Metric on moduli of Higgs bundles

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These are notes for a pair of lectures, given at the second Simons Workshop in Mathematics, July 15 and 17, 2024.

1 Introduction

The aim of these lectures is to describe a *conjectural* approach to more explicitly understanding the hyperkähler metrics on moduli spaces of Higgs bundles. I will specialize to $\mathrm{SL}(2)$-Higgs bundles, but the picture is similar for any reductive G.

Basic philosophy: replace $\mathrm{SL}(2)$-Higgs bundles and flat connections over a Riemann surface C by $\mathrm{GL}(1)$-Higgs bundles and flat connections over a branched double cover Σ_{ϕ_2} of C, called *spectral curve*. The surprise is how well this works. Indeed, the conjecture says that the metric on $\mathcal{M}$ is constructed from two ingredients:

1. The *periods* Z_γ of the spectral curves Σ_{ϕ_2},

2. A collection of integer "Donaldson-Thomas invariants" $\mathrm{DT}(\gamma)$, which count (tropicalizations of) special Lagrangian discs in T^*C with boundary on Σ_{ϕ_2}.

Although ultimately the metrics we want to understand are smooth and have no obvious integrality properties, this way of understanding them involves Donaldson-Thomas invariants, cluster transformations, wall-crossing phenomena. It is most digestible near asymptotic infinity, but in principle gives information everywhere on $\mathcal{M}$.

The strategy of the lectures will be roughly:

1. The moduli space $\mathcal{M}$, its basic features and hyperkähler structure; the weak form of the metric conjecture (now almost proven).

2. The strong form of the conjecture (idea: calculate the nonabelian Hodge map), and the available evidence that the conjecture is correct.

This is a review of joint work with Davide Gaiotto and Greg Moore, follow-up work with Lotte Hollands and David Dumas; closely related work by Mazzeo-Swoboda-Weiss-Witt, Fredrickson, Mochizuki; inspired by Fock-Goncharov, Kontsevich-Soibelman, Hitchin, Corlette, Donaldson, Simpson, Biquard-Boalch.

1.1 References

The conjecture reviewed in these notes is mostly contained in the papers [1, 2], which are joint work of mine with Davide Gaiotto and Greg Moore. In [3] I reviewed some parts of the conjecture, focusing on the abstract construction of hyperkähler metrics from a special Kähler base and Donaldson-Thomas invariants; in these lectures I focus more on the examples provided by moduli spaces of Higgs bundles, and even more specifically, on one specific moduli space of Higgs bundles with irregular singularity.

These works depend on many prior developments in physics and mathematics. Here I can only single out a few which were of singular importance (for more, see the references in [1, 2]):

- The work [1] originated in an attempt to understand the physical meaning of the remarkable wall-crossing formula for generalized Donaldson-Thomas invariants, given by Kontsevich-Soibelman [4].
- Many of the key constructions in [1] can be understood as infinite-dimensional analogues of constructions used by Cecotti-Vafa and Dubrovin in tt^* geometry [5, 6], with additional inspiration from work of Bridgeland and Toledano Laredo [7].
- The application to Hitchin systems in [2] depended importantly on the work of Fock-Goncharov on moduli spaces of local systems over surfaces [8], as well as the foundational work of Hitchin [9] and Corlette, Donaldson, Simpson [10, 11, 12] on Higgs bundles without singularities, Simpson's extension to Higgs bundles with regular singularities [13], and Biquard-Boalch for Higgs bundles with wild ramification [14].

2 Background on Hitchin system

2.1 The unpunctured case

Let me first recall the simplest case to define (though not the simplest case to study!) The most fundamental reference is [9]. A very useful review can be found in [15] and references therein.

We fix a compact Riemann surface C of genus $g_C \geq 2$.

Definition 2.1 ($\mathrm{SL}(2)$-Higgs bundles). A $\mathrm{SL}(2)$-*Higgs bundle* is a pair (E, φ), where:

- E is a holomorphic vector bundle of rank 2 over C, with $\det E = \mathcal{O}$,
- φ is a holomorphic section of $\operatorname{End} E \otimes K_C(P)$, with $\operatorname{Tr} \varphi = 0$.

