

Metric on moduli of Higgs bundles

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These are notes for a lecture series on moduli of Higgs bundles, originally delivered at the mini-workshop “Gauge theory and Geometry in Göttingen” March 1-4, 2018.

1 Introduction

The aim of these lectures is to describe a *conjectural* approach to “more explicitly” understanding the hyperkähler metric g on the moduli space \mathcal{M} of Higgs bundles.

1.1 Overview

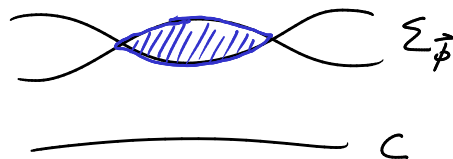
\mathcal{M} is a complex integrable system: this means that it admits a holomorphic fibration $\pi : \mathcal{M} \rightarrow \mathcal{B}$ with complex Lagrangian fibers. There is a complex codimension 1 “singular locus” $\mathcal{B}_{\text{sing}} \subset \mathcal{B}$. Let $\mathcal{B}_{\text{reg}} = \mathcal{B} \setminus \mathcal{B}_{\text{sing}}$, and $\mathcal{M}_{\text{reg}} = \pi^{-1}(\mathcal{B}_{\text{reg}})$. On \mathcal{M}_{reg} , the fibers of π are compact complex tori, roughly the Jacobians of a family of smooth *spectral curves* $\Sigma_{\vec{\phi}} \subset T^*C$ parameterized by $\vec{\phi} \in \mathcal{B}_{\text{reg}}$.

The conjectural picture we are aiming for is that the hyperkähler metric g on \mathcal{M} is constructed from two ingredients:

1. The special Kähler structure on \mathcal{B}_{reg} , constructed from *periods* Z_γ of the spectral curves $\Sigma_{\vec{\phi}}$,



2. A collection of integer “Donaldson-Thomas invariants” $DT(\gamma)$, which count (tropicalizations of) special Lagrangian discs in T^*C with boundary on $\Sigma_{\vec{\phi}}$.



So, what we will describe is a conjectural recipe which uses Z_γ and $DT(\gamma)$ to build the hyperkähler metric on \mathcal{M} .

If one sets all $DT(\gamma) = 0$ in this recipe, one obtains a simple and explicit hyperkähler metric g^{sf} on \mathcal{M}_{reg} , the “semiflat metric,” so called because it is flat (and translation invariant) on the torus fibers. g^{sf} however does not extend over $\mathcal{M}_{\text{sing}}$, so it cannot be equal to Hitchin’s metric on \mathcal{M} . The effect of the nonzero $DT(\gamma)$ is to add corrections which

break the translation invariance. Away from $\mathcal{M}_{\text{sing}}$, these corrections are of the order e^{-2M} where M is the area of the smallest special Lagrangian disc, i.e. we predict

$$g = g^{\text{sf}} + O(e^{-2M}). \quad (1.1)$$

In particular, if we follow some path to infinity in \mathcal{M} along which $M \rightarrow \infty$, we expect to see g converging to g^{sf} . On the other hand, as we approach $\mathcal{M}_{\text{sing}}$ the effect of these corrections is large: it smoothes out the singularity of g^{sf} .¹

The strategy of the lectures will be roughly:

1. The moduli space \mathcal{M} , its fibration $\mathcal{M} \rightarrow \mathcal{B}$ and special Kähler structure.
2. The conjectural metric construction.
3. The available evidence that the conjecture is correct.

1.2 References

The conjecture reviewed in these notes is mostly contained in the papers [1, 2], which are joint work of mine with Davide Gaiotto and Greg Moore. In [3] I reviewed some parts of the conjecture, focusing on the abstract construction of hyperkähler metrics from a special Kähler base and Donaldson-Thomas invariants; in these lectures I focus more on the specific example of moduli spaces of parabolic Higgs bundles.

These works depend on many prior developments in physics and mathematics. Here I can only single out a few which were of singular importance (for more, see the references in [1, 2]):

- The work [1] originated in an attempt to understand the physical meaning of the remarkable wall-crossing formula for generalized Donaldson-Thomas invariants, given by Kontsevich-Soibelman [4].
- Many of the key constructions in [1] can be understood as infinite-dimensional analogues of constructions used by Cecotti-Vafa and Dubrovin in tt^* geometry [5, 6], with additional inspiration from work of Bridgeland and Toledano Laredo [7].
- The application to Hitchin systems in [2] depended importantly on the work of Fock-Goncharov on moduli spaces of local systems over surfaces [8], as well as the foundational work of Hitchin [9] and Corlette, Donaldson, Simpson [10, 11, 12] on Higgs bundles without singularities, Simpson's extension to Higgs bundles with regular singularities [13], and Biquard-Boalch for Higgs bundles with wild ramification [14].

¹In some cases the hyperkähler metric g on \mathcal{M} actually has singularities. The singularities of g are milder than those of g^{sf} — in particular, the singularities of g occur in complex codimension at least 2. In these cases, the corrections do not completely *smooth* the singularity of g^{sf} , but they still *improve* it.

2 Background on Hitchin system

2.1 Data

Throughout these lectures we will fix data (G, C, \vec{m}) as follows:

- A group $G = \mathrm{SU}(K)$ or $\mathrm{U}(K)$, with diagonal subgroup $T \subset G$,
- A compact Riemann surface C , equipped with a finite subset $P \subset C$,
- A vector $\vec{m}_p = (m_p^{\mathbb{C}}, m_p^{\mathbb{R}}) \in \mathbb{C}^K \oplus \mathbb{R}^K$ for each $p \in P$, with the additional constraint that all components of $m_p^{\mathbb{R}}$ lie in $[0, 2\pi)$.

It is often convenient to think of $m_p^{\mathbb{C}}$ and $m_p^{\mathbb{R}}$ as diagonal $K \times K$ matrices rather than as vectors. They must satisfy a further constraint:

- If $G = \mathrm{U}(K)$, then $\sum_{p \in P} \mathrm{Tr} m_p^{\mathbb{C}} = 0$, and $\sum_{p \in P} \mathrm{Tr} m_p^{\mathbb{R}} \in 2\pi\mathbb{Z}$.
- If $G = \mathrm{SU}(K)$, then $\mathrm{Tr} m_p^{\mathbb{C}} = 0$, and $\mathrm{Tr} m_p^{\mathbb{R}} \in 2\pi\mathbb{Z}$.

Finally, we require that $2g_C + |P| - 2 > 0$; equivalently, if C has genus 0 we require $|P| \geq 3$, and if C has genus 1 we require $|P| \geq 1$.



Definition 2.1 (Generic puncture data). We say \vec{m} is *generic* if, for all $p \in P$, each $m_p^{\mathbb{C}} \in \mathbb{C}^K$ has all entries distinct; equivalently, as a diagonal matrix, $m_p^{\mathbb{C}}$ is a regular element. The generic case is the simplest case, and for the main purposes of these lectures, it is fine to restrict to the generic case throughout.

Example 2.2 (The case of $G = \mathrm{SU}(2)$). A good case to keep in mind is the case $G = \mathrm{SU}(2)$. In that case our data reduces to a Riemann surface C , a finite subset $P \subset C$, and numbers $m_p^{\mathbb{C}} \in \mathbb{C}$, $m_p^{\mathbb{R}} \in \mathbb{R}$ for each $p \in P$. The generic case is the case when all $m_p^{\mathbb{C}} \neq 0$.

Remark 2.3. For any $p \in P$, the outputs of all constructions to be discussed below are invariant under the symmetric group S_K acting on \vec{m}_p .

2.2 Moduli of Higgs bundles

Usually one would start out with ordinary Higgs bundles, but with an eye to what will come later, we go straight to the parabolic case. Roughly this means that we will consider Higgs fields φ which, rather than being holomorphic, are allowed to have simple poles at the points $p \in P$.²

²The parameter $m_p^{\mathbb{C}}$ will control the eigenvalues of the residue of φ ; the “strongly parabolic” case of nilpotent residues is thus the case $m_p^{\mathbb{C}} = 0$, which we usually avoid.

The original reference for the material in this section is Simpson [13]. A very useful review of the unpunctured case can be found in [15] and references therein.

