then it is also zero for $\bar{\partial}_{D_\infty}$), so $\bar{\partial}_{D_\infty^*} \otimes \bar{\partial}_D$ also has a nontrivial kernel. This gives the desired γ .

Since Hom(*E*, *E*) has degree zero and γ is holomorphic, either det $\gamma = 0$ or det γ is nowhere vanishing. However, by assumption γ is not an isomorphism, so we must have det $\gamma = 0$. Now let us consider the sheaves ker γ and Im γ . Since degree is additive in exact sequences of sheaves we have

$$\deg \operatorname{Im} \gamma = -\deg \ker \gamma. \tag{5.76}$$

The sheaf ker γ is a nontrivial proper holomorphic subbundle *K* of $(E, \bar{\partial}_D)$. The sheaf Im γ is not necessarily a subbundle, but using Lemma 5.28 we can saturate it to a holomorphic subbundle *I* of $(E, \bar{\partial}_{D_{\infty}})$, which has the same rank as Im γ but perhaps higher degree. Thus we have subbundles

$$K \subset (E, \bar{\partial}_D), \quad I \subset (E, \bar{\partial}_{D_{\infty}}), \quad \deg I \ge -\deg K.$$
 (5.77)

We now derive a contradiction from this state of affairs.

Since $(E, \bar{\partial}_D)$ was assumed stable, we must have deg K < 0, so deg I > 0. In particular, this means $(E, \bar{\partial}_{D_{\infty}})$ is unstable. (That is as we should expect: D_{∞} is in the closure of the $\mathfrak{G}_{\mathbb{C}}$ -orbit \mathcal{O} , so if it were stable, it would lead to some non-Hausdorff behavior in the quotient.) Consider a connection D' in the orbit of $(E, \bar{\partial}_{D_{\infty}})$. Because $(E, \bar{\partial}_{D_{\infty}})$ is unstable we cannot find such a D' with J(D') = 0; indeed the destabilizing subbundle I implies the lower bound

$$J(D') \ge 2 \deg I, \tag{5.78}$$

as we now show. Let *H* be the orthocomplement of *I* in *E*. We use the shape of the curvature of a connection in $(E, \overline{\partial}_{D_{\infty}})$ as we used above,

$$F_{D'} = \begin{pmatrix} F_{D'_{I}} - \beta^{\dagger} \wedge \beta & -D'_{\text{Hom}(H,I)}\beta^{\dagger} \\ D'_{\text{Hom}(I,H)}\beta & F_{D'_{H}} - \beta \wedge \beta^{\dagger} \end{pmatrix},$$
(5.79)

and the fact that the matrix norm ν behaves well:

$$\nu \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \ge |\operatorname{Tr} a| + |\operatorname{Tr} d|.$$
(5.80)

Applying this to (5.79) the key point is that the extra terms involving β have the same sign as the curvatures; thus the equation

$$J(D') \ge \int_{C} \nu\left(\frac{\star F_{D'}}{2\pi i}\right) \tag{5.81}$$

gives $J(D') \ge |\deg I| + |\deg H| = 2 \deg I$ as desired.

On the other hand, using the subbundle *K* one can construct a connection $D'' \in \mathcal{O}$ such that

$$J(D'') < -2 \deg K.$$
(5.82)

For this, one works by induction. Assume that the result has been proven for all ranks K' < K. (The case K' = 1 was Lemma 5.19.) Then the idea is to use the existence of Einstein connections on smaller bundles to cook up a connection on the full $(E, \bar{\partial}_D)$ which does the job. [...]

Combining (5.82), (5.78), (5.77) we have

$$m \le J(D'') < J(D_{\infty}) \le m \tag{5.83}$$

which is the desired contradiction.

[comment on other approaches, Yang-Mills flow]

Exercise 5.22. Verify the formula (5.67). It is convenient to think of (5.66) as shifting a connection by a 1-form,

$$D = \begin{pmatrix} D_{E'} & 0\\ 0 & D_{E''} \end{pmatrix} + \begin{pmatrix} 0 & -\beta^{\dagger}\\ \beta & 0 \end{pmatrix}$$
(5.84)

and then use the formula for the perturbed curvature, $F_{D+A} = F_D + DA + A \wedge A$.

Exercise 5.23. Over \mathbb{CP}^1 , show that $\mathcal{O}(m) \oplus \mathcal{O}(m)$ admits a holomorphic subbundle isomorphic to $\mathcal{O}(n)$ if and only if $n \leq m$. (This is an illustration of the principle that curvature decreases in holomorphic subbundles, but it is instructive to do it directly in the holomorphic language without using connections.)

5.8 Bundles with fixed determinant

[should say this more systematically]

One way to focus attention on only the interesting directions is to consider a smaller moduli space, consisting of bundles with "fixed determinant." For this, we fix some line bundle *L* of degree *d*, use it to define the moduli space $\mathcal{N}_{1,d}(C)$, and then fix an element $\tilde{d} \in \mathcal{N}_{1,d}(C)$. If d = 0 then it would be natural to take *L* and \tilde{d} to be trivial. Then we fix a bundle *E* together with an isomorphism det $E \simeq L$, and construct moduli spaces $S\mathcal{N}_{K,\tilde{d}}(C)$ and $S\mathcal{N}_{K,\tilde{d}}^s(C)$ by repeating all our previous constructions, now with the extra condition that the connections *D* which we consider have det $D = \tilde{d}$, and the gauge group is restricted to $g \in U(E)$ such that det *g* acts trivially.

As before, we have:

Proposition 5.36 (Moduli of stable bundles with fixed determinant is Kähler). $SN^s_{K,\tilde{d}}(C)$ is a Kähler manifold, of complex dimension $(g-1)(K^2-1)$.

Exercise 5.24. In what sense is $SN_{K,\tilde{d}}(C)$ "independent" of the choice of \tilde{d} ?

Exercise 5.25. State the analogue of the Narasimhan-Seshadri theorem for $SN^s_{K\tilde{d}}$.

Example 5.37 (Bundles of rank 2 with fixed determinant over a genus 2 curve). Consider the case where the rank K = 2 and C has genus g = 2. In this case we have

$$\dim_{\mathbb{C}} \mathcal{SN}^{s}_{2\tilde{d}}(C) = 3.$$
(5.85)

These spaces are described concretely in the paper [38].

It turns out that $SN_{2,0}(C)$ can be naturally identified with \mathbb{CP}^3 . The vague idea is as follows. Let $P = N_{1,1}(C)$ (a torus of complex dimension 2.) Suppose given a polystable rank 2 holomorphic bundle E on C, with det E holomorphically trivial. We consider the set $\Theta_E \subset P$ consisting of all degree 1 line bundles L such that $E \otimes L$ has nontrivial holomorphic sections. This subset turns out to be the zero locus of a holomorphic section s_E of a certain canonically defined line bundle $\mathcal{L} \to P$. s_E is determined up to overall rescaling by E, and s_E determines E up to equivalence; thus $SN_{2,0}(C)$ gets identified with the projectivization of the vector space of sections of \mathcal{L} .

(Warning: it is a bit of an accident that $SN_{2,0}(C)$ has the structure of complex manifold even though 0 and 2 are not relatively prime; this doesn't happen in most examples. Also, as far as I know, the Kähler structure on the open subset $SN_{2,0}^s(C)$ does not extend smoothly over the whole \mathbb{CP}^3 , so you probably shouldn't think of this as being the standard Kähler structure on \mathbb{CP}^3 .)

The description of $SN^s_{2,\tilde{d}}(C)$ when \tilde{d} has degree 1 is equally concrete but more complicated to state; see [38] for that.

Exercise 5.26. Suppose a rank 2 bundle *E* over *C* is given holomorphically as a nontrivial (i.e. non-split) extension

$$0 \to L^* \to E \to L \to 0 \tag{5.86}$$

where *L* has degree 1. Show that *E* is semistable and det *E* is holomorphically trivial.

6 Higgs bundles

Finally we are ready to treat the moduli space of Higgs bundles. The fundamental reference for this subject is [5]; strictly speaking that paper treats only the case of rank 2 bundles with fixed determinant, but most of the fundamental issues and constructions appear already there.

As before we fix a compact Riemann surface *C*, with a Kähler metric of total volume 1, and two integers $K \ge 2$ and *d*. We are going to define a moduli space $\mathcal{M}_{K,d}(C)$ which can be studied, and thought of, in several different ways.

6.1 Basic definitions

Definition 6.1 (Higgs bundle). A *Higgs bundle* of rank *K* over *C* is a tuple $(E, \bar{\partial}_E, \varphi)$ where $(E, \bar{\partial}_E)$ is a holomorphic vector bundle of rank *K* over *C*, and

$$\varphi \in H^0(C, \operatorname{End} E \otimes K_C). \tag{6.1}$$

 φ is called the *Higgs field*.

Thus relative to a local holomorphic trivialization of $(E, \bar{\partial}_E)$ and local coordinate on *C*, φ would be written as a matrix of holomorphic 1-forms: something like

$$\varphi = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} dz, \tag{6.2}$$

where all f_i are holomorphic functions of z.

Definition 6.2 (Sub-Higgs bundle). If $(E, \bar{\partial}_E, \varphi)$ is a Higgs bundle over *C*, a *sub-Higgs bundle* thereof is a subbundle $E' \subset E$ which is preserved by $\bar{\partial}_E$ and has

$$\varphi(E') \subset E' \otimes K_C. \tag{6.3}$$

6.2 Moduli space as holomorphic symplectic quotient

Now suppose we want to construct a moduli space parameterizing Higgs bundles modulo equivalence — perhaps with some stability condition, yet to be described. To get some crude idea of what will happen, let us imagine that we take all stable vector bundles and then equip them with arbitrary Higgs fields: in other words we take our "moduli space" to be a vector bundle over $\mathcal{N}_{K,d}^s(C)$, with fiber $H^0(C, \operatorname{End} E \otimes K_C)$.



What structure does it carry?

Fix a point $D \in \mathcal{N}_{K,d}^s(C)$. The space of Higgs fields on the holomorphic bundle $(E, \bar{\partial}_D)$ is the same as the space of harmonic forms $\varphi = \Phi_z \in \mathcal{H}_D^{1,0}(\text{End } E)$. On the other hand, the tangent space to $\mathcal{N}_{K,d}^s(C)$ at D, considered as a complex vector space, is the space of harmonic forms $\dot{A}_{\bar{z}} \in \mathcal{H}_D^{0,1}(\text{End } E)$. The two are dual via the pairing

$$(\dot{A}_{\bar{z}}, \Phi_z) \mapsto \int_C \operatorname{Tr}(\dot{A}_{\bar{z}} \wedge \Phi_z).$$
 (6.4)

Thus the moduli space we obtain in this way is simply $T^*\mathcal{N}_{K,d}(C)$. In particular this is a *holomorphic symplectic* manifold.

There is another way of viewing this space which will be enlightening. We recall that, ignoring questions of stability, $N_{K,d}$ just consists of all holomorphic structures on E, up to equivalence,

$$\mathcal{N}_{K,d}(C) = \mathcal{A}^{\partial} / \mathfrak{G}_{\mathbb{C}}.$$
(6.5)

This suggests (following the theme of Exercise 2.4) that we might profitably view $T^*\mathcal{N}_{K,d}(C)$ as a holomorphic symplectic quotient,

$$T^* \mathcal{N}_{K,d}(C) = (T^* \mathcal{A}^{\partial}) // \mathfrak{G}_{\mathbb{C}}.$$
(6.6)

Let us see how this works.