With respect to a local trivialization of E, then, φ is represented by a traceless 2×2 matrix whose entries are holomorphic 1-forms.

Definition 2.2 (Stability of $\mathrm{SL}(2)$-Higgs bundles). We say (E, φ) is *stable* if, for all φ-invariant $E' \subset E$, we have $\deg E' < 0$. We say (E, φ) is *strictly polystable* if it is a direct sum of two degree-zero φ-invariant line bundles. We say (E, φ) is *polystable* if it is either stable or strictly polystable.

Proposition 2.3 (Moduli space of Higgs bundles). There is a moduli space $\mathcal{M} = \mathcal{M}(G, C)$ parameterizing polystable $\mathrm{SL}(2)$-Higgs bundles (E, φ) up to equivalence. $\mathcal{M}$ is a manifold away from the locus of strictly-polystable Higgs bundles. It has complex dimension $6g_C - 6$. It carries a natural complex structure I_1 and holomorphic symplectic form Ω_1.

The holomorphic symplectic form Ω_1 comes from the fact that variations of the bundle E are valued in $H^1(\operatorname{End} E)$, while variations of the Higgs field φ are valued in $H^0(\operatorname{End} E \otimes K_C)$, and the two are Serre dual.

2.2 The Hitchin map

Next we exhibit $\mathcal{M}$ as a complex integrable system, i.e. a holomorphic Lagrangian fibration over a base $\mathcal{B}$.

Given a Higgs bundle $(E, \varphi) \in \mathcal{M}$ and $z \in C$, we can consider the eigenvalues of $\varphi(z)$. As z varies these sweep out a curve Σ:

$$\Sigma = \{(z, \lambda) : \lambda^2 + \phi_2(z) = 0\} \subset T^*C\,, \tag{2.1}$$

where

$$\phi_2 = -\frac{1}{2}\operatorname{Tr}\varphi^2\,. \tag{2.2}$$

The projection $\rho : \Sigma \to C$ given by $\rho(z, \lambda) = z$ is a branched double cover, ramified at the zeroes of ϕ_2.

Definition 2.4 (Hitchin base and Hitchin map). Define the *Hitchin base* $\mathcal{B} = H^0(C, K_C^2)$. The *Hitchin map* is the map $\pi : \mathcal{M} \to \mathcal{B}$ given by

$$(E, \varphi) \mapsto \phi_2 = -\frac{1}{2}\operatorname{Tr}\varphi^2\,. \tag{2.3}$$

Proposition 2.5 (Hitchin map has Lagrangian fibers). The fibers $\mathcal{M}_{\phi_2} = \pi^{-1}(\vec{\phi})$ are compact complex Lagrangian subsets of $(\mathcal{M}, I_1, \Omega_1)$. (In particular, $\dim_{\mathbb{C}} \mathcal{B} = \frac{1}{2}\dim_{\mathbb{C}} \mathcal{M}$.)

We can say more precisely what the fibers are, over most of the Hitchin base:

Definition 2.6 (Singular locus and smooth locus). The *singular locus* $\mathcal{B}_{\mathrm{sing}} \subset \mathcal{B}$ is the set of $\vec{\phi} \in \mathcal{B}$ for which $\Sigma_{\vec{\phi}}$ is singular. The *smooth locus* is $\mathcal{B}_{\mathrm{reg}} = \mathcal{B} \setminus \mathcal{B}_{\mathrm{sing}}$. It consists of those ϕ_2 which have only simple zeroes ($4g_C - 4$ of them). Also let $\mathcal{M}_{\mathrm{reg}} = \pi^{-1}(\mathcal{B}_{\mathrm{reg}})$.

Proposition 2.7 (Fibers of the Hitchin map over $\mathcal{B}_{\mathrm{reg}}$). Suppose $\vec{\phi} \in \mathcal{B}_{\mathrm{reg}}$. Then $\mathcal{M}_{\phi_2}$ is a torsor over the compact complex torus $\mathrm{Prym}(\rho : \Sigma_{\phi_2} \to C)$. After choosing spin structures on C and Σ_{ϕ_2}, we can identify $\mathcal{M}_{\phi_2}$ with the space of flat U(1)-connections ∇ over Σ_{ϕ_2}, equipped with a trivialization of $\det \rho_* \nabla$.