Definition 2.4 (Parabolic Higgs bundles for $G = \mathrm{U}(K)$). When $G = \mathrm{U}(K)$, a (G, C, \vec{m}) -Higgs bundle is a pair (E, φ) , where:

- E is a holomorphic vector bundle of rank K over C ,
- φ is a holomorphic section of $\mathrm{End} E \otimes K_C(P)$,

with additional “parabolic structure” at the points $p \in P$ as follows. Each E_p carries a decreasing filtration with weights in $[0, 2\pi)$, where

$$\dim \mathrm{Gr}_\alpha E_p = \text{multiplicity of } \alpha \text{ in } m_p^{\mathbb{R}}. \quad (2.1)$$

The residue $\mathrm{Res}_p \varphi$ preserves the filtration on E_p and thus descends to act on the graded pieces $\mathrm{Gr}_\alpha E_p$, with generalized eigenvalues determined by $m_p^{\mathbb{C}}$. Altogether then,

$$\mathrm{Gr} E_p = \bigoplus_{\vec{\lambda}} E_{p, \vec{\lambda}} \quad (2.2)$$

where $E_{p, \vec{\lambda}}$ is in grade $\lambda^{\mathbb{R}}$, and $\mathrm{Res}_p \varphi$ acts on $E_{p, \vec{\lambda}}$ with generalized eigenvalue $\lambda^{\mathbb{C}}$.

Remark 2.5 (Parabolic structure in case of generic puncture data). The case of generic puncture data is simpler: then $E_p = \bigoplus_{\vec{\lambda}} E_{p, \vec{\lambda}}$ with all $E_{p, \vec{\lambda}}$ one-dimensional, $\mathrm{Res}_p \varphi$ acting by $\lambda^{\mathbb{C}}$ on $E_{p, \vec{\lambda}}$, and filtration weight given by $\lambda^{\mathbb{R}}$ on $E_{p, \vec{\lambda}}$.

Definition 2.6 (Parabolic Higgs bundles for $G = \mathrm{SU}(K)$). When $G = \mathrm{SU}(K)$, a (G, C, \vec{m}) -Higgs bundle is a pair (E, φ) as in Definition 2.4, obeying two additional conditions:

- $\det E$ is trivial in the parabolic sense: this means that the holomorphic line bundle $(\det E) \otimes \mathcal{O} \left(\sum_{p \in P} n_p p \right)$ over C is trivial, where $n_p = \frac{1}{2\pi} \mathrm{Tr} m_p^{\mathbb{R}}$.
- $\mathrm{Tr} \varphi = 0$.

Definition 2.7 (Parabolic degree). Let E be a (G, C, \vec{m}) -Higgs bundle, and $E' \subset E$ any holomorphic subbundle preserved by φ . Then E'_p also gets a filtration with weights in $[0, 2\pi)$, and we define

$$\mathrm{pdeg} E' = \deg E' + \frac{1}{2\pi} \sum_P \sum_{\mu \in [0, 2\pi)} \mu \dim \mathrm{Gr}_\mu E'_p. \quad (2.3)$$

Example 2.8 (Parabolic degrees in the simplest case). With generic puncture data and $G = \mathrm{SU}(2)$, the two weights which occur in the decomposition of E_p are either $(0, 0)$ or of the form $(\alpha, 2\pi - \alpha)$, for some $\alpha \in (0, \pi]$. Assume we are in the latter case. Now suppose $E' \subset E$ is a line subbundle. If E'_p is the line $E_{p, \vec{\lambda}}$ with $\lambda^{\mathbb{R}} = \alpha$, then the contribution to $\mathrm{pdeg} E'$ from $p \in P$ is $1 - \frac{\alpha}{2\pi}$; otherwise the contribution is $1 - \frac{\alpha}{2\pi}$.

Remark 2.9 (Integrality of $\text{pdeg } E$). Our conditions on $m^{\mathbb{R}}$ imply that the whole bundle E has $\text{pdeg } E \in \mathbb{Z}$, either for $G = \text{U}(K)$ or $G = \text{SU}(K)$.

Definition 2.10 (Stability of parabolic Higgs bundles). We say E is *stable* if for all $E' \subset E$ preserved by φ we have

$$\frac{\text{pdeg } E'}{\text{rank } E'} < \frac{\text{pdeg } E}{\text{rank } E}. \quad (2.4)$$

We say E is *polystable* if it is a direct sum of stable Higgs bundles with the same $\frac{\text{pdeg}}{\text{rank}}$.

Proposition 2.11 (Moduli space of parabolic Higgs bundles exists). There is a moduli space $\mathcal{M} = \mathcal{M}(G, C, \vec{m})$ parameterizing polystable (G, C, \vec{m}) -Higgs bundles (E, φ) , with $\text{pdeg } E = 0$, up to equivalence. \mathcal{M} is a manifold away from the locus of unstable (but still polystable) Higgs bundles. It carries a natural complex structure I_1 and holomorphic symplectic form Ω_1 .

The holomorphic symplectic form Ω_1 comes from the fact that variations of the parabolic bundle E are valued in (for $G = \text{U}(K)$) $H^1(\text{ParEnd } E)$, while variations of the Higgs field φ are valued in $H^0(\text{SParEnd } E \otimes K_C(P))$, and the two are Serre dual.³

Remark 2.12 (Dimension of \mathcal{M}). The complex dimension of \mathcal{M} is

$$\dim_{\mathbb{C}} \mathcal{M} = \begin{cases} (2g_C - 2)K^2 + 2 + |P|K(K - 1), & G = \text{U}(K), \\ (2g_C - 2)(K^2 - 1) + |P|K(K - 1), & G = \text{SU}(K). \end{cases} \quad (2.5)$$

From now on we write $\dim_{\mathbb{C}} \mathcal{M} = 2r$.

Example 2.13 (The abelian case). $G = \text{U}(1)$ is the abelian case, in which $\dim_{\mathbb{C}} \mathcal{M} = 2g_C$ irrespective of $|P|$. When $P = \emptyset$ we have simply $\mathcal{M} = T^* \text{Jac } C$, the cotangent bundle to a compact complex torus. When $P \neq \emptyset$, \mathcal{M} is a torsor over $T^* \text{Jac } C$. This is the only example of \mathcal{M} which is so “linear” in nature: for nonabelian G the space \mathcal{M} will be much more interesting.

Example 2.14 (Some low-dimensional nonabelian cases). Here are a few examples:

- If $G = \text{SU}(2)$ and C is a genus 0 curve with $|P| = 4$, then $\dim_{\mathbb{C}} \mathcal{M} = 2$.
- If $G = \text{SU}(3)$ and C is a genus 0 curve with $|P| = 3$, then again $\dim_{\mathbb{C}} \mathcal{M} = 2$.
- If we want to take $P = \emptyset$, then the simplest nonabelian case is $G = \text{SU}(2)$ and C a genus 2 curve, in which case $\dim_{\mathbb{C}} \mathcal{M} = 6$.

Remark 2.15 (Restriction of Jordan form). When some $\vec{\lambda}$ occurs with multiplicity greater than 1 in \vec{m}_p , there is a natural way of getting a subspace of \mathcal{M} : we can restrict the Jordan block structure of the endomorphism $\text{Res}_p \varphi$ acting in each $E_{p, \vec{\lambda}}$ (e.g. if $\vec{\lambda}$ has multiplicity 2 we can require that $\text{Res}_p \varphi$ acts by the scalar $\lambda^{\mathbb{C}}$ in $E_{p, \vec{\lambda}}$ instead of a nontrivial Jordan

³Here ParEnd means endomorphisms preserving the filtration, and $\text{SParEnd} \subset \text{ParEnd}$ means endomorphisms which in addition act as 0 on the associated graded (i.e. they are *strictly* upper triangular, not just upper triangular).

block). In the physics literature it is proposed that this subspace of \mathcal{M} can be considered as a moduli space of Higgs bundles in its own right, with all the structure we will discuss in the rest of these lectures; in particular we can get more 2-dimensional examples in this way. I am not sure whether this construction has yet been carefully developed in the mathematics literature.

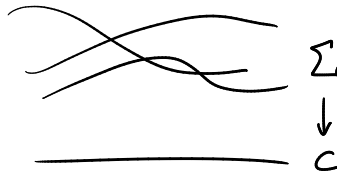
2.3 The Hitchin map

Now we want to exhibit \mathcal{M} as a complex integrable system, i.e. a holomorphic Lagrangian fibration.

Given a Higgs bundle $(E, \varphi) \in \mathcal{M}$ and $z \in C$, we can consider the eigenvalues of $\varphi(z)$. As z varies these sweep out a curve $\Sigma \subset \text{Tot}[K_C(P)]$:

$$\Sigma = \{(z, \lambda) : \det(\lambda - \varphi(z)) = 0\} \subset \text{Tot}[K_C(P)]. \quad (2.6)$$

We call Σ the *spectral curve* associated to the Higgs bundle (E, φ) . The projection $\text{Tot}[K_C(P)] \rightarrow C$ restricts to $\rho : \Sigma \rightarrow C$ which is a K -fold branched covering.