As before, we fix a smooth bundle *E* over *C*, of degree *d*. Now we define a "doubled" version of the space of $\bar{\partial}$ -operators on *E*,

$$\mathcal{A}_{1}^{\mathbb{C}} = \mathcal{A}^{\bar{\partial}} \times \Omega^{1,0}(\operatorname{End} E).$$
(6.7)

 $\mathcal{A}_1^{\mathbb{C}}$ is a complex affine space over $\Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$. We have

$$\mathcal{A}_1^{\mathsf{C}} = T^* \mathcal{A}^{\bar{\partial}} \tag{6.8}$$

and

$$T\mathcal{A}_1^{\mathbb{C}} = \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E).$$
(6.9)

The space $\mathcal{A}_1^{\mathbb{C}}$ carries an obvious complex structure (since $T\mathcal{A}_1^{\mathbb{C}}$ is naturally a complex vector space). It also carries a natural holomorphic symplectic form:

$$\Omega_1((\dot{A}_{1\bar{z}}, \dot{\Phi}_{1z}), (\dot{A}_{2\bar{z}}, \dot{\Phi}_{2z})) = 2i \int_C \operatorname{Tr}(\dot{A}_{1\bar{z}} \wedge \dot{\Phi}_{2z} - \dot{A}_{2\bar{z}} \wedge \dot{\Phi}_{1z}).$$
(6.10)

We denote points of $\mathcal{A}_1^{\mathbb{C}}$ as $(\bar{\partial}_E, \varphi) \in \mathcal{A}_1^{\mathbb{C}}$. As with $\mathcal{A}^{\bar{\partial}}$, $\mathcal{A}_1^{\mathbb{C}}$ has a natural action of $\mathfrak{G}_{\mathbb{C}}$, by

$$(\bar{\partial}_E, \varphi) \mapsto (\bar{\partial}_E^g = g\bar{\partial}_E g^{-1}, \varphi^g = g\varphi g^{-1}).$$
(6.11)

Now we want to take the holomorphic symplectic quotient by this action. So first we need to know the moment map:

Proposition 6.3 (Moment map for $\mathfrak{G}_{\mathbb{C}}$ **-action on** $\mathcal{A}_{1}^{\mathbb{C}}$ **).** The $\mathfrak{G}_{\mathbb{C}}$ -action on $\mathcal{A}_{1}^{\mathbb{C}}$ has a complex moment map with respect to Ω_{1} ,

$$M_1 = 2i\bar{\partial}_E \varphi. \tag{6.12}$$

More precisely, this means: given $Z \in \text{Lie } \mathfrak{G}_{\mathbb{C}} = \Omega^0(\text{End } E)$ we have

$$M_{1,Z}(\bar{\partial}_E,\varphi) = 2i \int_C \operatorname{Tr}(Z\bar{\partial}_E\varphi).$$
(6.13)

Proof. This follows the pattern of proof of Proposition 5.8. Each $Z \in \Omega^0(\text{End } E)$ generates a vector field on $\mathcal{A}_1^{\mathbb{C}}$,

$$\rho(Z) = (-\bar{\partial}_E Z, [Z, \varphi]), \tag{6.14}$$

and for $(\dot{A}_{\bar{z}}, \dot{\Phi}_z) \in \Omega^{0,1}(\text{End } E) \times \Omega^{1,0}(\text{End } E) = T\mathcal{A}_1^{\mathbb{C}}$, we compute by differentiating (6.13)

$$dM_{1,Z}(\dot{A}_{\bar{z}},\dot{\Phi}_z) = 2i \int_C \operatorname{Tr}(Z[\dot{A}_{\bar{z}},\varphi] + Z\bar{\partial}_E \dot{\Phi}_z).$$
(6.15)

Now we can check directly that M_1 is indeed a moment map, using (6.10):

$$\Omega_1(\rho(Z), (\dot{A}_{\bar{z}}, \dot{\Phi}_z)) = 2i \int_C \operatorname{Tr}(-\bar{\partial}_E Z \wedge \dot{\Phi}_z - \dot{A}_{\bar{z}} \wedge [Z, \varphi])$$
(6.16)

$$= 2i \int_{C} \operatorname{Tr}(Z\bar{\partial}_{E}\dot{\Phi}_{z} + Z[\dot{A}_{\bar{z}}, \varphi])$$
(6.17)

$$= dM_{1,Z}(\dot{A}_{\bar{z}}, \dot{\Phi}_{z})$$
(6.18)

as desired.

So, the zero set of the moment map is simply

$$M_1^{-1}(0) = \{ (\bar{\partial}_E, \varphi) \in \mathcal{A}_1^{\mathbb{C}} : \bar{\partial}_E \varphi = 0 \}.$$
(6.19)

This means that the holomorphic symplectic quotient

$$\mathcal{A}_1^{\mathbb{C}} / / \mathfrak{G}_{\mathbb{C}} = M_1^{-1}(0) / \mathfrak{G}_{\mathbb{C}}$$

$$(6.20)$$

consists of equivalence classes of holomorphic bundles equipped with *holomorphic* Higgs fields, just as we wanted. This is encouraging. On the other hand, by itself, it doesn't give us anything really new about the space.

6.3 Moduli space as hyperkähler quotient

As we have noted, though, a holomorphic symplectic quotient $X//G_{\mathbb{C}}$ often has another interpretation: if the space X is actually hyperkähler, then we can try to identify $X//G_{\mathbb{C}} = X////G$, and thus get a hyperkähler structure on the quotient. Let us try to do that here. As we did before, we begin by introducing a Hermitian metric *h* on the bundle *E*.

Definition 6.4 (Doubled connection). A *doubled connection* on (E, h) is a pair (D, Φ) where D is a unitary connection on (E, h) and $\Phi \in \Omega^1(\mathfrak{u}(E))$.

Let \mathcal{A}^H denote the space of all doubled connections on (E,h). \mathcal{A}^H is naturally an affine space over the real vector space $\Omega^1(\mathfrak{u}(E)) \oplus \Omega^1(\mathfrak{u}(E))$. The group \mathfrak{G} of unitary gauge transformations acts on \mathcal{A}^H by

$$(D, \Phi) \mapsto (D^g = gDg^{-1}, \Phi^g = g\Phi g^{-1}).$$
 (6.21)

We have an isomorphism of real affine spaces

$$\mathcal{A}_{1}^{\mathbb{C}} \xrightarrow{\sim} \mathcal{A}^{H} \tag{6.22}$$

$$(\bar{\partial}_E, \varphi) \mapsto (D = \bar{\partial}_E + \partial_E, \Phi = \varphi - \varphi^{\dagger})$$
 (6.23)

where $\bar{\partial}_E + \partial_E$ denotes the Chern connection.

 \mathcal{A}^{H} carries the complex form Ω_{1} which we break into components $\Omega_{1} = \omega_{2} + i\omega_{3}$, plus another real symplectic form ω_{1} , naturally extending the one (5.7) which we had on \mathcal{A}^{h} :

$$\omega_1((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_C \operatorname{Tr}(-\dot{A}_1 \wedge \dot{A}_2 + \dot{\Phi}_1 \wedge \dot{\Phi}_2), \qquad (6.24a)$$

$$\omega_2((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_C \operatorname{Tr}(\dot{\Phi}_1 \wedge \star \dot{A}_2 - \dot{A}_1 \wedge \star \dot{\Phi}_2), \tag{6.24b}$$

$$\omega_3((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_C \operatorname{Tr}(\dot{\Phi}_1 \wedge \dot{A}_2 + \dot{A}_1 \wedge \dot{\Phi}_2).$$
(6.24c)

Note that the ω_i are translation invariant forms on \mathcal{A}^H and hence closed.

Exercise 6.1. Verify that indeed $\Omega_1 = \omega_2 + i\omega_3$. (Hint: it is easiest to compute $\omega_2 + i\omega_3$ and then compare it to Ω_1 . For example, in the first term you will get $\star \dot{A}_2 + i\dot{A}_2$ appearing. Then use the fact that $\dot{A}_2 = \dot{A}_{2z} + \dot{A}_{2\bar{z}}$.)

Proposition 6.5 (Hyperkähler structure on \mathcal{A}^{H} **).** The forms $(\omega_1, \omega_2, \omega_3)$ on \mathcal{A}^{H} are the symplectic forms for a hyperkähler structure (I_1, I_2, I_3, g) on \mathcal{A}^{H} , with hyperkähler metric

$$g((\dot{A}_{1}, \dot{\Phi}_{1}), (\dot{A}_{2}, \dot{\Phi}_{2})) = -\int_{C} \operatorname{Tr}(\dot{A}_{1} \wedge \star \dot{A}_{2} + \dot{\Phi}_{1} \wedge \star \dot{\Phi}_{2})$$
(6.25)

The complex structures act by:

$$I_1(\dot{A}, \dot{\Phi}) = (\star \dot{A}, -\star \dot{\Phi}), \tag{6.26}$$

$$I_2(\dot{A}, \dot{\Phi}) = (-\dot{\Phi}, \dot{A}),$$
 (6.27)

$$I_3(\dot{A}, \dot{\Phi}) = (-\star \dot{\Phi}, -\star \dot{A}). \tag{6.28}$$

Proof. Just compute directly that $I_i^2 = -1$, $I_1I_2 = I_3$, and $\omega_i(\cdot, \cdot) = g(I_i \cdot, \cdot)$. For example, when i = 3 this amounts to checking that

$$g((-\star\dot{\Phi}_1,-\star\dot{A}_1),(\dot{A}_2,\dot{\Phi}_2)) = \omega_3((\dot{A}_1,\dot{\Phi}_1),(\dot{A}_2,\dot{\Phi}_2)).$$
(6.29)

There is a more abstract way of viewing this construction:

Exercise 6.2. If (V, I, ω) is a Kähler vector space, show that $V \oplus V^*$ admits a canonical hyperkähler structure, for which $I_1 = I \oplus I^T$, $\omega_1 = \omega \oplus -\omega^{-1}$, and $\Omega_1(v \oplus \alpha, v' \oplus \alpha') = 2i(\alpha'(v) - \alpha(v'))$.

Exercise 6.3. Show that the hyperkähler structure on \mathbb{R}^4 from Example 3.7 arises from the construction of Exercise 6.2, where *V* is the first \mathbb{R}^2 and V^* the second \mathbb{R}^2 in \mathbb{R}^4 , and the duality pairing is $(x_2, x_3) \cdot (x_0, x_1) = x_3 x_0 + x_2 x_1$.

Exercise 6.4. Show that the hyperkähler structure on \mathcal{A}^H introduced above arises from the construction of Exercise 6.2, applied to the vector space $V = \Omega^1(\mathfrak{u}(E))$.

In order to take a hyperkähler quotient $\mathcal{A}^H / / / \mathfrak{G}$ we need first to have a moment map. To write it we will use a bit of notation which is convenient but also confusing, so let's spell its meaning out. In local real coordinates (x, y) on C, for $\alpha, \beta \in \Omega^1(\text{End } E)$, we have

$$\alpha \wedge \beta = (\alpha_x dx + \alpha_y dy) \wedge (\beta_x dx + \beta_y dy)$$
(6.30)

$$= (\alpha_x \beta_y - \alpha_y \beta_x) dx \wedge dy.$$
 (6.31)

Thus if we define (note the tricky sign)

$$[\alpha,\beta] = \alpha \wedge \beta + \beta \wedge \alpha \tag{6.32}$$

we have

$$[\alpha,\beta] = ([\alpha_x,\beta_y] - [\alpha_y,\beta_x])dx \wedge dy.$$
(6.33)

In particular,

$$\Phi \wedge \Phi = \frac{1}{2} [\Phi, \Phi] = [\Phi_x, \Phi_y] dx \wedge dy, \qquad (6.34)$$

and recalling $\Phi = \varphi - \varphi^{\dagger}$,

$$\Phi \wedge \Phi = -[\varphi_z, \varphi_{\bar{z}}^{\dagger}] dz \wedge d\bar{z}$$
(6.35)

$$= -[\varphi, \varphi^{\dagger}]. \tag{6.36}$$

Now we can write the moment map:

Proposition 6.6 (Hyperkähler moment map for \mathfrak{G} **action on** \mathcal{A}^H **).** The \mathfrak{G} action on \mathcal{A}^H admits a hyperkähler moment map $\vec{\mu}$, given by:

$$\mu_2 + i\mu_3 = 2i\bar{\partial}_D\varphi \tag{6.37}$$

$$\mu_1 = -F_D + \Phi \wedge \Phi - 2\pi i \frac{d}{K} \mathbf{1} \omega_C.$$
(6.38)

Proof. We have already computed the holomorphic moment map $M_1 = \mu_2 + i\mu_3$, in Proposition 6.3. All that remains is to check the formula for μ_1 . We have

$$d\mu_{1,Z}(\dot{A},\dot{\Phi}) = \int_{C} \operatorname{Tr} Z\left(-D\dot{A} + [\dot{\Phi},\Phi]\right)$$
(6.39)

and

$$\rho(Z) = (-DZ, [Z, \Phi])$$
(6.40)

giving

$$\omega_1(\rho(Z), (\dot{A}, \dot{\Phi})) = -\int_C \operatorname{Tr}(-DZ \wedge \dot{A} - [Z, \Phi] \wedge \dot{\Phi})$$
(6.41)

$$= -\int_{C} \operatorname{Tr}(ZD\dot{A} - Z[\dot{\Phi}, \Phi])$$
(6.42)

$$= \mathrm{d}\mu_{1,Z}(\dot{A},\dot{\Phi}) \tag{6.43}$$

as desired. (Note that μ_1 comes as a sum of two pieces, one involving *D* and one involving Φ . This happens because the gauge group acts separately on the two and the symplectic form ω_1 is a sum of one piece involving *D* and one involving Φ . Moreover, if we look only at the part involving *D*, then our computations just reduce to those in Proposition 5.8.)

Exercise 6.5. The statement that the \mathfrak{G} action on \mathcal{A}^H admits a moment map implies in particular that the forms $\omega_i \in \Omega^2(\mathcal{A}^H)$ are \mathfrak{G} -invariant. Verify this invariance directly from the formulas defining ω_i .

It is occasionally useful to write $\vec{\mu}$ in a more symmetric fashion, without combining μ_2 and μ_3 into $\mu_2 + i\mu_3$; for that purpose the next exercise is helpful:

Exercise 6.6. Show that

$$\mu_2 = -D \star \Phi, \qquad \mu_3 = D\Phi. \tag{6.44}$$

The vanishing of the hyperkähler moment map, $\vec{\mu} = 0$, gives *Hitchin's equations* [5]:

$$\bar{\partial}_D \varphi = 0,$$
 (6.45a)

$$F_D + [\varphi, \varphi^{\dagger}] = -2\pi i \frac{d}{K} \mathbf{1} \omega_C.$$
(6.45b)

Solutions of these equations are important enough that we give them a name:

Definition 6.7 (Harmonic doubled connection). A *harmonic doubled connection* on *E* is a doubled connection $(D, \Phi) \in A^H$ obeying Hitchin's equations (6.45).