So we reach the following picture: a point $\vec{\phi} \in \mathcal{B}_{\mathrm{reg}}$ gives a smooth spectral curve Σ_{ϕ_2}; the torus $\mathcal{M}_{\phi_2}$ is a space of flat $\mathrm{U}(1)$-connections over Σ_{ϕ_2}. When $\vec{\phi} \in \mathcal{B}_{\mathrm{sing}}$, $\mathcal{M}_{\phi_2}$ is compact, but generally singular.

2.3 The hyperkähler metric

A key fact about $\mathcal{M}$ is that it carries a canonically defined hyperkähler metric g. However, g is not easily written in closed form. To construct g, one needs to consider *Hitchin's equation*: given a Higgs bundle (E, φ) this is a PDE for a Hermitian metric h in E, written

$$F_{D_h} + [\varphi, \varphi^{\dagger_h}] = 0. \tag{2.4}$$

Here D_h denotes the Chern connection in (E, h), the unique h-unitary connection compatible with the holomorphic structure of E.

Now there is the following key theorem:

Theorem 2.8 (Existence of harmonic metrics). The equation (2.4) has a solution h for each (E, φ); this h is unique up to scalar multiple. We call h the *harmonic metric*.

Using Theorem 2.8 one can define Hitchin's metric on $\mathcal{M}$, as follows. Given a tangent vector v to $\mathcal{M}$ whose norm we wish to calculate, we represent v by a family of Higgs bundles (E_t, φ_t), with harmonic metrics h_t. Identifying the underlying Hermitian bundles with a single (E, h) we have an arc of unitary connections D_t and skew-Hermitian Higgs fields $\Phi_t = \varphi_t - \varphi_t^\dagger$ on (E, h), determined up to gauge transformations i.e. automorphisms of (E, h). In particular, differentiating at $t = 0$ gives a pair

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (D_t, \Phi_t) = (\dot{A}, \dot{\Phi}) \in \Omega^1(\mathfrak{su}(E)) \oplus \Omega^1(\mathfrak{su}(E)), \tag{2.5}$$

defined up to gauge transformations. Then the norm of v is the L^2 norm

$$g(v, v) = \int_C \|\dot{A}\|^2 + \|\dot{\Phi}\|^2 \tag{2.6}$$

where for $(\dot{A}, \dot{\Phi})$ we choose the representative *minimizing* the norm.

Remark 2.9 (Hyperkähler quotient). I have not really explained why the metric g constructed in this way turns out to be hyperkähler, or even Kähler. The most conceptual explanation of this comes by viewing the construction in terms of an infinite-dimensional hyperkähler quotient. This was explained by Hitchin in [9].

2.4 The semiflat metric

Now we want to describe a first approximation to Hitchin's metric.

The regular part $\mathcal{B}_{\mathrm{reg}}$ of the Hitchin base carries a (rigid) special Kähler structure in the sense of [16], as follows.

The deck transformation $\sigma : \Sigma_{\phi_2} \to \Sigma_{\phi_2}$ induces an action on $H_1(\Sigma, \mathbb{Z})$. Let $H_1(\Sigma_{\phi_2}, \mathbb{Z})^\pm$ denote the ± 1-eigenspaces.

Definition 2.10 (Charge lattice). The *charge lattice* is

$$\Gamma_{\phi_2} = H_1(\Sigma_{\phi_2}, \mathbb{Z})^- . \tag{2.7}$$

Let $\langle , \rangle$ denote the intersection pairing on Γ and $\langle\langle , \rangle\rangle$ its inverse on $\Gamma^*_{\mathbb{R}}$. These lattices make up a local system Γ over $\mathcal{B}_{\mathrm{reg}}$. We write local formulas using a local trivialization of $\Gamma_{\mathbb{R}}$ by "A and B cycles" obeying

$$\langle A^I, A^J \rangle = 0, \quad \langle B_I, B_J \rangle = 0, \quad \langle A^I, B_J \rangle = \delta^I_J. \tag{2.8}$$

Definition 2.11 (Period map). Let λ denote the tautological (Liouville) holomorphic 1-form on $\mathrm{Tot}[K_C]$. The *period map* is the map

$$Z : \Gamma_{\phi_2} \to \mathbb{C}, \qquad Z_\gamma = \oint_\gamma \lambda \tag{2.9}$$

which we could also view as an element $Z \in \Gamma^*_{\mathbb{C}}$.