Now what are all the curves Σ we can get in this way? We can describe them by their coefficients, ie write

$$\det(\lambda - \varphi) = \lambda^K + \sum_{n=1}^K \lambda^{K-n} \phi_n = 0, \quad \phi_n \in K_C(P)^n. \quad (2.7)$$

The coefficients $\vec{\phi} = (\phi_1, \dots, \phi_K)$ lie in the Hitchin base:

Definition 2.16 (Hitchin base). If $G = \text{U}(K)$, the *Hitchin base* $\mathcal{B} = \mathcal{B}(G, C, m_p^{\mathbb{C}})$ is the space of tuples $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_K)$ where ϕ_n is a holomorphic section of $K_C(P)^n$, and $m_p^{\mathbb{C}}$ controls the residues $\phi_n(p)$ via the equation⁴

$$\det(\lambda - m_p^{\mathbb{C}}) = \lambda^K + \sum_{n=1}^K \lambda^{K-n} \phi_n(p). \quad (2.8)$$

If $G = \text{SU}(K)$ then we make the same definition except that $\phi_1 = 0$ everywhere.

\mathcal{B} is a complex affine space, a torsor for the complex vector space of $\vec{\phi}$ vanishing at P .

⁴Recall that for a section of K_C^n the residue at a pole is the coefficient of $\frac{(dz)^n}{z^n}$: this is a well defined complex number. This generalizes the case of a meromorphic 1-form where the residue at a pole is the coefficient of $\frac{dz}{z}$. Said otherwise, the fiber of $K_C(P)$ over $p \in P$ is canonically trivial, and likewise for $K_C(P)^n$.

Definition 2.17 (Hitchin map). The *Hitchin map* is the map $\pi : \mathcal{M} \rightarrow \mathcal{B}$ given by

$$(E, \varphi) \mapsto (\phi_1, \dots, \phi_K) \quad (2.9)$$

where the ϕ_n are defined by (2.7).

Proposition 2.18 (Hitchin map has Lagrangian fibers). The fibers $\mathcal{M}_{\vec{\phi}} = \pi^{-1}(\vec{\phi})$ are compact complex Lagrangian subsets of (\mathcal{M}, Ω_1) . (In particular, $\dim_{\mathbb{C}} \mathcal{B} = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{M}$.)

[say why?]

We can say more precisely what the fibers are, over most of the Hitchin base:

Definition 2.19 (Singular locus and smooth locus). The *singular locus* $\mathcal{B}_{\text{sing}} \subset \mathcal{B}$ is the set of $\vec{\phi} \in \mathcal{B}$ for which $\Sigma_{\vec{\phi}}$ is singular. $\mathcal{B}_{\text{sing}}$ has complex codimension 1 in \mathcal{B} . The *smooth locus* is $\mathcal{B}_{\text{reg}} = \mathcal{B} \setminus \mathcal{B}_{\text{sing}}$. We also let $\mathcal{M}_{\text{reg}} = \pi^{-1}(\mathcal{B}_{\text{reg}})$.

Proposition 2.20 (Fibers of the Hitchin map over \mathcal{B}_{reg}). Suppose $\vec{\phi} \in \mathcal{B}_{\text{reg}}$. Let $\mathcal{M}_{\vec{\phi}} = \pi^{-1}(\vec{\phi})$. Then:

- If $G = \text{U}(K)$, then $\mathcal{M}_{\vec{\phi}}$ is a torsor over $\text{Jac } \Sigma_{\vec{\phi}}$. After choosing spin structures on C and on $\Sigma_{\vec{\phi}}$, in the case of generic puncture data, we can identify $\mathcal{M}_{\vec{\phi}}$ with the space of flat $\text{U}(1)$ -connections over $\Sigma_{\vec{\phi}} \setminus \rho^{-1}(P)$ with holonomy around $\rho^{-1}(p)$ given by $\exp(im_p^{\mathbb{R}})$.
- If $G = \text{SU}(K)$, then $\mathcal{M}_{\vec{\phi}}$ is a torsor over $\text{Prym}(\rho : \Sigma_{\vec{\phi}} \rightarrow C)$. After choosing spin structures on C and on $\Sigma_{\vec{\phi}}$, in the case of generic puncture data, we can identify $\mathcal{M}_{\vec{\phi}}$ with the space of flat $\text{U}(1)$ -connections ∇ over $\Sigma_{\vec{\phi}} \setminus \rho^{-1}(P)$ with holonomy around $\rho^{-1}(p)$ given by $\exp(im_p^{\mathbb{R}})$, equipped with a trivialization of $\det \rho_* \nabla$.

So we reach the following picture: a point $\vec{\phi} \in \mathcal{B}_{\text{reg}}$ gives a smooth spectral curve $\Sigma_{\vec{\phi}}$; the torus $\mathcal{M}_{\vec{\phi}}$ is a space of flat $\text{U}(1)$ -connections over $\Sigma_{\vec{\phi}}$, with fixed holonomies around the punctures.

Remark 2.21 (Concrete description of the singular locus). What is $\mathcal{B}_{\text{sing}}$ concretely? The branch locus of the covering $\rho : \Sigma_{\vec{\phi}} \rightarrow C$ is the zero locus of the discriminant $\Delta_{\vec{\phi}}$ of the equation (2.7). $\Delta_{\vec{\phi}}$ is a holomorphic section of $K_C(P)^{K(K-1)}$. $\vec{\phi} \in \mathcal{B}_{\text{reg}}$ iff $\Delta_{\vec{\phi}}$ has only simple zeroes; in this case it has $K(K-1)(2g_C + |P| - 2)$ of them, and the genus of $\Sigma_{\vec{\phi}}$ is

$$g_{\Sigma} = 1 + K^2(g_C - 1) + \frac{1}{2}K(K-1)|P|. \quad (2.10)$$

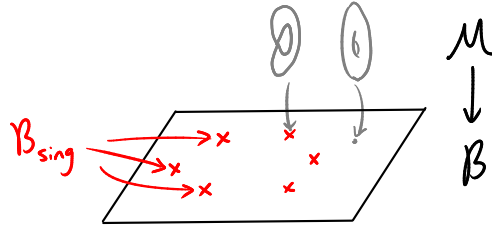
Example 2.22 (Hitchin base and spectral curves for $G = \text{SU}(2)$). When $G = \text{SU}(2)$, \mathcal{B} is the space of meromorphic quadratic differentials ϕ_2 on C , with a pole of order ≤ 2 at each $p \in P$, of residue $\text{Res}_p \phi_2 = (m_p^{\mathbb{C}})^2$. It has complex dimension $\dim_{\mathbb{C}} \mathcal{B} = 3g_C - 3 + |P|$. The spectral curve for a given $\phi_2 \in \mathcal{B}$ is

$$\Sigma_{\phi_2} = \{(z, \lambda) : \lambda^2 + \phi_2(z) = 0\} \subset \text{Tot}[K_C(P)]. \quad (2.11)$$

Σ_{ϕ_2} is a 2-fold branched covering of C , branched at the zeroes of ϕ_2 . The regular locus $\mathcal{B}_{\text{reg}} \subset \mathcal{B}$ consists of those ϕ_2 which have only simple zeroes ($4g_C + 2|P| - 4$ of them).

Example 2.23 (A one-dimensional Hitchin base). Suppose $G = \text{SU}(2)$, $|P| = 4$, $g_C = 0$.

- If $m^{\mathbb{C}} = 0$, \mathcal{B} is a complex vector space of dimension 1, and $\mathcal{B}_{\text{sing}} \subset \mathcal{B}$ is the origin; for generic $m^{\mathbb{R}}$, the fiber $\mathcal{M}_{\vec{\phi}=0}$ consists of five $\mathbb{C}\mathbb{P}^1$'s arranged in an affine D_4 configuration.
- If $m^{\mathbb{C}} \neq 0$, \mathcal{B} is a complex affine space of dimension 1. If $m^{\mathbb{C}}$ is completely generic, $\mathcal{B}_{\text{sing}} \subset \mathcal{B}$ consists of 6 points, and the fiber $\mathcal{M}_{\vec{\phi}}$ over any $\vec{\phi} \in \mathcal{B}_{\text{sing}}$ is a nodal torus. (For special choices of $m^{\mathbb{C}}$ some of these discriminant points may collide.)



[also do the 3-punctured $\text{SU}(3)$ case?]

We should emphasize that this one-dimensional example can lead to the wrong mental picture about the generic case: generally, when $m^{\mathbb{C}} = 0$, $\mathcal{B}_{\text{sing}}$ is some codimension-1 cone inside \mathcal{B} , and in particular, when $\dim \mathcal{B} > 1$, $\mathcal{B}_{\text{sing}}$ is not compact. For $m^{\mathbb{C}} \neq 0$, $\mathcal{B}_{\text{sing}} \subset \mathcal{B}$ is not a cone anymore, but near asymptotic infinity of \mathcal{B} , it still looks asymptotically like a cone.