A harmonic doubled connection (D, Φ) with $\Phi = 0$ reduces to an Einstein connection.

Thus the moduli space we are after, $A^H / / / \mathfrak{G} = \vec{\mu}^{-1}(0) / \mathfrak{G}$, is the space of harmonic doubled connections on *E* modulo gauge equivalence. As before, to study this hyperkähler quotient, we need to understand the extent to which \mathfrak{G} acts freely:

Definition 6.8 (Irreducible doubled connections). A doubled connection $(D, \Phi) \in \mathcal{A}^H$ is called *irreducible* if there exist no proper subbundles $E' \subset E$ which are preserved by both D and Φ . Let $\mathcal{A}^{H,s} \subset \mathcal{A}^H$ be the set of irreducible doubled connections.

In parallel to Proposition 5.11, we have:

Proposition 6.9 (Gauge group acts almost freely on irreducible doubled connections). If $g \in \mathfrak{G}$ and $(D, \Phi) \in \mathcal{A}^{H,s}$, then $(D^g, \Phi^g) = (D, \Phi)$ if and only if g acts on E by multiplication by a constant scalar.

Proof. This is essentially the same as the proof of Proposition 5.11.

Now finally we define:

Definition 6.10 (Moduli space of harmonic doubled connections).

$$\mathcal{M}_{K,d}(C) = \mathcal{A}^H / / / \mathfrak{G}, \qquad \mathcal{M}^s_{K,d}(C) = \mathcal{A}^{H,s} / / / / \mathfrak{G}.$$
(6.46)

As we did with $\mathcal{N}_{K,d}^s(C)$, we first discuss the formal picture. The tangent space to $\mathcal{M}_{K,d}^s(C)$ at a given harmonic doubled connection (D, Φ) should formally be the joint kernel of the three linearized moment maps $d\mu_i$, modulo the space spanned by infinitesimal gauge transformations. That is, we consider cohomology of the complex

$$0 \to \Omega^0(\mathfrak{u}(E)) \to (\Omega^1(\mathfrak{u}(E)))^2 \to (\Omega^2(\mathfrak{u}(E)))^3 \to 0$$
(6.47)

where the first arrow is

$$Z \mapsto \rho(Z) = (\dot{A} = -DZ, \dot{\Phi} = [Z, \Phi]) \tag{6.48}$$

and the second is

 $(\dot{A}, \dot{\Phi}) \mapsto (D\dot{A} + [\Phi, \dot{\Phi}], -D \star \dot{\Phi} - [\dot{A}, \star \Phi], D\dot{\Phi} + [\dot{A}, \Phi]).$ (6.49)

As before, this is indeed a complex, when (D, Φ) is a harmonic pair. Also as before, the hyperkähler quotient construction dictates that we should interpret the quotient by Im ρ by taking the orthocomplement, [make this sentence clearer] [double-check signs and factors here!]

$$(\operatorname{Im} \rho)^{\perp} = \{ D \star \dot{A} - [\star \Phi, \dot{\Phi}] = 0 \} \subset \Omega^{1}(\mathfrak{u}(E))^{\oplus 2}.$$
(6.50)

So altogether the tangent space we are after is the kernel of the operator

$$\hat{D}: \Omega^{1}(\mathfrak{u}(E))^{\oplus 2} \to \Omega^{2}(\mathfrak{u}(E))^{\oplus 4}$$

$$(6.51)$$

$$(\dot{A}, \dot{\Phi}) \mapsto \begin{pmatrix} DA + [\Phi, \Phi] \\ -D \star \dot{\Phi} - [\star \Phi, \dot{A}] \\ D\dot{\Phi} + [\Phi, \dot{A}] \\ D \star \dot{A} - [\star \Phi, \dot{\Phi}] \end{pmatrix}$$
(6.52)

Note that \hat{D} is *elliptic*, just as before: the principal symbol is $2K^2$ copies of $d \oplus d^*$. Thus, as before, elliptic theory in the sense of subsection 4.6 says that the extension \hat{D}_k of \hat{D} to the Banach space of L_k^2 doubled connections has a finite-dimensional kernel, and that kernel consists of smooth sections.

Again as before, we have:

Lemma 6.11 (Almost-vanishing for the doubled gauge complex). If $(D, \Phi) \in \mathcal{A}^{H,s}$, then coker \hat{D} is 4-dimensional, spanned by multiples of the identity in each summand of $\Omega^2(\mathfrak{u}(E))^{\oplus 4}$.

Proof. [...]

Theorem 6.12 ($\mathcal{M}^{s}_{K,d}(C)$ **is hyperkähler).** $\mathcal{M}^{s}_{K,d}(C)$ is a hyperkähler manifold.

Proof. [...]

This already has the remarkable consequence that the metric on $\mathcal{M}^{s}_{K,d}(C)$ is Ricci-flat.

Proposition 6.13 (Dimension of $\mathcal{M}^{s}_{K,d}(C)$ **).** The quaternionic dimension of $\mathcal{M}^{s}_{K,d}(C)$ is $(g-1)K^{2}+1$.

Proof. [...]

6.4 The twistor family, formally

We have been studying the hyperkähler space \mathcal{A}^{H} , which in its complex structure I_1 is identified with the space $\mathcal{A}_{1}^{\mathbb{C}}$ of pairs $(\bar{\partial}_{E}, \varphi)$. Recall that modulo questions of stability we have

$$\mathcal{A}^H / / / \mathfrak{G} = \mathcal{A}_1^{\mathbb{C}} / / \mathfrak{G}_{\mathbb{C}}, \tag{6.53}$$

and the RHS at least formally deserves the name "moduli space of Higgs bundles."

Note as usual that the complex description is simpler than the unitary one — to construct examples of Higgs bundles, we do not have to solve the complicated PDE (6.45)!

Now, since \mathcal{A}^H is hyperkähler, it has plenty of other complex structures I_{ζ} . We may ask: do those too have simple descriptions? Let us begin with structure I_2 . Since I_2 acts by $(\dot{A}, \dot{\Phi}) \mapsto (-\dot{\Phi}, \dot{A})$ we see that the combination $\dot{A} + i\dot{\Phi}$ is holomorphic. Said otherwise, $\mathcal{A}_2^{\mathbb{C}}$ is the space of all (generally *complex*, i.e. non-unitary) connections in *E*, which we split into their unitary and self-adjoint parts by writing

$$\nabla_2 = D + i\Phi. \tag{6.54}$$

The action of \mathfrak{G} on $\mathcal{A}_2^{\mathbb{C}}$ takes $\nabla_2 \to \nabla_2^g = g \nabla_2 g^{-1}$, i.e. it acts by the standard action of gauge transformations on connections. This action thus complexifies to the standard action of $\mathfrak{G}_{\mathbb{C}}$ by *complex* gauge transformations on ∇_2 . (We emphasize that this is *not* the same as the action of $\mathfrak{G}_{\mathbb{C}}$ on $\mathcal{A}_1^{\mathbb{C}}$ which we considered above. Indeed, a real group action on a hyperkähler space complexifies differently in each complex structure.)

The holomorphic moment map for this action is

$$M_2 = \mu_3 + i\mu_1 = D\Phi + i\left(-F_D + \Phi \wedge \Phi - 2\pi i \frac{d}{K} \mathbf{1}\omega_C\right).$$
(6.55)

This funny-looking combination has a nice interpretation: the curvature of ∇_2 is

$$F_{\nabla_2} = F_D + iD\Phi - \Phi \wedge \Phi, \tag{6.56}$$

so the condition $M_2 = 0$ says that

$$F_{\nabla_2} = -2\pi \mathrm{i} \frac{d}{K} \mathbf{1} \omega_C. \tag{6.57}$$

When d = 0 this says that ∇_2 is a complex *flat* connection. More generally, ∇_2 is a complex Einstein connection.

Summing up: modulo questions of stability, we have in complex structure I_2

$$\mathcal{A}^H / / / \mathfrak{G} = \mathcal{A}_2^{\mathbb{C}} / / \mathfrak{G}_{\mathbb{C}}, \tag{6.58}$$

and the RHS deserves the name "moduli space of complex Einstein connections." Thus we have the remarkable situation that the single hyperkähler space $\mathcal{M}_{K,d}(C)$ has two very diffrent interpretations: in one complex structure it is a moduli space of Higgs bundles, in another it is a moduli space of complex Einstein connections.

What about structure I_3 ? In this structure we have a very similar story, except that now the holomorphic combination is $\dot{A} - i \star \dot{\Phi}$, so the complex connection we consider is

$$\nabla_3 = D + \mathbf{i} \star \Phi. \tag{6.59}$$

Exercise 6.7. Check that the vanishing of the moment map $M_3 = \mu_1 + i\mu_2$ implies that ∇_3 is a complex Einstein connection.

In fact this is the tip of a bigger iceberg: for any $\zeta \in \mathbb{C}^{\times}$ we can consider the connection

$$\nabla_{\zeta} = \zeta^{-1} \varphi + D + \zeta \varphi^{\dagger} \tag{6.60}$$

Exercise 6.8. Check that the full Hitchin equations (6.45) imply that ∇_{ζ} is a complex Einstein connection, for any $\zeta \in \mathbb{C}^{\times}$.

Exercise 6.9. Check that (again ignoring stability) we have, in complex structure I_{ζ} ,

$$\mathcal{A}^{H} / / / \mathcal{G} = \mathcal{A}_{\zeta}^{\mathbb{C}} / / \mathfrak{G}_{\mathbb{C}}.$$
(6.61)

So, ignoring stability, it seems that $\mathcal{M}_{K,d}(C)$ can be identified with a moduli space of complex Einstein connections *in many different ways*: indeed each complex structure I_{ζ} gives such an identification.

6.5 The case of Higgs line bundles

Example 6.14 (Moduli of degree zero Higgs line bundles). Now let us see how this works in the concrete example K = 1.

In this case things are particularly simple: End(E) is trivial, so the Higgs field $\varphi \in \Omega^{1,0}(C)$, and the brackets in the Hitchin equations (6.45) drop out, giving two decoupled equations:

$$\bar{\partial}\varphi = 0,$$
 (6.62a)

$$F_D = -2\pi i \frac{d}{K} \mathbf{1} \omega_C. \tag{6.62b}$$

Thus $\vec{\mu}^{-1}(0)$ just consists of pairs (D, φ) where *D* is an Einstein connection in the line bundle *E* and φ is a holomorphic 1-form on *C*.

Moreover the action of \mathfrak{G} on φ is trivial. Thus, after taking the quotient by \mathfrak{G} we just get

$$\mathcal{M}_{1,d}(C) = \mathcal{N}_{1,d}(C) \times H^{1,0}(C).$$
 (6.63)

We can view this as a trivial holomorphic vector bundle over $\mathcal{N}_{1,d}(C)$ whose fiber is $H^{1,0}(C)$. Moreover this trivial bundle has another name:

$$\mathcal{M}_{1,d}(C) = T^* \mathcal{N}_{1,d}(C).$$
 (6.64)

We have already analyzed $\mathcal{N}_{1,d}(C)$ at some length: it is a compact Kähler torus, which we described in various ways in subsection 5.4. What we have found here is that the cotangent bundle to this torus is canonically hyperkähler, and arises as the simplest example of a moduli space of Higgs bundles.

Example 6.15 (Moduli of degree zero Higgs line bundles over a torus). When *C* is a torus, and d = 0, all this becomes even more concrete. Extending our description of Jac *C* from Example 5.18, a general $(D, \Phi) \in \mathcal{M}_{1,0}(C)$ is gauge equivalent to

$$D = \mathbf{d} + \mathbf{i}\theta_A \mathbf{d}x + \mathbf{i}\theta_B \mathbf{d}y = \mathbf{d} + (2\operatorname{Im}\tau)^{-1}(\bar{\alpha}\mathbf{d}z - \alpha \mathbf{d}\bar{z}), \tag{6.65}$$

$$\Phi = a \mathrm{d}z - \bar{a} \mathrm{d}\bar{z},\tag{6.66}$$

where

$$\alpha = \theta_B - \tau \theta_A. \tag{6.67}$$

The coordinates $(\theta_A, \theta_B, \alpha)$ identify $\mathcal{M}_{1,0}(C)$ with $U(1)^2 \times \mathbb{C}$.