Then we have $\mathrm{d}Z \in \Omega^1(\mathcal{M}) \otimes \Gamma^*_{\mathbb{C}}$, and we can define a 2-form $\langle\langle \mathrm{d}Z, \mathrm{d}\overline{Z} \rangle\rangle$, given concretely by

$$\langle\langle \mathrm{d}Z, \mathrm{d}\overline{Z} \rangle\rangle = \sum_{I=1}^{r} \mathrm{d}Z_{A^I} \wedge \mathrm{d}\overline{Z}_{B_I} - \mathrm{d}Z_{B_I} \wedge \mathrm{d}\overline{Z}_{A^I} \,. \tag{2.10}$$

Proposition 2.12 (Positivity). $\langle\langle \mathrm{d}Z, \mathrm{d}\overline{Z} \rangle\rangle$ is a positive $(1,1)$-form on $\mathcal{B}_{\mathrm{reg}}$.

Thus it defines a Kähler metric on $\mathcal{B}_{\mathrm{reg}}$.

Now we consider the fiber directions. As we have said, $\mathcal{M}_{\phi_2}$ is a space of flat $U(1)$-connections over Σ_{ϕ_2}. In particular, for each $\gamma \in \Gamma_{\phi_2}$ there is a corresponding holonomy $\theta_\gamma : \mathcal{M}_{\phi_2} \to \mathbb{R}/2\pi\mathbb{Z}$. Their differentials can be assembled into $\mathrm{d}\theta \in \Omega^1(\mathcal{M}) \otimes \Gamma^*_{\mathbb{R}}$. If we choose a basis for Γ_{ϕ_2} then we get $\mathbb{R}/2\pi\mathbb{Z}$-valued coordinates $\theta_1, \ldots, \theta_{2r}$ on $\mathcal{M}_{\phi_2}$.

Definition 2.13 (Semiflat metric). The *semiflat metric* g^{sf} on $\mathcal{M}_{\mathrm{reg}}$ is the metric whose Kähler form in structure I_1 is

$$\omega_1^{\mathrm{sf}} = 2\langle\langle \mathrm{d}Z, \mathrm{d}\overline{Z} \rangle\rangle - \langle\langle \mathrm{d}\theta, \mathrm{d}\theta \rangle\rangle. \tag{2.11}$$

Now we can formulate the weak version of our conjecture. It says that, away from the singular fibers, Hitchin's metric is exponentially close to the semiflat metric.

Definition 2.14 (Threshold). Fix $\phi_2 \in \mathcal{B}_{\mathrm{reg}}$. Then $|\phi_2|$ is a singular flat metric on C, with singularities at the zeroes of ϕ_2. The *threshold* $M(\phi_2)$ is twice the length of the shortest inextendible geodesic (saddle connection) in this metric.[1]

Conjecture 2.15 (Weak metric conjecture). Fix $\phi_2 \in \mathcal{B}_{\mathrm{reg}}$. As we rescale $\phi_2 \to R^2\phi_2$ with $R > 0$, Hitchin's metric approaches the semiflat metric exponentially fast:

$$g = g^{\mathrm{sf}} + O(\mathrm{e}^{-2RM(\phi_2)})\,. \tag{2.12}$$

The meaning of the conjecture is that the naive cartoon drawing of the torus fibration is exponentially close to being the correct metric picture. It is an instance of the Strominger-Yau-Zaslow picture of Calabi-Yau manifolds.