2.4 The hyperkähler metric

A key fact about \mathcal{M} is that it carries a canonically defined hyperkähler metric g . However, g is not easily written in closed form.

To construct g , one needs to consider *Hitchin's equation*: given a Higgs bundle (E, φ) this is a PDE for a Hermitian metric h in E , written

$$F_{D_h} + [\varphi, \varphi^{\dagger h}] = 0. \quad (2.12)$$

Here D_h denotes the Chern connection in (E, h) , the unique h -unitary connection compatible with the holomorphic structure of E . One considers (2.12) for metrics h which are smooth on $C - P$ and have a prescribed singular behavior near each $p \in P$.

Definition 2.24 (Adapted metrics for generic puncture data). In the case of generic puncture data, a Hermitian metric h in E is *adapted* if, for a holomorphic section s where $s(p) \in E_p$ has grade α , we have $h(s, s) \sim |z|^{\frac{2\alpha}{2\pi}}$ near p .

Remark 2.25 (Holonomy of D_h around punctures for generic puncture data). For an adapted metric h on E , and generic puncture data, the holonomy of D_h around p is just $\exp(im^{\mathbb{R}})$. This is one of the most concrete ways of understanding the role of $m^{\mathbb{R}}$ in the story.

For more general puncture data, the situation is a bit more complicated: in addition to polynomial growth we need to allow logarithmic behavior, in a way dictated by the Jordan block structure of φ . The precise statement can be found in [13].

A basic fact from [13] is:

Theorem 2.26 (Existence of harmonic metrics). The equation (2.12) has an adapted solution h for each (E, φ) ; this h is unique up to scalar multiple. We call h the *harmonic metric*.

Using Theorem 2.26 one can define Hitchin's metric on \mathcal{M} , as follows. Given a tangent vector v to \mathcal{M} whose norm we wish to calculate, we represent v by a family of Higgs bundles (E_t, φ_t) , with harmonic metrics h_t . Identifying the underlying Hermitian bundles with a single (E, h) we have an arc of unitary connections D_t and skew-Hermitian Higgs fields $\Phi_t = \varphi_t - \varphi_t^\dagger$ on (E, h) , determined up to gauge transformations i.e. automorphisms of (E, h) . In particular, differentiating at $t = 0$ gives a pair

$$\left. \frac{d}{dt} \right|_{t=0} (D_t, \Phi_t) = (\dot{A}, \dot{\Phi}) \in \Omega^1(\mathfrak{u}(E)) \oplus i\Omega^1(\mathfrak{u}(E)), \quad (2.13)$$

defined up to gauge transformations. Then the norm of v is the L^2 norm

$$g(v, v) = \int_C \|\dot{A}\|^2 + \|\dot{\Phi}\|^2 \quad (2.14)$$

where for $(\dot{A}, \dot{\Phi})$ we choose the representative *minimizing* the norm.

Remark 2.27 (Hyperkähler quotient). I have not really explained why the metric g constructed in this way turns out to be hyperkähler, or even Kähler. The most conceptual explanation of this comes by viewing the construction in terms of an infinite-dimensional hyperkähler quotient. In the original context of Higgs bundles without singularities, this was explained by Hitchin in [9]. In the case of parabolic Higgs bundles which we are considering here, the construction of \mathcal{M} and g by hyperkähler quotient was given by Konno in [16].

2.5 The special Kähler structure

The regular part \mathcal{B}_{reg} of the Hitchin base carries a (rigid) special Kähler structure in the sense of [17], as follows.

Definition 2.28 (Charge lattices for $G = \text{U}(K)$). Suppose $G = \text{U}(K)$ and $\vec{\varphi} \in \mathcal{B}_{\text{reg}}$. Then define:

- $\Gamma^{\text{flavor}} = \bigoplus_{p \in P} \Gamma^{\text{flavor}, p}$ where $\Gamma^{\text{flavor}, p}$ is the weight lattice of the centralizer of $m_p^{\mathbb{C}}$ in G ,
- $\Gamma_{\vec{\varphi}}^{\text{gauge}} = H_1(\Sigma_{\vec{\varphi}}, \mathbb{Z})$,
- $\Gamma_{\vec{\varphi}} = \left(\Gamma^{\text{flavor}} \oplus H_1(\Sigma'_{\vec{\varphi}}, \mathbb{Z}) \right) / \sim$

where $\Sigma'_{\vec{\phi}} = \Sigma_{\vec{\phi}} \setminus \rho^{-1}(P)$, and the relation \sim is as follows. A point of $\pi^{-1}(p)$ with ramification index ν corresponds to a factor $U(\nu)$ in the centralizer of $m_p^{\mathbb{C}}$. We identify a clockwise loop around this point with the weight of the determinant representation of this $U(\nu)$ factor.

Example 2.29 (Charge lattices for $G = U(K)$ with generic puncture data). When the puncture data is generic, we can say all this more simply:

- Γ^{flavor} is the free \mathbb{Z} -module generated by loops around the points of $\rho^{-1}(P)$,
- $\Gamma_{\vec{\phi}}^{\text{gauge}} = H_1(\Sigma_{\vec{\phi}}, \mathbb{Z})$,
- $\Gamma_{\vec{\phi}} = H_1(\Sigma'_{\vec{\phi}}, \mathbb{Z})$.

[discuss $SU(K)$ case]

In any case, these lattices assemble into an exact sequence of local systems of lattices over \mathcal{B}_{reg} ,

$$0 \rightarrow \Gamma^{\text{flavor}} \rightarrow \Gamma \rightarrow \Gamma^{\text{gauge}} \rightarrow 0. \quad (2.15)$$

Γ^{gauge} has a nondegenerate skew pairing, the intersection pairing on $H_1(\Sigma_{\vec{\phi}}, \mathbb{Z})$. We will sometimes write local formulas using a local trivialization of Γ^{gauge} by “ A and B cycles” obeying

$$\langle A^I, A^J \rangle = 0, \quad \langle B_I, B_J \rangle = 0, \quad \langle A^I, B_J \rangle = \delta_J^I. \quad (2.16)$$

Definition 2.30 (Period map). Let λ denote the meromorphic 1-form on $\text{Tot}[K_{\mathbb{C}}(P)]$, induced by the tautological (Liouville) holomorphic 1-form on $\text{Tot}[K_{\mathbb{C}}]$. λ has poles along $\rho^{-1}(P)$. The *period map* is the map

$$Z : \Gamma_{\vec{\phi}} \rightarrow \mathbb{C}, \quad Z_{\gamma} = \oint_{\gamma} \lambda \quad (2.17)$$

which we could also view as an element $Z \in \Gamma_{\mathbb{C}}^*$.

The restriction of Z to the image of Γ^{flavor} is constant. It follows that the derivative $dZ : \Gamma \rightarrow T^*\mathcal{B}_{\text{reg}}$ descends to $dZ : \Gamma^{\text{gauge}} \rightarrow T^*\mathcal{B}_{\text{reg}}$, which we can also view as

$$dZ \in T^*\mathcal{B}_{\text{reg}} \otimes (\Gamma_{\mathbb{C}}^{\text{gauge}})^*. \quad (2.18)$$

Let \langle, \rangle denote the intersection pairing on Γ^{gauge} and $\langle\langle, \rangle\rangle$ its inverse on $(\Gamma^{\text{gauge}})^*$. Then we can define a 2-form $\langle\langle dZ, dZ \rangle\rangle \in \Omega^{2,0}(\mathcal{B}_{\text{reg}})$. Using a local trivialization of Γ^{gauge} ,

$$\langle\langle dZ, dZ \rangle\rangle = \sum_{I=1}^n dZ_{A^I} \wedge dZ_{B_I}. \quad (2.19)$$

Proposition 2.31 (Lagrangian property). We have

$$\langle\langle dZ, dZ \rangle\rangle = 0. \quad (2.20)$$

Note that (2.20) is automatic in case $\dim_{\mathbb{C}} \mathcal{B} = 1$, but otherwise it is a nontrivial constraint on Z . The idea of the proof of (2.20) (in the unpunctured case) is to consider two tangent vectors to \mathcal{B} i.e. infinitesimal variations of Σ , and study the corresponding variations δ_1, δ_2 of the cohomology class $[\lambda] \in H^1(\Sigma, \mathbb{C})$ by integrating along arbitrary 1-cycles on Σ . δ_1, δ_2 turn out to be of type $(1, 0)$ (you can get them by pairing the normal variation of Σ with the holomorphic symplectic form on T^*C) and thus $\int_{\Sigma} \delta_1 \wedge \delta_2 = 0$.