This looks a lot like the situation of Example 3.36: we are considering a translation invariant hyperkähler metric on $T^2 \times \mathbb{R}^2$. In fact, it is *exactly* the situation of Example 3.36. To see this, we can compute directly from (6.24) and (6.10):

$$\omega_1 = \mathrm{d}\theta_A \wedge \mathrm{d}\theta_B + 2\mathrm{i}(\mathrm{Im}\,\tau)\mathrm{d}a \wedge \mathrm{d}\bar{a},\tag{6.68}$$

$$\Omega_1 = 2\mathrm{d}\alpha \wedge \mathrm{d}a. \tag{6.69}$$

This indeed matches with Example 3.36, when we identify our coordinates $\theta_{A,B}$ with those in (3.120), and identify the functions

$$Z_A = a, \qquad Z_B = \tau a, \tag{6.70}$$

with the $Z_{A,B}$ in (3.121).

Now, we already studied the whole family of complex structures I_{ζ} in Example 3.36: for any $\zeta \in \mathbb{C}^{\times}$, we have an identification

$$(\mathcal{M}_{1,0}(C), I_{\zeta}) \simeq \mathbb{C}^{\times} \times \mathbb{C}^{\times}$$
(6.71)

given by the explicit functions

$$\mathcal{X}_{A,B} = \exp\left(\zeta^{-1}Z_{A,B} + \mathrm{i}\theta_{A,B} + \zeta\bar{Z}_{A,B}\right).$$
(6.72)

These formulas have a simple interpretation. $Z_{A,B}$ are the integrals of $\varphi = adz$ over the cycles A, B; similarly $\overline{Z}_{A,B}$ are the integrals of φ^{\dagger} . Thus, if we consider the connection ∇_{ζ} given by (6.60), its holonomy around the cycles A, B is $\mathcal{X}_{A,B}$.

Exercise 6.10. Verify (6.68) and (6.69).

6.6 Stability for Higgs bundles and flat connections

Definition 6.16 (Stable Higgs bundle). A Higgs bundle $(E, \bar{\partial}_E, \varphi)$ over C is called:

- *stable* if, for every sub-Higgs bundle $E' \subset E$, we have $\mu(E') < \mu(E)$,
- *polystable* if *E* is a direct sum of stable Higgs bundles of the same slope,
- *semistable* if, for every every φ -invariant holomorphic subbundle $E' \subset E$, we have $\mu(E') \leq \mu(E)$.

Note that the slope μ is defined just as it was for a holomorphic vector bundle; the only role of the Higgs field is to restrict the allowed subbundles.

There are a few Higgs bundles which we can describe particularly concretely:

Example 6.17 (Zero Higgs field). For any holomorphic vector bundle $(E, \bar{\partial})$ we obtain a trivial example of a Higgs bundle by taking $(E, \bar{\partial}, \varphi = 0)$. It is a stable Higgs bundle if and only if $(E, \bar{\partial})$ is stable.

Example 6.18 (Higgs line bundles). If K = 1 then a Higgs bundle just means a holomorphic line bundle plus a holomorphic 1-form, $\varphi \in H^0(K_C)$. This is always stable.

Example 6.19 ("Hitchin section" for GL(2)**).** Fix a line bundle \mathcal{L} on C. Then consider the holomorphic vector bundle

$$E = \mathcal{L} \otimes K_C \oplus \mathcal{L}. \tag{6.73}$$

For any $(\phi_1, \phi_2) \in H^0(K_C) \oplus H^0(K_C^2)$, we can equip *E* with the Higgs field

$$\varphi = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & \phi_1 \end{pmatrix} \in \begin{pmatrix} \operatorname{Hom}(\mathcal{L} \otimes K_C, \mathcal{L} \otimes K_C) \otimes K_C & \operatorname{Hom}(\mathcal{L}, \mathcal{L} \otimes K_C) \otimes K_C \\ \operatorname{Hom}(\mathcal{L} \otimes K_C, \mathcal{L}) \otimes K_C & \operatorname{Hom}(\mathcal{L}, \mathcal{L}) \otimes K_C \end{pmatrix} = \begin{pmatrix} K_C & K_C^2 \\ \mathcal{O} & K_C \end{pmatrix}$$
(6.74)

and thus obtain a Higgs bundle.

Exercise 6.11. Show that the Higgs bundles in Example 6.19 are stable (despite the fact that the underlying vector bundle *E* is unstable), for any (ϕ_1, ϕ_2) , if *C* has genus $g \ge 2$.

For the next example we need a little bit of setup:

Definition 6.20 (Spin structure). A *spin structure* on *C* is a holomorphic line bundle \mathcal{L} equipped with an isomorphism $\mathcal{L}^2 \simeq K_C$.

Exercise 6.12. Show that the set of spin structures on *C* (up to the natural notion of equivalence) is a torsor for $H^1(C, \mathbb{Z}/2\mathbb{Z})$. In particular, there are 2^{2g} inequivalent spin structures on a genus *g* surface.

Exercise 6.13. Show that Definition 6.20 is equivalent to your favorite definition of spin structure, if you have one.

Example 6.21 ("Hitchin section" for SL(2)**).** Fix a spin structure \mathcal{L} on C. Then consider the holomorphic vector bundle

$$E = \mathcal{L} \oplus \mathcal{L}^{-1}. \tag{6.75}$$

For any $\phi_2 \in H^0(K_C^2)$, we can equip *E* with the Higgs field

$$\varphi = \begin{pmatrix} 0 & \phi_2 \\ 1 & 0 \end{pmatrix} \in \begin{pmatrix} \operatorname{Hom}(\mathcal{L}, \mathcal{L}) \otimes K_C & \operatorname{Hom}(\mathcal{L}^{-1}, \mathcal{L}) \otimes K_C \\ \operatorname{Hom}(\mathcal{L}, \mathcal{L}^{-1}) \otimes K_C & \operatorname{Hom}(\mathcal{L}^{-1}, \mathcal{L}^{-1}) \otimes K_C \end{pmatrix} = \begin{pmatrix} K_C & K_C^2 \\ \mathcal{O} & K_C \end{pmatrix}$$
(6.76)

and thus obtain a Higgs bundle.

Definition 6.22 (Reductive complex connections). A connection *D* in *E* is *reductive* if, whenever $E' \subset E$ is *D*-invariant, there is a decomposition $E = E' \oplus E''$ where E'' is also *D*-invariant.

(So "reductive" is the complex-connection analogue of "polystable," in the same way as "irreducible" is the analogue of "stable." Note that the phenomenon of non-reductivity

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is peculiar to complex connections: for unitary ones we could always take E'' to be the orthocomplement of E'.)

Exercise 6.14. Suppse $\rho : \pi_1(C) \to GL(K, \mathbb{C})$ is a representation for which all $\rho(\wp)$ are upper-triangular matrices, at least one of which is not diagonal. Show that ρ corresponds to a flat complex connection which is not reductive.

6.7 Gauge-theoretic meaning of stability

Now let us consider the case of general ranks K > 1. We would like to compare $\mathcal{A}^H / / / \mathfrak{G}$ to $\mathcal{A}_{\zeta}^{\mathbb{C}} / / \mathfrak{G}_{\mathbb{C}}$. The problem we face is familiar: for each complex structure I_{ζ} we have a complex moment map M and a real moment map μ_{ζ} ; after restricting to the locus $M^{-1}(0)$, we need to understand the intersection between the $\mathfrak{G}_{\mathbb{C}}$ orbits and the locus $\mu_{\zeta}^{-1}(0)$. This amounts to proving an analogue of Theorem 5.35 in this doubled context. More precisely we need two such theorems, one for $\zeta = 0$ and one for $\zeta \neq 0$: these are Theorem 6.23 and Theorem 6.25 below.

Theorem 6.23 (Polystable Higgs bundles admit harmonic doubled connections). We have:

- For any [(D,Φ)] ∈ M^s_{K,d}(C) the pair (∂̄_D, φ) induces the structure of *stable* Higgs bundle on *E*. Conversely, any stable Higgs bundle structure on *E* is equivalent to (∂̄_D, φ) for a unique [(D,Φ)] ∈ M^s_{K,d}(C).
- For any $[(D, \Phi)] \in \mathcal{M}_{K,d}(C)$ the pair $(\bar{\partial}_D, \varphi)$ induces the structure of *polystable* Higgs bundle on *E*. Conversely, any polystable Higgs bundle structure on *E* is equivalent to $(\bar{\partial}_D, \varphi)$ for a unique $[(D, \Phi)] \in \mathcal{M}_{K,d}(C)$.

Proof. This is proven by analytic means, broadly similar to those in the proof of Theorem 5.35. I will not try to treat it here. The case of K = 2 can be found in [5]; for general K see [39].

Example 6.24 (Uniformization theorem via Higgs bundle). Fix a spin structure \mathcal{L} on C. We consider the special case of Example 6.21 with $\phi_2 = 0$:

$$E = \mathcal{L} \oplus \mathcal{L}^{-1}, \qquad \varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
 (6.77)

According to Theorem 6.23 there exists a harmonic doubled connection in the $\mathfrak{G}_{\mathbb{C}}$ -orbit of this Higgs bundle, or equivalently, a Hermitian metric *h* on *E* such that

$$F_{D_h} + [\varphi, \varphi^{\dagger_h}] = 0.$$
 (6.78)

One can show [...] that this metric is actually diagonal with respect to the decomposition of *E*: we can write it

$$h = \begin{pmatrix} g^{-\frac{1}{2}} & 0\\ 0 & g^{\frac{1}{2}} \end{pmatrix}$$
(6.79)

where g is a metric on TC. Then in a local coordinate system, (6.78) becomes

$$-\frac{1}{2}\bar{\partial}_{\bar{z}}\partial_{z}\log g_{z\bar{z}} + g_{z\bar{z}} = 0.$$
(6.80)

This equation says that the metric *g* has constant curvature -4. So Theorem 6.23 applied to this Higgs bundle implies the uniformization theorem.

Let us explore this example a bit further. The harmonic doubled connection (D, Φ) induces a flat connection

$$\nabla = D + i\Phi \tag{6.81}$$

[...]

Theorem 6.25 (Reductive connections can be made harmonic). Fix some $\zeta \in \mathbb{C}^{\times}$.

- For any $[(D, \Phi)] \in \mathcal{M}^{s}_{K,d}(C)$ the complex Einstein connection $\nabla = \zeta^{-1}\varphi + D + \zeta \varphi^{\dagger}$ is irreducible. Conversely, any irreducible complex Einstein connection ∇ on E arises as $\nabla = \zeta^{-1}\varphi + D + \zeta \varphi^{\dagger}$ for a unique $[(D, \Phi)] \in \mathcal{M}^{s}_{K,d}(C)$.
- For any [(D, Φ)] ∈ M_{K,d}(C) the complex Einstein connection ∇ = ζ⁻¹φ + D + ζφ[†] is reductive. Conversely, any reductive complex Einstein connection ∇ on *E* arises as ∇ = ζ⁻¹φ + D + ζφ[†] for a unique [(D, Φ)] ∈ M_{K,d}(C).

Proof. This is also an analytic problem: see [40] for the case K = 2 and [41] more generally. (More precisely, they prove it for $\zeta = 1$, but the extension to $\zeta \in \mathbb{C}^{\times}$ should be straightforward.)

[summarize the situation!]

6.8 The Hitchin fibration

Definition 6.26 (Hitchin base). The Hitchin base is the complex vector space

$$\mathcal{B} = \mathcal{B}_K(C) = \bigoplus_{i=1}^K H^0(C, K_C^{\otimes i}).$$
(6.82)

Definition 6.27 (Hitchin fibration). The Hitchin fibration is the map

$$\rho: \mathcal{M}_{K,d}(C) \to \mathcal{B}_K(C) \tag{6.83}$$

defined as follows. Given $([D, \varphi]) \in \mathcal{M}$ we consider the characteristic polynomial

$$\det(\lambda - \varphi) = \lambda^{K} + \sum_{i=1}^{K} \phi_{i} \lambda^{K-i}.$$
(6.84)

Then

$$\rho([(D,\varphi)]) = (\phi_1,\phi_2,\ldots,\phi_K) \in \mathcal{B}.$$
(6.85)

Definition 6.28 (Spectral curve). Given a point $\vec{\phi} \in \mathcal{B}$ we define $\Sigma_{\vec{\phi}}$ to be the curve

$$\Sigma_{\vec{\phi}} = \{\lambda^{K} + \sum_{i=1}^{K} \phi_{i} \lambda^{K-i} = 0\} \subset T^{*}C.$$

$$(6.86)$$

Abusing notation we also write

$$\Sigma_{\varphi} = \Sigma_{\rho([D,\varphi])}.$$
(6.87)

Then informally $\Sigma_{\varphi} \subset T^*C$ is a *K*-sheeted covering of *C*, whose sheets over $z \in C$ are the *K* eigenvalues λ_i of $\varphi(z)$.

Proposition 6.29 (Hitchin fibration is I_1 **-holomorphic).** The Hitchin fibration ρ is holomorphic as a map $(\mathcal{M}, I_1) \rightarrow \mathcal{B}$.

Proof. [...]