This conjecture is now almost proven [18] following earlier work [19, 17, 20]: on the section $\theta_\gamma = 0$ it is fully proven, on the full moduli space it is proven with the exponent RM instead of the desired $2RM$.

There is a similar conjecture for higher rank G. Here too it is proven that g approaches g^{sf} exponentially fast [20] but the conjectured coefficient has not been verified.

3 Our strategy

Now let us explain our strategy for describing the hyperkähler metric on $\mathcal{M}$.

[1] We redefined it by a factor of 2 relative to [17]; this accounts for the fact that some factors of 2 here differ from those in the references.

3.1 The other complex structures of $\mathcal{M}$

So far we have focused on just one of the complex structures of $\mathcal{M}$, which we called I_1, and its holomorphic symplectic form Ω_1. We have also said that $\mathcal{M}$ is hyperkähler, which means it has more structure. Our construction of the metric needs to use all that structure.

The definition of hyperkähler manifold says that $\mathcal{M}$ has complex structures I_1, I_2, I_3 obeying $I_1 I_2 = -I_2 I_1 = I_3$ and cyclic permutations, with corresponding Kähler forms $\omega_1, \omega_2, \omega_3$. The holomorphic symplectic forms are related to the Kähler forms by $\Omega_1 = \omega_2 + \mathrm{i}\omega_3$ and cyclic permutations. In fact $\mathcal{M}$ has a whole family of complex structures I^ζ, $\zeta \in \mathbb{CP}^1$ (where $I^{\zeta=0} = I_1, I^{\zeta=\mathrm{i}} = I_2, I^{\zeta=1} = I_3$), and corresponding holomorphic symplectic forms Ω^ζ. What are they?

Given a SL(2)-Higgs bundle (E, φ) and solution h of Hitchin's equations (2.4) there is a corresponding 1-parameter family of flat $\mathrm{SL}(2,\mathbb{C})$-connections over C:

$$\nabla(\zeta) = \zeta^{-1}\varphi + D + \zeta\varphi^\dagger. \tag{3.1}$$

Let $\mathcal{M}^\flat$ be the moduli space of flat reductive $\mathrm{SL}(2,\mathbb{C})$-connections over C. $\mathcal{M}^\flat$ carries a complex structure $I^\flat$ and a holomorphic symplectic structure Ω^{ABG} (Atiyah-Bott-Goldman).

Proposition 3.1. For any $\zeta \in \mathbb{C}^\times$, the map $(E, \varphi) \mapsto \nabla(\zeta)$ identifies

$$(\mathcal{M}, I^\zeta, \Omega^\zeta) \xrightarrow{\sim} (\mathcal{M}^\flat, I^\flat, \Omega^{ABG}). \tag{3.2}$$

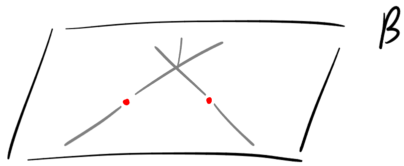
So the complex structures I^ζ, $\zeta \in \mathbb{C}^\times$, look very different from I^0 and I^∞.

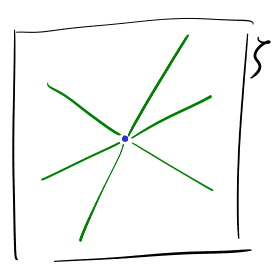
3.2 Local Darboux coordinates

Proposition 3.1 implies that any holomorphic function $\mathcal{X}$ on $\mathcal{M}^\flat$, when applied to the flat connection $\nabla(\zeta)$, becomes a holomorphic function on $(\mathcal{M}, I^\zeta)$. Extending this to coordinate systems, any holomorphic Darboux coordinate system $\{\mathcal{X}_i\}$ on $(\mathcal{M}^\flat, \Omega^{ABG})$ becomes a holomorphic Darboux coordinate system on $(\mathcal{M}, I^\zeta)$.

Our aim is to calculate holomorphic Darboux coordinates $\mathcal{X}_\gamma(\zeta)$ of a given fixed Higgs bundle. Since $\nabla(\zeta)$ varies holomorphically with ζ, the coordinates $\mathcal{X}_\gamma(\zeta)$ do as well; so they can be thought of as functions on the *twistor space* of $\mathcal{M}$.