Definition 2.32 (Special Kähler form on \mathcal{B}_{reg}). Next we define a 2-form on \mathcal{B}_{reg} by

$$\omega = \langle\langle dZ, d\bar{Z} \rangle\rangle. \quad (2.21)$$

In terms of A and B cycles, $\omega = \sum_{I=1}^r dZ_{A^I} \wedge d\bar{Z}_{B_I}$.

Proposition 2.33 (Positivity of ω). ω is a positive $(1, 1)$ -form on \mathcal{B}_{reg} , and thus it defines a Kähler metric on \mathcal{B}_{reg} .

From the existence of Z with the above properties one can deduce all the structure of special Kähler manifold on \mathcal{B}_{reg} . In particular, for any choice of linearly independent “ A cycles” $A^1, \dots, A^r \in \Gamma^{\text{gauge}}$ with $\langle A^I, A^J \rangle = 0$, lifted to $\tilde{A}^I \in \Gamma$, the functions $a^I = Z_{\tilde{A}^I}$ give a local coordinate system, so-called “special coordinates.” It is conventional to also define dual coordinates $a_{D,J} = Z_{\tilde{B}_J}$.

2.6 The semiflat metric

As we have said, $\mathcal{M}_{\vec{\phi}}$ is a space of flat $U(1)$ -connections over $\Sigma_{\vec{\phi}} \setminus \rho^{-1}(P)$, with fixed holonomies around $\rho^{-1}(P)$. In particular, for each $\gamma \in \Gamma_{\vec{\phi}}$ there is a corresponding holonomy $\theta_{\gamma} : \mathcal{M}_{\vec{\phi}} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$. Their differentials can be assembled into

$$d\theta \in \Omega^1(\mathcal{M}) \otimes (\Gamma^{\text{gauge}})^*. \quad (2.22)$$

If we choose a basis for $\Gamma_{\vec{\phi}}$ then we get $\mathbb{R}/2\pi\mathbb{Z}$ -valued coordinates $\theta_1, \dots, \theta_{2r}$ on $\mathcal{M}_{\vec{\phi}}$.

Definition 2.34 (Semiflat metric). The *semiflat metric* g^{sf} on \mathcal{M}_{reg} is the metric whose Kähler form in structure I_1 is

$$\omega_1^{\text{sf}} = 2\langle\langle dZ, d\bar{Z} \rangle\rangle - \langle\langle d\theta, d\theta \rangle\rangle. \quad (2.23)$$

Relative to special coordinates, (2.23) becomes

$$\omega_1^{\text{sf}} = -4i(\text{Im } \tau)_{IJ}(da^I \wedge d\bar{a}^J) - 2d\theta_{A^I} \wedge d\theta_{B_I}, \quad (2.24)$$

where $\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a^J}$.

As we will show below, ω_1^{sf} is the Kähler form for the hyperkähler metric on \mathcal{M} in the case $G = U(1)$.

2.7 The Hitchin section

Definition 2.35 (Hitchin section, for $G = \mathrm{SU}(2)$). Choose a spin structure on C and thus a line bundle $K_C^{\frac{1}{2}}$. Given $\phi_2 \in \mathcal{B}$ we consider the Higgs bundle (E, φ) :

$$E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}(-P), \quad \varphi = \begin{pmatrix} 0 & \phi_2 \\ 1 & 0 \end{pmatrix}. \quad (2.25)$$

Note ϕ_2 is a section of $\mathrm{Hom}(K_C^{-\frac{1}{2}}(-P), K_C^{\frac{1}{2}}) \otimes K_C(P) = K_C(P)^2$ as needed, and by 1 we mean the canonical section of $\mathrm{Hom}(K_C^{\frac{1}{2}}, K_C^{-\frac{1}{2}}(-P)) \otimes K_C(P) = \mathcal{O}$. If at each $p \in P$ we set $m_p^{\mathbb{R}} = (\pi, \pi)$ and $m_p^{\mathbb{C}} = (\sqrt{\mathrm{Res}_p \phi_2}, -\sqrt{\mathrm{Res}_p \phi_2})$, then (E, φ) is a stable (C, G, \vec{m}) -Higgs bundle lying in the fiber \mathcal{M}_{ϕ_2} . This gives a section of the Hitchin map for this (C, G, \vec{m}) .

There is a similar construction of sections of the Hitchin map for other G ; for the unpunctured case it originates in [18].

3 Hyperkähler structure

3.1 The hyperkähler structure of \mathcal{M}

So far we have focused on just one of the complex structures of \mathcal{M} . Now let us look at the other complex structures I^ζ , $\zeta \in \mathbb{C}^\times$. (note $I^{\zeta=0} = I_1, I^{\zeta=i} = I_2, I^{\zeta=1} = I_3$.)

Given a Higgs bundle (E, φ) and solution h of Hitchin's equations (2.12) there is a corresponding 1-parameter family of flat G_C -connections over C :

$$\nabla(\zeta) = \zeta^{-1} \varphi + D + \zeta \varphi^\dagger. \quad (3.1)$$

Proposition 3.1. For any $\zeta \in \mathbb{C}^\times$, the map $(E, \varphi) \rightarrow \nabla(\zeta)$ identifies

$$(\mathcal{M}, I^\zeta, \Omega^\zeta) \xrightarrow{\sim} (\mathcal{M}^b, \Omega^{ABG}) \quad (3.2)$$

Here $\mathcal{M}^b = \mathcal{M}^b(G, C, m^\zeta)$ is the moduli space of flat reductive G_C -connections over $C \setminus P$ with some prescribed structure at the points of P — e.g. in the case of generic puncture data, the holonomy around $p \in P$ has to be conjugate to the diagonal matrix $\exp(m_p^\zeta)$ with

$$m_p^\zeta = \zeta^{-1} m_p^{\mathbb{C}} + i m_p^{\mathbb{R}} + \zeta \bar{m}_p^{\mathbb{C}}. \quad (3.3)$$

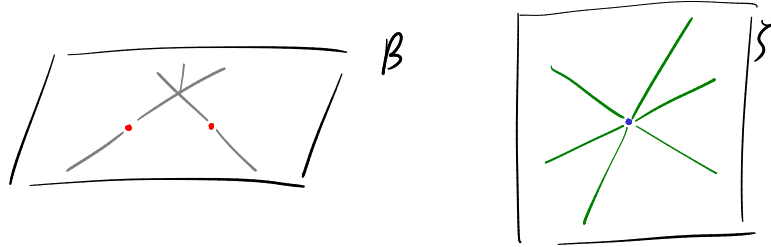
Ω^{ABG} is the standard ‘‘Atiyah-Bott-Goldman’’ symplectic structure on \mathcal{M}^b .

3.2 Our strategy

Proposition 3.1 implies that any holomorphic function \mathcal{X} on \mathcal{M}^b , when applied to the flat connection $\nabla(\zeta)$, becomes a holomorphic function on (\mathcal{M}, I^ζ) . Extending this to coordinate systems, any holomorphic Darboux coordinate system $\{\mathcal{X}_i\}$ on $(\mathcal{M}^b, \Omega^{ABG})$ becomes a holomorphic Darboux coordinate system on (\mathcal{M}, I^ζ) .

Our aim is to use this idea to calculate holomorphic Darboux coordinates $\mathcal{X}_\gamma(\zeta)$ of a given fixed Higgs bundle, in an “explicit” way, in terms of the data (Z, θ) . Note that since $\nabla(\zeta)$ varies *holomorphically* with ζ , the coordinates $\mathcal{X}_\gamma(\zeta)$ do as well.

- Q: Which holomorphic Darboux coordinate system on $(\mathcal{M}^b, \Omega^{ABG})$ will you use?
A: We actually will not use just one: instead, as we move around on the Hitchin base \mathcal{B} and/or vary the argument of ζ , we will choose different coordinate systems in different regions, separated by codimension-1 “walls.”



- Q: Why will you do that? A: Because we want to study these coordinates through their analytic properties in the ζ -plane, and only certain coordinates are “good” as $\zeta \rightarrow 0, \infty$; moreover, which ones are “good” changes as we move around on \mathcal{B} or vary the argument of ζ .
- Q: How does this help you get the metric? A: On the moduli space of Higgs bundles (\mathcal{M}, I_1) we already have the holomorphic symplectic form $\Omega_1 = \omega_2 + i\omega_3$. All that is missing is the third symplectic form ω_1 . Once we have holomorphic Darboux coordinate functions $\mathcal{X}_\gamma(\zeta)$, we can specialize them to say $\zeta = 1$ and get a formula for the holomorphic symplectic form $\Omega^{\zeta=1} = \Omega_3 = \omega_1 + i\omega_2$; then the desired ω_1 is just $\text{Re } \Omega^{\zeta=1}$.
- Q: Won’t the jumping of the $\mathcal{X}_\gamma(\zeta)$ at the walls cause a problem? A: No, the jumps are always by *symplectomorphisms*, so that even though $\mathcal{X}_\gamma(\zeta)$ jumps, Ω^ζ doesn’t.