6.9 The smooth locus

There is a large domain inside of \mathcal{B} where the Hitchin fibration is "nice". To describe it we consider the *discriminant* of the equation (6.86). This is a polynomial $\Delta_{\vec{\phi}}$ in the ϕ_i , with the property

$$\Delta = \prod_{i>j} (\lambda_i - \lambda_j)^2 \tag{6.88}$$

where λ_i are the roots of (6.86). For example, if K = 2 then (6.86) becomes $\lambda^2 + \phi_1 \lambda + \phi_2 = 0$, which has discriminant

$$\Delta_{\vec{\phi}} = \phi_1^2 - 4\phi_2. \tag{6.89}$$

Globally over *C*, (6.89) is a section of $K_C^{\otimes 2}$. For general *K*, (6.88) implies that $\Delta_{\vec{\phi}}$ is a section of $K_C^{\otimes K(K-1)}$. The zeroes of $\Delta_{\vec{\phi}}$ are the places on *C* where the sheets λ_i collide, i.e. they are the places where the curve $\Sigma_{\vec{\phi}}$ is ramified as a cover of *C*.

Let us explore a bit more closely what happens near the ramification points. The simplest behavior occurs at a simple zero of $\Delta_{\vec{\phi}}$. Indeed at such a zero the local behavior of (6.86) is like that of the equation $y^2 = z$, which has two solutions $y = \pm \sqrt{z}$, colliding at the ramification point y = z = 0; note that the curve $y^2 = z$ is smooth even at this point. This motivates the following definition:

Definition 6.30 (Smooth locus and discriminant locus). Let $\Delta_{\vec{\phi}} \in H^0(C, K_C^{\otimes K(K-1)})$ denote the discriminant of the equation (6.86). The *smooth locus* $\mathcal{B}' \subset \mathcal{B}$ consists of all $\vec{\phi}$ for which $\Delta_{\vec{\phi}}$ has only simple zeroes. The *discriminant locus* is the complement of the smooth locus.

Exercise 6.15. Show that, if $\vec{\phi} \in \mathcal{B}'$, $\Sigma_{\vec{\phi}}$ is a smooth curve, which is a branched *K*-fold cover of *C*, and all branch points have ramification index 2.

Proposition 6.31 (Generic spectral curves have 2K(K-1)(g-1) **branch points and genus** $1 + K^2(g-1)$). If $\vec{\phi} \in \mathcal{B}'$ then the covering $\Sigma_{\vec{\phi}} \to C$ has $n_b = 2K(K-1)(g-1)$ branch points and has genus $1 + K^2(g-1)$.

Proof. To count the number of branch points, use the fact that deg $K_C = -\chi(C) = 2g - 2$ (Gauss-Bonnet theorem) and the discriminant $\Delta_{\vec{\phi}}$ is a holomorphic section of $K_C^{K(K-1)}$, with only simple zeroes.

To get the genus, use the Riemann-Hurwitz formula which says

$$\chi(\Sigma_{\vec{\phi}}) = K\chi(C) - n_b = K^2\chi(C).$$
(6.90)

Example 6.32 (Simplest spectral curves). The simplest nontrivial case is the case K = 2, g = 2. In this case Proposition 6.31 says the generic spectral curves have genus 5. Thus even in this simple case we are dealing with a somewhat complicated family of curves.

Proposition 6.33 (Generic fibers of the Hitchin fibration are shifted Jacobians). If $\vec{\phi} \in \mathcal{B}'$, then the fiber $\rho^{-1}(\vec{\phi}) \subset \mathcal{M}_{K,d}(C)$ is the compact torus $\mathcal{N}_{1,d'}(\Sigma_{\vec{\phi}})$ where

$$d' = d - K(K - 1)(g - 1).$$
(6.91)

This identification is holomorphic in structure I_1 .

Proof. First we construct a map

$$\pi^{-1}(\vec{\phi}) \to \mathcal{N}_{1,d'}(\Sigma_{\vec{\phi}}). \tag{6.92}$$

So, suppose given a Higgs bundle structure $(\bar{\partial}_E, \varphi) \in \pi^{-1}(\vec{\phi})$. Then the spectral curve $\Sigma_{\vec{\phi}}$ is given by the characteristic polynomial of φ . Away from the branch locus $\pi^{-1}(\Delta_{\varphi}^{-1}(0)) \subset \Sigma_{\vec{\phi}}$, we can define a holomorphic line bundle \mathcal{L} over $\Sigma_{\vec{\phi}}$ by

$$\mathcal{L}_{\lambda} = \ker(\varphi - \lambda) \subset \pi^* E. \tag{6.93}$$

More simply put: $\Sigma_{\vec{\phi}}$ consists of the *eigenvalues* of φ , and it carries a line bundle \mathcal{L} consisting of the *eigenspaces*.

The tricky point is to extend \mathcal{L} to a line bundle over the whole $\Sigma_{\vec{\phi}}$ including the branch locus. If we are algebraically minded we can consider the sheaf ker($\varphi - \lambda$) and verify directly that it is locally free of rank 1, thus it is the sheaf of sections of a holomorphic line bundle. I will describe the same thing in a more analytic language. For notational simplicity I consider the special case where K = 2. Then around a branch point we can always find a local coordinate and gauge in which

$$\varphi = \left[f(z)\mathbf{1} + \begin{pmatrix} 0 & 1\\ z & 0 \end{pmatrix} \right] \mathrm{d}z. \tag{6.94}$$

Then $\Sigma_{\vec{\phi}}$ is locally given by $\{y^2 = z\}$ (where *y* is a local coordinate on T^*C , $\lambda = (y + f(z)) dz$). z = 0 is a branch point. The line bundle \mathcal{L} away from y = 0 can be written as

$$\mathcal{L}_{y} = \left\langle \begin{pmatrix} 1 \\ y \end{pmatrix} \right\rangle \subset E_{y^{2}} \tag{6.95}$$

and thus it extends just fine over the point y = 0.

Finally we want to compute the degree of \mathcal{L} . For this we could use the Grothendieck-Riemann-Roch theorem, but it will be useful to get it in a more hands-on way. So, fix some connection D in E. By projection on the two eigenspaces, D induces a new connection D^{\parallel} in E, defined away from the branch points. Equivalently we can view D^{\parallel} as a connection in \mathcal{L} , again away from the branch points. We have

$$\operatorname{Tr} F_{D^{\parallel}} = \operatorname{Tr} F_D. \tag{6.96}$$

It follows that

$$\int_{\Sigma'} F_{D\parallel} = \int_C \operatorname{Tr} F_D, \tag{6.97}$$

where Σ' is Σ with the branch points deleted. However, D^{\parallel} does not extend over the branch points, so we cannot conclude from (6.97) that deg $E = \text{deg } \mathcal{L}$.

Let us see how to repair this difficulty. For simplicity suppose that *D* is trivial in a neighborhood of each branch point, in our local gauge above. Then the projected connection has

$$D_y^{\parallel} \begin{pmatrix} 1\\ y \end{pmatrix} = \begin{pmatrix} 0\\ 1 \end{pmatrix}^{\parallel} = \begin{pmatrix} 0\\ 1 \end{pmatrix} + \frac{1}{2y} \begin{pmatrix} 1\\ -y \end{pmatrix} = \frac{1}{2y} \begin{pmatrix} 1\\ y \end{pmatrix}$$
(6.98)

Thus, relative to the local trivialization of \mathcal{L} by the vector $\begin{pmatrix} 1 \\ y \end{pmatrix}$, D_y^{\parallel} is given by $d + \frac{1}{2} \frac{dy}{y}$. In particular, its holonomy around the branch point y = 0 is -1. By modifying D^{\parallel} in a small disc around y = 0 we we obtain a connection which does extend over y = 0 and has $\int F = \frac{1}{2} \oint \frac{dy}{y} = \pi i$ in that disc. Make this modification at all of the 2K(K-1)(g-1) branch points, to get a new connection D' in \mathcal{L} ; then (6.97) is replaced by

$$\int_{\Sigma} F_{D'} = \int_{C} \operatorname{Tr} F_{D} + \pi i K (K-1)(g-1)$$
(6.99)

which gives the desired formula for d'.

The inverse map

$$\mathcal{N}_{1,d'}(\Sigma_{\vec{\phi}}) \to \pi^{-1}(\vec{\phi}) \tag{6.100}$$

is similar: given the line bundle \mathcal{L} over $\Sigma_{\vec{\phi}}$ we construct a Higgs bundle away from the branch locus by pushforward, and construct directly its extension over the branch locus.

Exercise 6.16. Verify the assertion above, that when K = 2 and $\vec{\phi} \in \mathcal{B}'$, around a branch point we can always find a local coordinate and gauge in which

$$\varphi = \left[f(z)\mathbf{1} + \begin{pmatrix} 0 & 1\\ z & 0 \end{pmatrix} \right] \mathrm{d}z. \tag{6.101}$$

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Exercise 6.17. Extend the proof of Proposition 6.33 to general *K*.

6.10 The other fibers

We have just explained that the generic fiber $\rho^{-1}(\vec{\phi})$ of the Hitchin fibration ρ is a compact complex torus, the Jacobian of a smooth spectral curve $\Sigma_{\vec{\phi}}$. The complex structures of these fibers vary as the complex structure of $\Sigma_{\vec{\phi}}$ varies.

It remains to understand the fibers lying over the discriminant locus. These fibers are generally *not* compact complex tori. When $\Sigma_{\vec{\phi}}$ is reduced and connected, $\rho^{-1}(\vec{\phi})$ is the *compactified Jacobian* of $\Sigma_{\vec{\phi}}$; see the Appendix of [42] for a useful summary and more precise statement.

The next two statements give at least a little information about the general fibers: they are compact and nonempty.

Proposition 6.34 (Hitchin fibration is proper). The Hitchin fibration is proper.

Proof. This is proven in [5] for K = 2, and in [43] for general K, by gauge-theoretic methods (using Uhlenbeck compactness). Also see [44] for a more algebraic method.

Proposition 6.35 (Hitchin fibration is surjective). The Hitchin fibration is surjective.

Proof. To prove this it is sufficient to exhibit one point in each fiber. For K = 2, the Higgs bundles given in Example 6.19 will do the job. For larger *K* there is a similar construction [...]

6.11 The nilpotent cone

The most interesting fiber is the one over $0 \in \mathcal{B}$:

Definition 6.36 (Nilpotent cone). The *nilpotent cone* is the fiber $\rho^{-1}(0) \subset \mathcal{M}_{K,d}(C)$.

The nilpotent cone in particular contains $\mathcal{N}_{K,d}(C)$ (polystable bundles with zero Higgs field), but it contains more.

Proposition 6.37 ($\mathcal{M}_{K,d}(C)$ **deformation retracts to the nilpotent cone).** When (K, d) = 1 the nilpotent cone is a deformation retract of $\mathcal{M}_{K,d}(C)$.

Proof. [Hausel thesis] Consider the function $\mu : \mathcal{M}_{Kd}(C) \to \mathbb{R}$ given by

$$\mu(D,\Phi) = \frac{1}{2} \int_C \operatorname{Tr}(\Phi \wedge \star \Phi).$$
(6.102)

 μ is a proper map [why?] with finitely many critical points and an absolute minimum at 0. Moreover we have

$$d\mu(\dot{A}, \dot{\Phi}) = \int_{C} \text{Tr}(\dot{\Phi} \wedge \star \Phi)$$
(6.103)

$$=\omega_1(X,(\dot{A},\dot{\Phi})) \tag{6.104}$$
where *X* is the vector field

$$X = (\dot{A} = 0, \dot{\Phi} = -\star \Phi).$$
(6.105)

Thus μ is a moment map generating (with respect to ω_1) an action of U(1) on $\mathcal{M}_{K,d}(C)$,

$$(D,\varphi) \to (D,e^{i\vartheta}\varphi).$$
 (6.106)

This action complexifies to an action of \mathbb{C}^{\times} , holomorphic in structure I_1 ,

$$(D, \varphi) \to (D, \lambda \varphi).$$
 (6.107)

The \mathbb{R}^{\times} part of this action is the gradient flow of μ .

The nilpotent cone can be characterized as the set of points for which this \mathbb{C}^{\times} action has limits both as $\lambda \to 0$ and as $\lambda \to \infty$. Then results of Kirwan [...] give the desired retraction.

Exercise 6.18. Suppose (X, ω, I) is a Kähler manifold, with a function $\mu : X \to \mathbb{R}$ generating a vector field *Z*. Show that *IZ* is the (Riemannian) gradient of μ .

Exercise 6.19. Prove that the nilpotent cone is the set of points for which the \mathbb{C}^{\times} action (6.107) has limits both as $\lambda \to 0$ and as $\lambda \to \infty$.

Exercise 6.20. Compute the function μ on $\mathcal{M}_{1,d}(C)$ explicitly when *C* is a torus, and describe its gradient flow.

But the nilpotent cone is also an incredibly singular fiber from our point of view. We will restrict our attention mainly to the "boring" smooth torus fibers over \mathcal{B}' .

6.12 More formalities

We can think of $\mathcal{M}_{K,d}^{s}(C)$ as a kind of partial compactification of $T^*\mathcal{N}_{K,d}^{s}(C)$:

Proposition 6.38. $T^*\mathcal{N}^s_{K,d}(C)$ is an open dense subset of $\mathcal{M}^s_{K,d}(C)$.

Proof. [...]