Remark 3.2 (The case of $G = \text{U}(1)$). A toy model for what we are doing arises in the case $G = \text{U}(1)$. In that case we can easily produce holomorphic Darboux coordinates: just take the \mathbb{C}^\times -valued *holonomies* of the complex flat connection $\nabla(\zeta)$, which are simply⁵

$$\mathcal{X}_\gamma(\zeta) = \exp \left(\zeta^{-1} Z_\gamma + i\theta_\gamma + \zeta \bar{Z}_\gamma \right). \quad (3.4)$$

These functions obey the relation

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}. \quad (3.5)$$

If we choose a basis $\{\gamma_1, \dots, \gamma_{2r}\}$ for Γ , then the corresponding functions $\{\mathcal{X}_{\gamma_1}, \dots, \mathcal{X}_{\gamma_{2r}}\}$ give coordinates on \mathcal{M} ; by abuse of language we refer to the whole collection \mathcal{X}_γ as a coordinate system.

⁵Note that in the case $G = \text{U}(1)$ we have $\Sigma = \mathbb{C}$, so $\Gamma = H_1(\mathbb{C}, \mathbb{Z})$.

Using these coordinates at $\zeta = 1$ we get

$$\omega_1 = \operatorname{Re} \Omega_3 = \operatorname{Re} \langle\langle d \log \mathcal{X}, d \log \mathcal{X} \rangle\rangle = 2 \langle\langle dZ, d\bar{Z} \rangle\rangle - \langle\langle d\theta, d\theta \rangle\rangle \quad (3.6)$$

which is the semiflat Kähler form (2.23). Thus we have proven that, in case $G = \mathrm{U}(1)$, Hitchin's metric g agrees with the semiflat metric g^{sf} on the nose!

We could try to do similarly for general G . Indeed, if we choose a local section γ of the local system Γ , then the formula (3.4) makes good local sense, and the functions $\mathcal{X}_\gamma(\zeta)$ so defined give an honest local coordinate system on \mathcal{M} . The trouble is that it is not a *holomorphic* coordinate system in complex structure I^ζ , so we cannot use it to compute the hyperkähler metric. Instead we will construct some other functions $\mathcal{X}_\gamma(\zeta)$ which are true holomorphic Darboux coordinates.

Happily, it will turn out that (3.4) does not have to be abandoned completely: it is true “asymptotically”, in two different senses — either as we go to infinity in \mathcal{M} , or as we take $\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$.

4 The coordinates

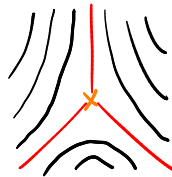
What are the true holomorphic coordinates $\mathcal{X}_\gamma(\zeta)$ on (\mathcal{M}, I^ζ) which we will use?

4.1 Defining the coordinates

Specialize to $G = \mathrm{SU}(2)$, and fix $\phi_2 \in \mathcal{B}$ and $\zeta \in \mathbb{C}^\times$. ϕ_2 is a holomorphic section of $K_C(P)^2$, i.e. a meromorphic quadratic differential on C with double poles at P , of residues $(m_p^C)^2$.

Definition 4.1 (ζ -trajectories of a quadratic differential). A ζ -trajectory of ϕ_2 is a path on C along which $\zeta^{-1} \sqrt{\phi_2}$ (with either choice of sign for $\sqrt{\phi_2}$) is a *real* and nowhere vanishing form.

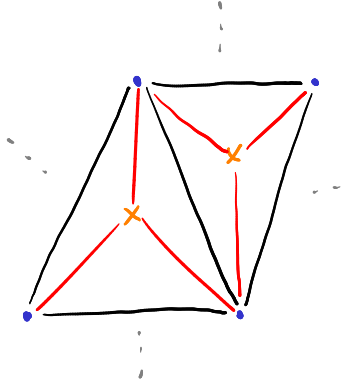
Proposition 4.2 (ζ -trajectories give a foliation). The ζ -trajectories are the leaves of a singular foliation of C , with singularities at the zeroes and poles of ϕ_2 . At each zero of ϕ_2 , the foliation by ζ -trajectories has a three-pronged singularity, as shown below.



At each pole p of ϕ_2 with $\zeta^{-1} m_p^C \notin \mathbb{R}$, the foliation has a “spiraling” singularity, as shown below.

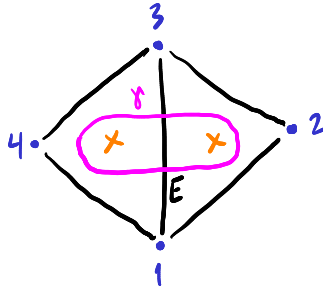


Proposition 4.3 (Ideal triangulation determined by the ζ -trajectories). Suppose (ϕ_2, ζ) is generic, in the sense that $\zeta^{-1}Z_\gamma \notin \mathbb{R}$ for all $\gamma \in \Gamma_{\phi_2}$. Then the ζ -trajectories determine an ideal triangulation $T(\phi_2, \zeta)$ of C , by the picture below.



The proof of [Proposition 4.3](#) is given in [2], leaning heavily on the analysis of trajectories given by Strebel [19].

Definition 4.4 (Fock-Goncharov coordinate attached to an edge). Fix an edge $E \in T(\phi_2, \vartheta)$. E determines a class $\gamma \in \Gamma_{\phi_2}$, shown below:⁶



To define $\mathcal{X}_\gamma(\zeta)$, we consider the connection $\nabla(\zeta)$ restricted to the quadrilateral shown. Its space of flat sections is a 2-dimensional vector space V , equipped with 4 distinguished lines $l_i \subset V$: l_i consists of the flat sections which have exponentially *decaying* norm as we go into the i -th puncture along a leaf of $T(\phi_2, \zeta)$. Said otherwise, the l_i give 4 points of $\mathbb{C}P^1$. We define $\mathcal{X}_\gamma(\zeta)$ to be the $SL(2, \mathbb{C})$ -invariant cross-ratio of these 4 points:

$$\mathcal{X}_\gamma(\zeta) = -\frac{(l_1 \wedge l_2)(l_3 \wedge l_4)}{(l_2 \wedge l_3)(l_4 \wedge l_1)}. \quad (4.1)$$

This definition comes essentially from the work of Fock-Goncharov [8]; it is a complexification of the notion of shear coordinate.

Applying [Definition 4.4](#) for all edges E of $T(\phi_2, \vartheta)$ gives functions $\mathcal{X}_\gamma(\zeta)$ with γ running over a basis for a finite-index sublattice of Γ . They are local Darboux coordinates:

$$\Omega^\zeta = \langle\langle d \log \mathcal{X}(\zeta), d \log \mathcal{X}(\zeta) \rangle\rangle. \quad (4.2)$$

⁶More precisely, the picture shows only the projection of γ to C , and does not show the orientation. The ambiguity can be fixed as follows: the intersection $\langle \gamma, \hat{E} \rangle$ should be positive, where \hat{E} denotes one of the lifts of E to Σ , oriented so that λ is a positive 1-form along \hat{E} .

4.2 Asymptotic behavior of the coordinates

The main asymptotic property of the coordinates $\mathcal{X}_\gamma(\zeta)$ is:

Conjecture 4.5. Fix a point of \mathcal{M} . Then, as $\zeta \rightarrow 0$ along any ray,

$$\mathcal{X}_\gamma(\zeta) \sim c_\gamma \exp\left(\zeta^{-1}Z_\gamma + i\theta_\gamma\right). \quad (4.3)$$

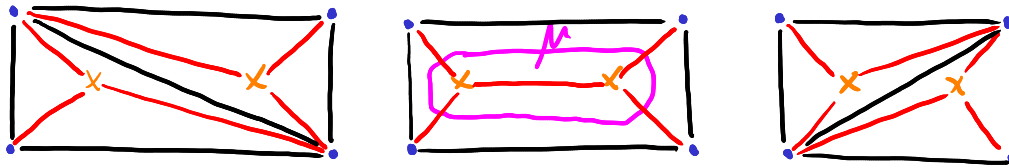
If our chosen point of \mathcal{M} has all $\theta_\gamma = 0$, then all $c_\gamma = 1$, so in that case

$$\mathcal{X}_\gamma(\zeta) \sim \exp\left(\zeta^{-1}Z_\gamma\right). \quad (4.4)$$

(The idea: it should follow from the exact WKB method. Morally, the connection $\nabla(\zeta) = \varphi/\zeta + \dots$ is dominated by the leading term. [\[explain a little more?\]](#))

4.3 Piecewise analytic behavior of the coordinates

As we vary (ϕ_2, ζ) , the function $\mathcal{X}_\gamma(\zeta)$ is only piecewise smooth: it suffers a jump whenever the triangulation $T(\phi_2, \zeta)$ changes. The simplest kind of jump is shown below:



This jump is associated with the “saddle connection” connecting two zeroes of ϕ_2 , appearing in the middle of the figure. Such a saddle connection can only appear when $\zeta^{-1}Z_\mu \in \mathbb{R}_-$. The coordinates on the two sides of the jump are related by:

$$\mathcal{X}_\gamma \rightarrow \mathcal{X}_\gamma(1 + \mathcal{X}_\mu)^{\langle \mu, \gamma \rangle}. \quad (4.5)$$

A similar (but more intricate) phenomenon occurs when we cross a (ϕ_2, ζ) for which an annulus of closed trajectories appears: then the \mathcal{X}_γ undergo a jump of the form

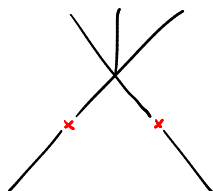
$$\mathcal{X}_\gamma \rightarrow \mathcal{X}_\gamma(1 - \mathcal{X}_\mu)^{-2\langle \mu, \gamma \rangle}. \quad (4.6)$$

Both of these are instances of the following general structure:

$$\mathcal{X}_\gamma \rightarrow \mathcal{X}_\gamma(1 - \sigma(\mu)\mathcal{X}_\mu)^{DT(\mu)\langle \mu, \gamma \rangle} \quad (4.7)$$

where for a saddle connection we have $DT(\mu) = +1$ and $\sigma(\mu) = -1$, while for a closed loop we have $DT(\mu) = -2$ and $\sigma(\mu) = +1$.

Remark 4.6 (Wall structure on \mathcal{B}). Suppose we fix ζ and move around in \mathcal{M} . We get an interesting structure on \mathcal{B} :



Each simple discriminant point (where ϕ_2 develops a double zero) emits two walls. Each of these two walls carries a transformation of the coordinates \mathcal{X}_γ , of the form (4.7), with $DT(\mu) = +1$.

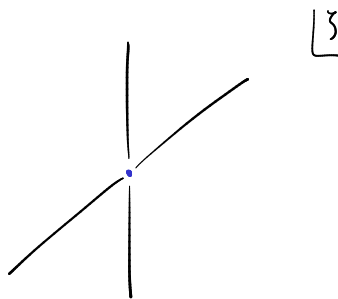
These two walls are “hyperplanes” in the sense of the special Kähler structure on \mathcal{B}_{reg} : they are of the form $\zeta^{-1}Z_{\pm\mu} \in \mathbb{R}_-$, where μ is the vanishing cycle. When walls collide, they can generate new walls. The new walls are also of the form $Z_\mu/\zeta \in \mathbb{R}_-$ for some μ , and carry transformations of the form (4.7). The precise structure of the new walls is completely determined by the requirement that the \mathcal{X}_γ are well defined; this is essentially an application of the Kontsevich-Soibelman wall-crossing formula [4].

Remark 4.7 (More interesting discriminant points). When $\mathcal{B}_{\text{sing}} \subset \mathcal{B}$ meets itself, there’s a more interesting structure of walls emanating. Many things can happen; here are two:



At left is what happens when three zeroes of ϕ_2 collide: this produces a discriminant point which emits 5 walls. Each of these walls carries a transformation of the form (4.7) with $DT(\mu) = 1$. At right is the example of $G = \text{SU}(2)$, $g_C = 0$, $|P| = 4$: as we adjust all m^C to zero, so that all zeroes of ϕ_2 move onto punctures, the six discriminant points collapse into one. The resulting point emits walls with every rational slope, each one carrying a product of two transformations (4.7), with $DT(\mu) = 8, DT(2\mu) = -2$.

For our purpose right now, we do not need to understand the details of the wall structure on \mathcal{B} : rather, what we need is to understand what happens when we fix the Higgs bundle and just let ζ vary. Then $\mathcal{X}_\gamma(\zeta)$ depends on ζ in a piecewise-analytic way: the collection $\{\mathcal{X}_\gamma(\zeta)\}_{\gamma \in \Gamma}$ jumps at various rays ℓ in the ζ -plane.



At each such ray, the jump is a product of transformations of the form (4.7), where the μ in (4.7) can be any $\mu \in \Gamma$ such that $Z_\mu/\zeta \in \mathbb{R}_-$ along ℓ .

Example 4.8 (Finite chamber for $G = \text{SU}(2)$, $g_C = 0$, $|P| = 4$). In case $G = \text{SU}(2)$, $g_C = 0$, $|P| = 4$, at least for some choices of m^C , there exists a domain $D \subset \mathcal{B}$ such that, when $\phi_2 \in D$, the function $\mathcal{X}_\gamma(\zeta)$ jumps at exactly 24 rays in the ζ -plane, corresponding to 24 lattice vectors $\gamma_1, \dots, \gamma_{24} \in \Gamma$ for which $DT(\gamma_i) = 1$. For all other $\gamma \in \Gamma$ we have

$\text{DT}(\gamma) = 0$. One concrete example of a $\phi_2 \in D$ is

$$\phi_2 = \frac{z^4 - \frac{1}{2}(z^4 - 1)}{(z^4 - 1)^2} dz^2. \quad (4.8)$$

Remark 4.9 (Even simpler cases). In “wildly ramified” examples (where we allow higher-order poles for the Higgs field) the analytic structure of the functions $\mathcal{X}_\gamma(\zeta)$ can be even simpler: in the simplest example, we can arrange that in some domain $D \subset \mathcal{B}$ there are jumps along just 4 rays, as shown in the picture above. We will discuss that example more below.

Remark 4.10 (Higher rank). So far we focused on $G = \text{SU}(2)$. The case of $G = \text{U}(2)$ is more or less the same. For $K > 2$ the situation becomes more interesting: instead of studying ideal triangulations $T(\phi_2, \zeta)$ one needs to study *WKB spectral networks* $\mathcal{W}(\vec{\phi}, \zeta)$, as defined in [20].⁷ Then again there is a recipe for invariants $\text{DT}(\gamma)$ and coordinates $\mathcal{X}_\gamma(\zeta)$. This recipe however depends on some conjectures: most importantly, to construct the $\mathcal{X}_\gamma(\zeta)$ one has to solve a certain linear algebra problem determined by the network $\mathcal{W}(\vec{\phi}, \zeta)$. Thus the construction has been carried out to the end in various special cases but not for arbitrary (G, C, \vec{m}) and $(\vec{\phi}, \zeta)$. In the special cases which have been worked out [22, 23] the $\mathcal{X}_\gamma(\zeta)$ turn out to be cluster coordinate systems on \mathcal{M} , i.e. they belong to the distinguished atlas constructed by Fock-Goncharov in [8].

5 The integral equation

Conjecture 5.1 (Integral equation for $\theta_\gamma = 0$). When all $\theta_\gamma = 0$,

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left[\frac{1}{4\pi i} \sum_{\mu \in \Gamma} \text{DT}(\mu) \langle \gamma, \mu \rangle \int_{Z_\mu \mathbb{R}_-} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \sigma(\mu) \mathcal{X}_\mu(\zeta')) \right] \quad (5.1)$$

where

$$\mathcal{X}_\gamma^{\text{sf}}(\zeta) = \exp \left(\zeta^{-1} Z_\gamma + \zeta \bar{Z}_\gamma \right) \quad (5.2)$$

and $\sigma(\mu) = \pm 1$ for each μ .

The functions $\mathcal{X}_\gamma(\zeta)$ appear on both sides of (5.1). Thus (5.1) is an integral equation, which needs to be solved for the whole collection $\{\mathcal{X}_\gamma(\zeta)\}_{\gamma \in \Gamma}$ at once, rather than an integral formula.

- Q: Why this equation? A: It is expected to lead to $\mathcal{X}_\gamma(\zeta)$ with the right analytic properties in the ζ -plane: asymptotics as $\zeta \rightarrow 0, \infty$ and jumps at the rays $Z_\mu / \zeta \in \mathbb{R}_-$ with $\text{DT}(\mu) \neq 0$. The optimistic hope is that these analytic properties are strong enough to *determine* $\mathcal{X}_\gamma(\zeta)$.

⁷In retrospect, essentially the same graphs had appeared earlier as *Stokes graphs* associated to the WKB analysis of linear scalar K -th order ODEs; see in particular [21].