Given the surjectivity of the Hitchin fibration it is evidently hopeless to ask for either $\mathcal{M}_{K,d}^{s}(C)$ or $\mathcal{M}_{K,d}(C)$ to be compact. A partial substitute is provided by the next two facts:

Exercise 6.21. Show that for (K, d) = 1 there are no strictly polystable Higgs bundles, i.e.

$$\mathcal{M}_{K,d}(C) = \mathcal{M}^{s}_{K,d}(C). \tag{6.108}$$

Proposition 6.39 (Completeness of hyperkähler metric on $\mathcal{M}^{s}_{K,d}(C)$ **when** (K,d) = 1**).** When (K,d) = 1 the hyperkähler metric on $\mathcal{M}^{s}_{K,d}(C)$ is complete.

Proof. A proof for K = 2 can be found in [5]. [more generally?]

6.13 The integrable system

Proposition 6.40 (\mathcal{M} is a complex integrable system). The fibers of the Hitchin fibration over \mathcal{B}' have dimension equal to that of the base \mathcal{B} , and are Lagrangian with respect to Ω_1 .

Proof. The fiber over $\vec{\phi} \in \mathcal{B}'$ is the Jacobian of the smooth spectral curve $\Sigma_{\vec{\phi}}$, and thus its dimension is the genus of $\Sigma_{\vec{\phi}}$, which we computed in Proposition 6.31 as

$$g_{\Sigma} = 1 + K^2(g - 1). \tag{6.109}$$

This is half the dimension of $\mathcal{M}_{K,d}(C)$. Since the fiber dimension and base dimension add to the full dimension, we get that the two are equal.

Next how do we see that the fibers are Lagrangian? Fix some *n* with $1 \le n \le K$ and some $\alpha \in \Omega^{0,1}(C, K^{-n+1})$, and consider the function

$$f_{\alpha} = \int_{C} \alpha \operatorname{Tr}(\varphi^{n}) \tag{6.110}$$

Then we have

$$df_{\alpha}(\dot{A}, \dot{\Phi}) = \int_{C} \alpha \operatorname{Tr}(\dot{\varphi} \varphi^{n-1}) = 2i \int_{C} \operatorname{Tr}\left(\dot{\varphi} \cdot -\frac{i}{2} \alpha \varphi^{n-1}\right)$$
(6.111)

$$= \Omega_1 \left((\dot{A}_{\bar{z}} = -\frac{i}{2} \alpha \varphi^{n-1}, \dot{\varphi} = 0), (\dot{A}, \dot{\Phi}) \right)$$
(6.112)

Said otherwise, up to $\mathfrak{G}_{\mathbb{C}}$ action we have

$$\Omega_1^{-1}(df_{\alpha}) = \left(\dot{A}_{\bar{z}} = -\frac{i}{2}\alpha\varphi^{n-1}, \dot{\varphi} = 0\right).$$
(6.113)

In particular, the vector fields $v_{\alpha} = \Omega_1^{-1}(df_{\alpha})$ obtained in this way are all tangent to the fiber (since they all have $\dot{\phi} = 0$) and all have $\Omega_1(v_{\alpha}, v_{\beta}) = 0$ (for the same reason).

Finally note that the df_{α} span $T^*\mathcal{B}$: indeed (6.110) is a nondegenerate pairing between $H^0(C, K^n)$ and $\mathcal{H}^{0,1}(K^{-n+1})$, so choosing enough α 's, the f_{α} give a global coordinate system on \mathcal{B} . Using this and the nondegeneracy of Ω_1 we get dually that the v_{α} span the tangent space to the fiber. Thus we conclude that $\Omega_1 = 0$ when restricted to the fiber, as desired.

Exercise 6.22. Use the Riemann-Roch formula and Kodaira vanishing to verify directly that the complex dimension of \mathcal{B} is $K^2(g-1) + 1$.

7 Metric formulas

In this section we will describe the proposal of [22, 45, 46], which is aimed at describing the hyperkähler metric on $\mathcal{M}_{K,d}(C)$ in a concrete way.

7.1 The semiflat picture

We first describe a certain simple, explicit hyperkähler metric, which is expected to be very close to the true hyperkähler metric "near the ends" of $\mathcal{M}_{K,d}(C)$, i.e. when we go out along a generic path to ∞ .

Over the smooth locus \mathcal{B}' we have a local system of lattices

$$\Gamma_{\vec{\phi}} = H_1(\Sigma_{\vec{\phi}}, \mathbb{Z})$$

with a canonical function

$$Z:\Gamma\to\mathbb{C}$$

given by

$$Z_{\gamma} = \oint_{\gamma} \lambda$$

(with λ the tautological 1-form on T^*C) and equipped with the intersection pairing

$$\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \to \mathbb{Z}.$$

By Poincare duality this pairing induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma^*$. Thus we also get a dual pairing

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle:\Gamma^*\times\Gamma^*\to\mathbb{Z}$$

Writing $\Gamma^*_{\mathbb{C}} = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}$, the dual pairing induces a *symmetric* pairing

$$\langle\!\langle \cdot \wedge \cdot \rangle\!\rangle : \Omega^1(\Gamma^*_{\mathbb{C}}) \times \Omega^1(\Gamma^*_{\mathbb{C}}) \to \Omega^2(\Gamma^*_{\mathbb{C}}).$$

Let $\mathcal{M}' = \rho^{-1}(\mathcal{B}') \subset \mathcal{M}$. We want to write coordinate formulas for the hyperkähler metric over \mathcal{M}' . There is an awkward point: the smooth torus fibers are not canonically trivialized, because of the degree $d' = d - K(K - 1)(g - 1) \neq 0$. In particular it is not immediately obvious how to identify nearby torus fibers, which we certainly need to do if we want to erect a coordinate system.

We know how to deal with this problem in certain special cases. One particularly interesting case is

$$d = -K(g-1) = \frac{1}{2}\chi(C).$$
(7.1)

So now we specialize to that case. Then $d' = -K^2(g-1) = \frac{1}{2}\chi(\Sigma)$. This is not 0 but it is "almost as good," in the following sense. Suppose we choose a spin structure $K_{\Sigma_{\vec{\phi}}}^{1/2}$ on some $\Sigma_{\vec{\phi}}$. This also induces spin structures on nearby $\Sigma_{\vec{\phi}}$, with $\vec{\phi}$ lying in a contractible neighborhood $U \subset \mathcal{B}'$. By tensoring with $K_{\Sigma_{\vec{\phi}}}^{1/2}$ we can identify the torus fibers over Uwith Jac $\Sigma_{\vec{\phi}} = \operatorname{Hom}(H_1(\Sigma_{\vec{\phi}}, \mathbb{Z}), U(1))$. Thus we have evaluation maps

$$\theta_{\gamma}: \rho^{-1}(U) \to \mathbb{R}/2\pi\mathbb{Z}.$$

These are angular coordinates on the torus fibers.

Fortunately, the dependence on the choice of spin structure is mild:

Exercise 7.1. Verify that changing the choice of spin structure on $\Sigma_{\vec{\phi}}$ shifts the coordinate functions θ_{γ} by constants (indeed integer multiples of π .)

In particular, despite this ambiguity, the 1-form $d\theta \in \Omega^1(\Gamma^*_{\mathbb{R}})$ is canonically and globally defined.

To construct our approximate hyperkähler metric on $\mathcal{M}_{K,d}(C)$ we need some warmups.

Proposition 7.1 (Positivity of semiflat 2-forms). The 2-form $\langle\!\langle d\theta \wedge d\theta \rangle\!\rangle$ is a positive form on each torus fiber $\rho^{-1}(\vec{\phi}), \vec{\phi} \in \mathcal{B}'$. The 2-form $-\langle\!\langle dZ \wedge d\bar{Z} \rangle\!\rangle$ is positive on \mathcal{B}' .

Proof. [...]

Proposition 7.2 ("Griffiths transversality" for spectral curves). The 1-form $dZ \in \Omega^1(\Gamma^*_{\mathbb{C}})$ obeys

$$\langle\!\langle \mathrm{d}Z \wedge \mathrm{d}Z \rangle\!\rangle = 0. \tag{7.2}$$

Proof. A tangent vector v to \mathcal{B} induces a holomorphic section s_v of the normal bundle $N(\Sigma_{\vec{\phi}})$. The variation

$$\mathrm{d} Z_{\gamma}(v) = \oint_{\gamma} \iota_{s_v} \Omega$$

where $\Omega = d\lambda$ is the holomorphic symplectic form on T^*C .

Now there is the "Riemann bilinear identity" for closed 1-forms on $\Sigma_{\vec{\phi}}$, which says that wedge product in de Rham cohomology is dual to intersection in homology, i.e.

$$\langle\!\langle \alpha,\beta\rangle\!\rangle = \int_{\Sigma_{\vec{\phi}}} \alpha \wedge \beta$$

where on the left we view α , β as elements in $\Gamma^*_{\mathbb{C}}$ by integration.

In our case what we have shown above is that $dZ \in \Gamma^*_{\mathbb{C}}$ corresponds to the 1-form $\iota_{s_n}\Omega$. Thus we get

$$\langle\!\langle \mathrm{d} Z(v),\mathrm{d} Z(v')
angle\!
angle = \int_{\Sigma_{ec{\phi}}} \iota_{s_v} \Omega \wedge \iota_{s_{v'}} \Omega$$

which vanishes for degree reasons.

Exercise 7.2. Interpret Proposition 7.2 as saying that *Z* locally embeds \mathcal{B} as a complex Lagrangian submanifold of a complex symplectic vector space.

Now we are ready to define an approximate version of the hyperkähler metric on $\mathcal{M}_{K,d}(C)$. Still working on the patch U, let Γ_U denote the lattice of global sections of Γ over U; then for $\gamma \in \Gamma_U$ introduce a function

$$\mathcal{X}_{\gamma}^{\mathrm{sf}}:\rho^{-1}(U)\times\mathbb{C}^{\times}\to\mathbb{C}^{\times}$$
(7.3)

by

$$\mathcal{X}_{\gamma}^{\mathrm{sf}}(\zeta) = \exp\left(\zeta^{-1}Z_{\gamma} + \mathrm{i}\theta_{\gamma} + \zeta\bar{Z}_{\gamma}\right). \tag{7.4}$$

Also introduce a complex 2-form

$$\Omega^{\rm st}(\zeta) = i \langle\!\langle d \log \mathcal{X}(\zeta), d \log \mathcal{X}(\zeta) \rangle\!\rangle.$$
(7.5)

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While the functions $\mathcal{X}_{\gamma}^{\text{sf}}$ depended on the patch U, the form $\Omega^{\text{sf}}(\zeta)$ does not: it is global on \mathcal{M}' (though it does not have any reason to extend to the singular fibers!)

Now we are in a situation very close to that of Example 3.39, with the difference that Γ now has rank 2*n* instead of 2, and we have introduced an additional global "twisting" from the local choices of spin structures.

Proposition 7.3 (Semiflat hyperkähler metric on \mathcal{M}' **exists).** There exists a hyperkähler metric on \mathcal{M}' for which the holomorphic symplectic form is $\Omega^{\text{sf}}(\zeta)$.

Proof. Expanding $\Omega^{sf}(\zeta)$ directly from (7.5) gives

$$\Omega(\zeta) = i\zeta^{-2} \langle\!\langle dZ \wedge dZ \rangle\!\rangle - \zeta^{-1} \langle\!\langle dZ \wedge d\theta \rangle\!\rangle + i \langle\!\langle dZ \wedge d\bar{Z} \rangle\!\rangle - i \langle\!\langle d\theta \wedge d\theta \rangle\!\rangle - \zeta \langle\!\langle d\bar{Z} \wedge d\theta \rangle\!\rangle + i\zeta^2 \langle\!\langle d\bar{Z} \wedge d\bar{Z} \rangle\!\rangle.$$
(7.6)

Fortunately, Proposition 7.2 says that the terms at order ζ^2 and ζ^{-2} vanish, so that we get

$$\Omega(\zeta) = -\zeta^{-1} \langle\!\langle dZ \wedge d\theta \rangle\!\rangle + i \langle\!\langle dZ \wedge d\bar{Z} \rangle\!\rangle - i \langle\!\langle d\theta \wedge d\theta \rangle\!\rangle - \zeta \langle\!\langle d\bar{Z} \wedge d\theta \rangle\!\rangle, \qquad (7.7)$$

which using (3.80) gives the candidate symplectic forms

$$\Omega_1 = -2\langle\!\langle dZ \wedge d\theta \rangle\!\rangle, \qquad \omega_1 = -\langle\!\langle dZ \wedge d\bar{Z} \rangle\!\rangle + \langle\!\langle d\theta \wedge d\theta \rangle\!\rangle. \tag{7.8}$$

Then the same arguments as in Exercise 3.32 show that these forms actually come from a hyperkähler structure. $\hfill \Box$

A useful way of thinking of the \mathcal{X}_{γ} is that they are the components of a single map

$$\mathcal{X}: U \times \mathbb{C}^{\times} \to T_U = \operatorname{Hom}(\Gamma_U, \mathbb{C}^{\times})$$
(7.9)

In other words: suppose we fix a point $(D, \Phi) \in \mathcal{M}_{K,d}(C)$. We know that (D, Φ) corresponds to a family of flat $GL(K, \mathbb{C})$ -connections in the bundle $E \otimes K_C^{1/2}$. Now we are assigning to it instead a family of flat \mathbb{C}^{\times} -connections $\mathcal{X}(\zeta)$ over a spectral curve $\Sigma_{\vec{\phi}}$.

Indeed there is a natural candidate way of doing this. The Higgs bundle $(\bar{\partial}_E, \varphi)$ has a corresponding spectral line bundle \mathcal{L} over $\Sigma_{\vec{\phi}}$, as in Proposition 6.33. We consider the line bundle $(\mathcal{L} \otimes K^{1/2}_{\Sigma_{\vec{\phi}}}, \lambda)$ as a Higgs bundle. Then it has a corresponding family of flat \mathbb{C}^{\times} -connections over $\Sigma_{\vec{\phi}}$. This is \mathcal{X}^{sf} .

Exercise 7.3. Check that \mathcal{X}^{sf} is indeed obtained by the above procedure.

However, the functions $\mathcal{X}_{\gamma}^{sf}$ are *not* generally holomorphic on $(\mathcal{M}_{K,d}(C), I_{\zeta})$. This corresponds to saying that the passage from the flat $GL(K, \mathbb{C})$ -connections $\nabla(\zeta)$ over C to the flat $GL(1, \mathbb{C})$ -connections $\mathcal{X}^{sf}(\zeta)$ over Σ is not a holomorphic map between moduli spaces of flat connections. Our next aim is to improve the functions $\mathcal{X}_{\gamma}^{sf}$ to true holomorphic Darboux coordinates \mathcal{X}_{γ} .

7.2 Metric comparison

One of the main claims of [22] is that the actual metric g on $\mathcal{M}_{K,d}(C)$ differs from the semiflat metric g^{sf} by a correction term which is "small on the smooth locus." A crude version of the estimate is to say that along any path $p(t) \in \mathcal{M}$ parameterized by $t \in \mathbb{R}_+$, with $\rho(p(t)) = t\vec{\phi}$, we should have

$$g = g^{\text{sf}} + O(e^{-2tM(\phi)})$$
 (7.10)

where

$$M(\vec{\phi}) = \min\{|Z_{\gamma}| : \gamma \in \Gamma_{\vec{\phi}}\}.$$
(7.11)

One approach to proving this kind of formula is to make a careful study of the asymptotic behavior of the corresponding harmonic doubled connections (D, Φ) . See [47] for this.

7.3 Twistorial construction, first steps

Here we follow an approach closer to the philosophy of [22]. The idea is to produce holomorphic functions $\mathcal{X}_{\gamma}(\zeta)$ on $\mathcal{M}_{K,d}(C)$ which are Darboux coordinates for $\Omega(\zeta)$ and which are also exponentially close to the simple functions $\mathcal{X}_{\gamma}^{sf}(\zeta)$ of (7.4). If we could do this completely rigorously it would establish the desired metric estimate (7.10), as well as various sharper statements. Along the way we will discover various extra interesting bits of structure.

[motivation for this picture from Stokes phenomena of irregular ODE]

A key lemma motivating our constructions is:

Lemma 7.4 (Uniqueness of solutions to Riemann-Hilbert problems). Suppose *G* is a complex Lie group, equipped with two antiholomorphic involutions *R* and ρ , with fixed loci $G_R, G_\rho \subset G$, and $G_R \cap G_\rho = \{1\}$. Fix a countable collection of rays $L = \{\ell_\mu\}_{\mu \in \Lambda}$ running from 0 to ∞ in \mathbb{C}^{\times} , and corresponding elements $S_\mu \in G$. Also fix a holomorphic map $\mathcal{X}^{\text{sf}} : \mathbb{C}^{\times} \to G$, obeying the "reality" condition

$$\mathcal{X}^{\mathrm{sf}}(-1/\bar{\zeta}) = \rho \mathcal{X}^{\mathrm{sf}}(\zeta). \tag{7.12}$$

Then there exists at most one map

$$\mathcal{X}:\mathbb{C}^{\times}\to G$$

with the properties:

- As $\zeta \to 0$ or $\zeta \to \infty$, $\mathcal{X}(\zeta)\mathcal{X}^{sf}(\zeta)^{-1}$ has a finite limit lying in G_R . (In other words, near $\zeta = 0$ we have $\mathcal{X}(\zeta) = F(\zeta)\mathcal{X}^{sf}(\zeta)$, where $F(0) \in G_R$, and similarly near $\zeta = \infty$.)
- X obeys the "reality" condition

$$\mathcal{X}(-1/\bar{\zeta}) = \rho \mathcal{X}(\zeta). \tag{7.13}$$

• On the ray ℓ_{μ} , the limits of \mathcal{X} from both sides exist, and are related by

$$\mathcal{X}^+_\mu = \mathcal{X}^-_\mu S_\mu. \tag{7.14}$$

Proof. Suppose \mathcal{X} and \mathcal{X}' obey all of these conditions. Then we consider the composite

$$Y: \mathbb{C}^{\times} \to G, \qquad Y(\zeta) = \mathcal{X}'(\zeta)\mathcal{X}(\zeta)^{-1}.$$
 (7.15)

The jumps S_{μ} cancel out in Y, so Y is continuous everywhere, and analytic away from the collection of rays ℓ_{μ} . It follows using Morera's theorem that Y is analytic everywhere in \mathbb{C}^{\times} . [even if the ℓ_{μ} are dense?] Moreover our asymptotic condition says Y is finite and G_R -valued in the limits $\zeta \to 0$ or $\zeta \to \infty$; but then the Riemann removable singularity theorem says that Y is analytic on the whole \mathbb{CP}^1 , and then Liouville's theorem says $Y(\zeta)$ is a ζ -independent element of G_R . Finally the reality condition (7.13) implies Y also belongs to G_{ρ} . So we conclude that Y = 1.

Example 7.5 (Riemann-Hilbert problems for G = GL(N)**).** [...] [cite Dubrovin, Cecotti-Vafa]

Before applying this to our situation we need one technical bit of preparation:

Definition 7.6 (Quadratic refinement of mod 2 pairing). Suppose given a lattice Γ with a bilinear pairing $\varepsilon : \Gamma \times \Gamma \to \mathbb{Z}/2\mathbb{Z}$. A *quadratic refinement* of ε is a map $\sigma : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ such that

$$\sigma(\gamma + \gamma') = \sigma(\gamma)\sigma(\gamma')\varepsilon(\gamma, \gamma'). \tag{7.16}$$

Proposition 7.7 (Spin structures on Σ **are quadratic refinements on** $H_1(\Sigma, \mathbb{Z})$). Suppose Σ is any compact surface. There is a canonical bijection between equivalence classes of spin structures on Σ and quadratic refinements of the pairing $\varepsilon(\gamma, \gamma') = (-1)^{\langle \gamma, \gamma' \rangle}$ on $H_1(\Sigma, \mathbb{Z})$.

Proof. This is proven in [48]. (The construction is completely topological.) [briefly explain it here?]

We would like to apply Lemma 7.4 in the following situation. Fix a point $\vec{\phi} \in \mathcal{B}$ and a spin structure on $\Sigma_{\vec{\phi}}$. Then we have a lattice $\Gamma = \Gamma_{\vec{\phi}}$ with:

- a skew pairing $\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \to \mathbb{Z}$,
- a homomorphism $Z : \Gamma \to \mathbb{C}$,
- a quadratic refinement σ of the pairing $\varepsilon(\gamma, \gamma') = (-1)^{\langle \gamma, \gamma' \rangle}$ on Γ .

Let *T* be the torus

$$T = \operatorname{Hom}(\Gamma, \mathbb{C}^{\times}), \tag{7.17}$$

and *G* the group of birational automorphisms of *T*. (One awkward point is that I do not know whether *G* can be thought of as an infinite-dimensional complex Lie group in any reasonable sense; thus it is not clear whether Lemma 7.4 can really be applied. Let us proceed optimistically nonetheless!)

Let *R* be conjugation by the map $T \to T$ given by $z \mapsto \overline{z}$ on \mathbb{C}^{\times} , and similarly ρ be conjugation by the map $T \to T$ given by $z \mapsto \overline{z}^{-1}$ on \mathbb{C}^{\times} . Define

$$\mathcal{X}^{\mathrm{sf}}: \mathbb{C}^{\times} \to G \tag{7.18}$$

by

$$\mathcal{X}^{\mathrm{sf}}(\zeta) = \exp(\zeta^{-1}Z + \zeta\bar{Z}) \in T \subset G.$$
(7.19)

Exercise 7.4. Check that \mathcal{X}^{sf} obeys the reality condition (7.12).

Finally, for each $\mu \in \Gamma$ primitive, let

$$\ell_{\mu} = \{\zeta : Z_{\mu}/\zeta \in \mathbb{R}_{-}\} \subset \mathbb{C}^{\times}$$

Assume for a moment that the ℓ_{μ} are all distinct. In this case, we are going to define one more bit of data:

• a map (not homomorphism) $DT : \Gamma \to \mathbb{Z}$.

Then we will let $S_{\mu} \in G$ be the birational map

$$X_{\gamma} \to X_{\gamma} \prod_{n \ge 1} (1 - \sigma(n\mu) X_{n\mu})^{n \langle \gamma, \mu \rangle DT(n\mu)}.$$
(7.20)

(In the case we consider below, this product is actually finite, so we do not have to worry about issues of convergence.)

Exercise 7.5. Show that S_{μ} is a birational symplectomorphism, with respect to the natural symplectic structure on *T* induced by the bilinear form $\langle \cdot, \cdot \rangle$ on Γ .

We should also remark that the name DT is meant to evoke "Donaldson-Thomas," and indeed the invariants appearing here are expected to be examples of generalized Donaldson-Thomas invariants. In this context the maps (7.20) appeared in the very important paper [49], and understanding their appearance was one of the original motivations for the picture of \mathcal{M} we are describing here.

7.4 Trajectories of quadratic differentials

The definition of *DT* is easiest to understand in the case K = 2, so let us consider that case first. The construction we will describe is closely related to one which appeared in the physics literature, first in [50]. It was later described in [45] where the application to the Hitchin system appeared.

Fix a point $\vec{\phi} = (\phi_1, \phi_2) \in \mathcal{B}'$. Most of what we say in this section is about the discriminant

$$\Delta = \phi_1^2 - 4\phi_2. \tag{7.21}$$

 Δ is a holomorphic quadratic differential on *C*.

Recall that the two sheets of $\Sigma_{\vec{\phi}}$ are given by (locally, in any domain where we can choose a single-valued $\sqrt{\Delta}$)

$$\lambda_{\pm} = \frac{1}{2}(-\phi_1 \pm \sqrt{\Delta}) \tag{7.22}$$

and in particular their difference is

$$\lambda_{+} - \lambda_{-} = \sqrt{\Delta}. \tag{7.23}$$

Definition 7.8 (ϑ -trajectories of a quadratic differential). A ϑ -trajectory of Δ is a connected real 1-manifold p on C such that $e^{-i\vartheta}\sqrt{\Delta}$ (with either choice of sign for $\sqrt{\Delta}$) is a *real* and nowhere vanishing form on p. Call a ϑ -trajectory *maximal* if it is not properly contained in any other.

Because of the sign ambiguity of $\sqrt{\Delta}$, there is no canonical way of orienting ϑ -trajectories. However, once we pass to the double cover, we do get an orientation:

Definition 7.9 (Orientation of lifted ϑ **-trajectories).** Given a ϑ -trajectory p, its lift $\tilde{p} = \pi^{-1}(p)$ to $\Sigma_{\vec{\phi}}$ is canonically oriented. Indeed, the two sheets of $\Sigma_{\vec{\phi}}$ correspond canonically to the two choices of $\sqrt{\Delta}$. Thus we can orient \tilde{p} by the condition that in the positive direction, $e^{-i\vartheta}\sqrt{\Delta}$ is positive.

Note that changing $\vartheta \to \vartheta + \pi$ preserves the notion of ϑ -trajectory but reverses the orientation of the lifts.



Let $C' = \{z : \Delta(z) \neq 0\} \subset C$.

Proposition 7.10 (ϑ **-trajectories give a foliation).** The ϑ -trajectories are the leaves of a foliation of *C*'.

Proof. Around any $z_0 \in C'$ we consider a local coordinate w given by $w(z) = \int_{z_0}^z \sqrt{\Delta}$. Then we have $\Delta = dw^2$. Thus in the coordinate w, ϑ -trajectories are just straight segments of inclination angle ϑ .

Proposition 7.11 (Local singularities of the foliation by ϑ **-trajectories).** Around each zero of Δ , the foliation by ϑ -trajectories has a three-pronged singularity, as shown below.



Proof. In a neighborhood of a zero of Δ , we may choose a local coordinate such that $\Delta = z \, dz^2$, using our assumption that all zeroes of Δ are simple.