- Q: How do you actually solve it? A: By iteration: pick $\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta)$ as initial guess, and then iterate.
- Q: Why would you think that that iteration would converge? A: If all $|Z_\gamma|$ are large enough, and $\text{DT}(\mu)$ doesn't grow too fast as a function of μ (e.g. if only finitely many are nonzero) saddle-point estimates show the iteration defines a contraction mapping, so it must converge to a (unique) fixed point.
- Q: How does this lead to asymptotic predictions? A: If we substitute $\mathcal{X} = \mathcal{X}^{\text{sf}}$, the log in the integrand is bounded above by $e^{-2|Z_\mu|}$; thus we expect that the first step of the iteration is already suppressed by e^{-2M} where M is the minimum $|Z_\mu|$ for which $\text{DT}(\mu) \neq 0$, and later steps should be further exponentially suppressed. That suggests that just truncating to the zeroth iteration (i.e. taking g^{sf}) would already give a result exponentially close to the true metric, and the accuracy will improve with each iteration we take. In particular we can truncate to the *first* iteration. Working this out leads to

$$g = g^{\text{sf}} - \frac{2}{\pi} \sum_{\mu \in \Gamma} \text{DT}(\mu) K_0(2|Z_\mu|) d|Z_\mu|^2 + \dots \quad (5.3)$$

where K_0 is the modified Bessel function. Note that $K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$, so $g - g^{\text{sf}}$ is already exponentially suppressed as e^{-2M} . The omitted terms \dots should be of order e^{-4M} .

6 Numerical tests

In this last section I want to explain some of the numerical evidence supporting [Conjecture 5.1](#). This evidence was obtained in ongoing joint work with David Dumas.

6.1 The simplest nontrivial Hitchin equations

We consider the case of $G = \text{SU}(2)$ and $C = \mathbb{CP}^1$, where we take ϕ_2 to be a *polynomial* quadratic differential of degree n , say

$$\phi_2 = P(z) dz^2, \quad (6.1)$$

and the Higgs bundles are of the form⁸

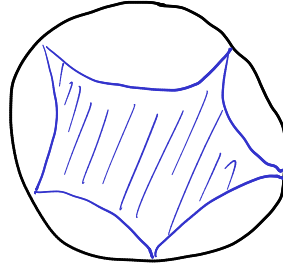
$$\mathcal{E} = \mathcal{O}\left(\frac{n}{4}\right) \oplus \mathcal{O}\left(-\frac{n}{4}\right), \quad \varphi = \begin{pmatrix} 0 & P(z) \\ 1 & 0 \end{pmatrix} dz. \quad (6.2)$$

Strictly speaking this case is outside the class of examples we have considered so far: the singularity of the Higgs field φ at $z = \infty$ is a non-simple pole. The theory of Higgs

⁸Here $\mathcal{O}(\pm \frac{n}{4})$ means a parabolic line bundle over \mathbb{CP}^1 of degree $\pm \frac{n}{4}$, trivialized away from $z = \infty$; see [24] for some more explanation.

bundles with non-simple poles has been worked out by Biquard-Boalch and Mochizuki, extending Simpson's work which we used above for simple poles. See e.g. [14]. In short, essentially all of the theory carries over to this situation, and indeed these examples turn out to be simpler in some respects.

Solving Hitchin's equation (2.12) in the specific case (6.2) is equivalent to finding a harmonic map from \mathbb{C} to the hyperbolic disc whose Hopf differential is ϕ_2 . Such a map exists and is unique up to $\mathrm{PSL}(2, \mathbb{R})$ action on the disc; its image turns out to be an ideal polygon with $n + 2$ vertices [25], again determined up to $\mathrm{PSL}(2, \mathbb{R})$ action.



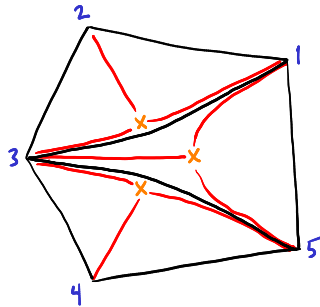
6.2 The \mathcal{X}_γ

All of the analysis we described in §4-§5 carries over to this case. The quantities $\mathcal{X}_\gamma(\zeta)$ in this case are monomials in cross-ratios r_{ijkl} of $n + 2$ flat sections s_i of $\nabla(\zeta)$, determined by their asymptotic behavior along $n + 2$ rays approaching $z \rightarrow \infty$.⁹

For example, when $n = 3$, we have 5 asymptotic rays and $\dim \mathcal{M} = 2$. The $\mathcal{X}_\gamma(\zeta)$ are monomials in 2 out of the 5 possible cross-ratios r_{ijkl} . Precisely *which* cross-ratios we take depends on P and ζ : it is determined by a triangulation $T(\phi_2, \zeta)$ of an $(n + 2)$ -gon, constructed similarly to the ideal triangulation $T(\phi_2, \zeta)$ of a punctured surface C constructed in Proposition 4.3. We take the concrete example

$$P(z) = R^2(z^3 - 1), \quad R \in \mathbb{R}_+. \quad (6.3)$$

In this case the triangulation $T(\phi_2, \zeta = 1)$ looks like:



From this picture we can read off that the relevant cross-ratios are r_{1235} and r_{1345} .

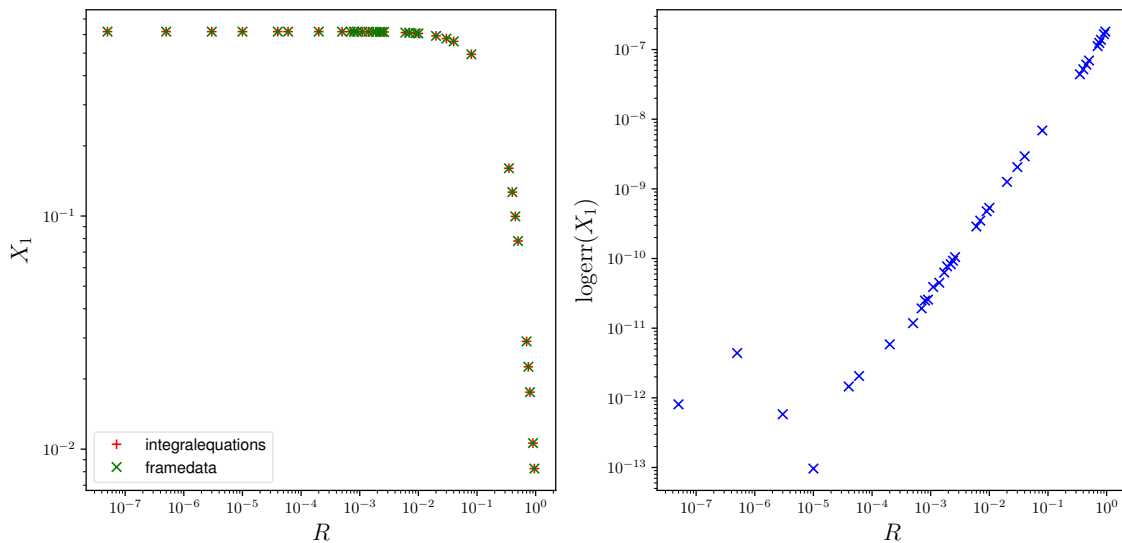
⁹When $\zeta = 1$ these quantities have a particularly simple geometric meaning: the flat sections s_i are the asymptotic vertices of the polygon above, so the r_{ijkl} are literally the cross-ratios of these asymptotic vertices. In particular they are *real*: this is a special feature arising for these particular Higgs bundles when $|\zeta| = 1$.

6.3 Numerical results

In joint work with David Dumas we have computed the $X_\gamma = \mathcal{X}_\gamma(\zeta = 1)$ numerically in this example, in two different ways:

- by directly solving Hitchin’s equation i.e. finding the harmonic maps,
- by solving the integral equations of §5.

Preliminary numerical results for the quantity $X_1 = r_{1235}$ are plotted below, for R ranging from $R = 10^{-8}$ to $R = 1$. On the left we show the values of X_1 computed by both methods, which we call X_1^i and X_1^f ; on the scale of that plot it appears as though $X_1^i = X_1^f$, as predicted by [Conjecture 5.1](#). On the right we plot the difference $\log(X_1^i) - \log(X_1^f)$, which never exceeds 2×10^{-7} over the range of R shown. We expect that this residual can be attributed to numerical error in the two computations (but this remains to be understood in detail.)



Dumas and I have also made similar numerical calculations in other cases:

- As above but with a polynomial $P(z)$ of degree 4,
- For Higgs bundles in the plane with $G = \text{SU}(3)$, with $\phi_2 = 0$ and $\phi_3 = P(z)dz^3$, for $P(z)$ of degree 2 or 3. (In this case the coordinates $\mathcal{X}_\gamma(\zeta)$ are not built from a triangulation, but rather from a “WKB spectral network” built from ϕ_3 , as we explained in [Remark 4.10](#). The construction in this case is described in [\[23\]](#).)

The results are qualitatively similar to those shown above: they appear to support [Conjecture 5.1](#).

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