Then in any simply connected domain away from z = 0 we have $w = \int \sqrt{\Delta} = \frac{2}{3}z^{\frac{3}{2}}$. In particular, each of the three domains

$$\frac{2}{3}(\vartheta + n\pi) < \arg z < \frac{2}{3}(\vartheta + (n+1)\pi), \qquad n = 0, 1, 2, \tag{7.24}$$

is mapped by w to a half-plane, whose boundary is the line through 0 of inclination ϑ . This gives the picture shown.

Exercise 7.6. Show that in a neighborhood of a zero of Δ , we may choose a local coordinate z such that $\Delta = z dz^2$.

Exercise 7.7. Suppose we consider a quadratic differential Δ which is allowed to have higher-order zeroes. What is the behavior of the foliation by ϑ -trajectories around such a zero?

Thus in the foliation by ϑ -trajectories, there are finitely many special leaves, namely those which are asymptotic in one or both directions to zeroes of Δ .

Definition 7.12 (Forward and backward asymptotics of trajectories). Given a maximal ϑ -trajectory p with a chosen orientation, we say p is *forward asymptotic* (resp. *backward asymptotic*) to z if, choosing any oriented parameterization of p, we have $\lim_{t\to\infty} p(t) = z$ (resp. $\lim_{t\to-\infty} p(t) = z$).

Definition 7.13 (Critical trajectories). A maximal ϑ -trajectory is called *critical* if it admits an orientation for which it is backward asymptotic to a zero of Δ . The ϑ -*critical graph* is the union of the critical trajectories and the zeroes of Δ .

Critical trajectories which are also *forward* asymptotic to zeroes of Δ are particularly special:

Definition 7.14 (Saddle connections). A ϑ -saddle connection is a maximal ϑ -trajectory p such that $\bar{p} \setminus p$ consists of two points of C.

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Definition 7.15 (Charge of a saddle connection). If *p* is a saddle connection, then the closure of \tilde{p} is an oriented loop on $\Sigma_{\vec{\phi}}$; the *charge of p* is the class of this loop in $\Gamma_{\vec{\phi}} = H_1(\Sigma_{\vec{\phi}}, \mathbb{Z})$.



There is one other way in which a trajectory can have finite length:

Definition 7.16 (Closed loops). A ϑ -closed loop is a maximal ϑ -trajectory p such that the closure of p has the topology of S^1 .

This includes the possibility that *p* begins and ends at the same branch point.

Definition 7.17 (Charge of a closed loop). If *p* is a closed loop, then \tilde{p} is the union of two oriented loops on Σ ; the *charge* of \tilde{p} is the class of this union in $\Gamma_{\vec{\phi}} = H_1(\Sigma_{\vec{\phi}}, \mathbb{Z})$.



Definition 7.18 (Finite ϑ **-trajectories).** A *finite* ϑ *-trajectory* is a ϑ -saddle connection or ϑ closed loop.

As we will now show, the existence of a finite ϑ -trajectory is a non-generic phenomenon: for "most" angles ϑ they do not occur.

Lemma 7.19 (Phase constraint for finite trajectories). If a ϑ -finite trajectory exists with charge γ , then $e^{-i\vartheta}Z_{\gamma} \in \mathbb{R}_+$.

Proof. Choose a $\sqrt{\Delta}$ over *p*, and orient *p* so that $e^{-i\vartheta}\sqrt{\Delta}$ is positive. Then

$$Z_{\gamma} = \int_{\tilde{p}} \lambda = \int_{p} \lambda_{+} - \lambda_{-} = \int_{p} \sqrt{\Delta}$$
(7.25)

so

$$e^{-\mathrm{i}\vartheta}Z_{\gamma} = \int_{p} e^{\mathrm{i}\vartheta}\sqrt{\Delta} > 0 \tag{7.26}$$

as desired.

In particular, finite ϑ -trajectories can only occur at countably many phases ϑ .

Proposition 7.20 (Landscape of finite ϑ **-trajectories).** Fix ϑ and $\overline{\phi} \in \mathcal{B}'$. Then the set of finite ϑ -trajectories is described as follows. There are finitely many subsets of *C* homeomorphic to open annuli ("ring domains"), foliated by ϑ -closed loops. Each boundary of a ring domain can be a ϑ -closed loop or a union of ϑ -saddle connections. In addition there can be finitely many ϑ -saddle connections elsewhere on *C*.



Proof. This is proven in [51].

Definition 7.21 (DT invariants). Fix $\vec{\phi} \in \mathcal{B}'$ and $\gamma \in \Gamma_{\vec{\phi}}$. $DT(\gamma) \in \mathbb{Z}$ is a count of finite ϑ -trajectories (of Δ), where $\vartheta = \arg Z_{\gamma}$:

 $DT(\gamma) = (\# \vartheta$ -saddle connections of charge $\gamma) - 2(\# \vartheta$ -ring domains of charge $\gamma)$. (7.27)

Exercise 7.8. Show that $DT(\gamma) = DT(-\gamma)$.

Note that *DT* is *not* a locally constant function of γ . Indeed, as we deform $\vec{\phi}$, saddle connections and/or ring domains may appear or disappear; this is the *wall-crossing* phenomenon. We will have more to say about this shortly.

7.5 Complexifying

In what follows it will be technically useful to complexify. Let \mathcal{Z} denote the twistor space of $\mathcal{M}^{s}_{K,d}(C)$, and let \mathcal{M}_{C} be the space of *complex* holomorphic sections of $\mathcal{Z} \to \mathbb{CP}^{1}$. \mathcal{M}_{C} has an antiholomorphic involution with \mathcal{M} as fixed locus. (I emphasize that the complex structure on \mathcal{M}_{C} is yet another one, not related to any of the I_{ζ} we already have on \mathcal{M} !)

Proposition 7.22 (Complex twistor lines for $\mathcal{M}_{K,d}^{s}(C)$). $\mathcal{M}_{\mathbb{C}}$ contains as a connected component the set of tuples $(D, \varphi, \overline{\varphi})$, where D is a complex connection in $E, \varphi \in \Omega^{1,0}(\operatorname{End} E)$, $\overline{\varphi} \in \Omega^{0,1}(\operatorname{End} E)$, obeying

$$\bar{\partial}_D \varphi = 0, \tag{7.28a}$$

$$\partial_D \bar{\varphi} = 0,$$
 (7.28b)

$$F_D + [\varphi, \bar{\varphi}] = -2\pi i \frac{d}{K} \mathbf{1} \omega_C, \qquad (7.28c)$$

modulo the gauge group $\mathfrak{G}_{\mathbb{C}} = \operatorname{Aut} E$.

Proof. Given such a tuple we can write a formula parallel to (6.60):

$$\nabla(\zeta) = \zeta^{-1}\varphi + D + \zeta\bar{\varphi}.$$
(7.29)

The conditions (7.28) imply as usual that $\nabla(\zeta)$ is a complex Einstein connection for all $\zeta \in \mathbb{C}^{\times}$. Since $\pi^{-1}(\mathbb{C}^{\times}) \subset \mathcal{Z}$ is isomorphic to $\mathbb{C}^{\times} \times \mathcal{M}_{K,d}^{Ein}(C)$, this gives a section over $\mathbb{C}^{\times} \subset \mathbb{CP}^1$. Moreover this section extends over $\zeta = 0$ and $\zeta = \infty$ since its behavior in these limits is just like that of the real twistor lines, which we already know extend.

Conversely, given a complex twistor line of the above form we need to show that it has a neighborhood consisting only of lines of the above form. [infinitesimal calculation using normal bundle]

Example 7.23 (Complex twistor lines for $\mathcal{M}_{1,0}(C)$). Suppose *E* is a degree 0 line bundle over *C*. Then given any $\varphi \in \Omega^{1,0}(\operatorname{End} E)$, $\bar{\varphi} \in \Omega^{0,1}(\operatorname{End} E)$ we can find [...]

Definition 7.24 (Complexified Hitchin fibration). By restricting to the fibers over 0 and ∞ , each point of $\mathcal{M}_{\mathbb{C}}$ gives a Higgs bundle and an anti-Higgs bundle. Thus we obtain a doubled Hitchin fibration,

$$\rho_{\mathbb{C}}: \mathcal{M}_{\mathbb{C}} \to \mathcal{B} \times \bar{\mathcal{B}}. \tag{7.30}$$

Proposition 7.25 (Diagonal fibers of complexified Hitchin fibration). Suppose $\vec{\phi} \in \mathcal{B}'$ and d' = 0. Then $\rho_{\mathbb{C}}^{-1}(\vec{\phi}, \vec{\phi})$ has a connected component which is $\operatorname{Hom}(H_1(\Sigma_{\vec{\phi}}), \mathbb{C}^{\times})$. [I hope!]

Proof. [...]

7.6 The punctured case

Now for each $\phi \in \mathcal{B}'$ we are going to construct a map

$$\mathcal{X}_{\vec{\phi}}: \mathbb{C}^{\times} \to G_{\vec{\phi}} \tag{7.31}$$

where $G_{\vec{\phi}}$ is the group of birational automorphisms of the torus $T_{\vec{\phi}}$. By Proposition 7.25, this is equivalent to giving a collection of functions $\mathcal{X}_{\gamma} : \mathbb{C}^{\times} \to \mathbb{C}^{\times}$ on the diagonal part of the complexified Hitchin fibration, obeying $\mathcal{X}_{\gamma}\mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$. We want to find \mathcal{X} such that it obeys the conditions of Lemma 7.4, and at the same time, gives a holomorphic Darboux coordinate system on \mathcal{M} .

The punctured case is technically much simpler and closer to being rigorously understood, so let's start there.

So, now let *C* denote a surface with *punctures*, such that $\chi(C) < 0$. Let \overline{C} be the original unpunctured surface, and $z_1, \ldots, z_n \in \overline{C}$ the punctures. Also fix additional "residue data" around the punctures: for each puncture z_ℓ give a semisimple conjugacy class $m_\ell \in gl(K, \mathbb{C})$ and $m_\ell^{\mathbb{R}} \in \mathfrak{u}(K)$.

Our constructions of moduli spaces can be extended to this punctured setting: the relevant moduli spaces involve (D, Φ) which are allowed to have first-order poles at the punctures, with residues controlled by $m^{\mathbb{R}}$ and m. For example, the connection form representing D near a puncture is like

$$D = d + A, \qquad A = im^{\mathbb{R}} d\theta + \text{regular}$$
 (7.32)

with θ the polar angle, and similarly

$$\varphi = m \frac{\mathrm{d}z}{z} + \text{ regular} \tag{7.33}$$

We get spaces $\mathcal{N}_{K,d,m^{\mathbb{R}}}(C)$, $\mathcal{M}_{K,d,m^{\mathbb{R}},m}(C)$. They have essentially all the same properties as their unpunctured cousins. [expand this a lot!]

Now let us assume that all eigenvalues of each m_{ℓ} are distinct, and also assume again that we are in the case K = 2. Then we have the following:

Proposition 7.26 (Good behavior of ϑ **-trajectories on punctured surfaces).** Suppose $\vec{\phi} \in \mathcal{B}'$ and $e^{-i\vartheta}Z_{\gamma}$ is real, (in particular no eigenvalue of $e^{-i\vartheta}m_{\ell}^{\mathbb{C}}$ is real). Then:

- Every θ-trajectory is forward asymptotic to a puncture,
- The complement of the θ-critical graph has finitely many connected components C_i; each C_i is foliated by θ-trajectories all of which are homotopic, and all of which are forward and backward asymptotic to punctures.



Proof. [...]

[WKB triangulation] [small flat sections] [definition of cross ratios] [jumps of cross ratios when saddle connection appears] [interpretation of cross ratios as abelianization]

Now how do we see that these cross ratios \mathcal{X}_{γ} have the asymptotic behavior we want as $\zeta \to 0, \infty$?

Proposition 7.27 (WKB evolution along trajectories). Fix K = 2. Suppose given a point of $\mathcal{M}_{\mathbb{C}}$, corresponding to a family of flat connections $\nabla(\zeta)$ as in (7.29). Suppose that p is a closed segment contained in a ϑ -trajectory. Choose a $\sqrt{\Delta}$ over p, and sections e_{\pm} of E with $\varphi e_{\pm} = \lambda_{\pm} e_{\pm}$. Let $\Psi_p(\zeta)$ denote the parallel transport of $\nabla(\zeta)$ from the beginning of p to the end. Then, if we define the "WKB remainder" by

$$r(\zeta) = \exp\left(-\zeta^{-1}\int_{p}\lambda_{+}\right)\Psi_{p}(\zeta)e_{+},$$
(7.34)

we have

$$\lim_{\zeta \to 0} r(\zeta) = ce_+ \tag{7.35}$$

for some $c \in \mathbb{C}^{\times}$.

What this lemma says is that the "leading part" of the propagation of $\nabla(\zeta)$ is given by the exponential growth one would naively expect.

7.7 Estimating X

For the purpose of computing metric estimates, we now want to see that \mathcal{X} becomes very close to \mathcal{X}^{sf} when we go out to infinity in \mathcal{B}' . For this the essential point is the uniqueness in Lemma 7.4.

[integral equations] [like Ooguri-Vafa!]

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