- **Q**: Which holomorphic Darboux coordinate system on $(\mathcal{M}^\flat, \Omega^{ABG})$ will we use? **A**: We actually will not use just one: instead, as we move around on the Hitchin base $\mathcal{B}$ and/or vary the argument of ζ, we will choose different coordinate systems in different regions, separated by codimension-1 "walls."





- **Q**: Why do you need to have such a complicated structure? **A**: Because we want to study these coordinates through their analytic properties in the ζ-plane, and only certain coordinates will be *good* in the sense of having simple analytic behavior as $\zeta \to 0, \infty$; moreover, which coordinates are good changes as we move around on $\mathcal{B}$ or vary the argument of ζ.

- **Q**: How does this help you get the metric? **A**: On $(\mathcal{M}, I_1)$ we already have the holomorphic symplectic form $\Omega_1 = \omega_2 + \mathrm{i}\omega_3$. All that is missing is the third symplectic form ω_1. Once we have holomorphic Darboux coordinate functions $\mathcal{X}_\gamma(\zeta)$, we can specialize them to say $\zeta = 1$ and get a formula for the holomorphic symplectic form $\Omega^{\zeta=1} = \Omega_3 = \omega_1 + \mathrm{i}\omega_2$; then the desired ω_1 is just $\operatorname{Re}\Omega^{\zeta=1}$.

- **Q**: Won't the jumping of the $\mathcal{X}_\gamma(\zeta)$ at the walls cause a problem? **A**: No, the jumps are always by *symplectomorphisms*, so that even though $\mathcal{X}_\gamma(\zeta)$ jumps, Ω^ζ doesn't.

4 An irregular extension

To go further, we introduce some of the simplest model examples, where we can actually describe the $\mathcal{X}_\gamma$ concretely. These examples however are slightly outside our original setup: we need to allow the Higgs bundles to have singularities. To get the very simplest setup we will actually allow *irregular* singularities.

4.1 Definitions

We let $C = \mathbb{CP}^1$, with the usual inhomogeneous coordinate z, and use objects which are *meromorphic*, with poles at $z = \infty$, rather than holomorphic.

Fix a polynomial q of degree $N \in 2\mathbb{N} + 1$. (In fact we only need to know q modulo polynomials of degree $\leq \frac{N-3}{2}$.)

Definition 4.1 (Irregular Higgs bundle). An *irregular* $\mathrm{SL}(2)$*-Higgs bundle* of type q is a pair (E, φ), where:

- E is a meromorphic vector bundle, holomorphic away from $z = \infty$, with a valuation ν_∞ on meromorphic sections, and a nowhere-vanishing section $\eta \in \det E$ with $\nu_\infty(\eta) = 0$,

- φ is a traceless meromorphic section of $\mathrm{End}(E) \otimes K_C$, holomorphic away from $z = \infty$, obeying
$$\phi_2 = -\frac{1}{2}\operatorname{Tr}\varphi^2 = (q(z) + l(z))\,\mathrm{d}z^2 \tag{4.1}$$
with $\deg l < \frac{N}{2} - 1$,

such that for any meromorphic section s of E,
$$\nu_\infty(z^m s) = \nu_\infty(s) + m, \qquad \nu_\infty(\varphi \cdot s) = \nu_\infty(s) + \frac{N}{2}\,. \tag{4.2}$$

Everything we said about ordinary SL(2)-Higgs bundles has an analogue for these irregular ones, as explained in [14] (plus epsilon). Here I write it in slightly different language. In particular:

Definition 4.2 (Adapted metric). Suppose (E, φ) is an irregular Higgs bundle. An *adapted metric* in (E, φ) is a Hermitian metric h in E, such that

$$\log h(s,s) \sim 2\nu_\infty(s) \log|z| \quad \text{as } |z| \to \infty . \tag{4.3}$$

Theorem 4.3. If (E, φ) is an irregular Higgs bundle, there is an adapted metric h in E which obeys Hitchin's equations (2.4), and h is unique up to scalar multiple.

Theorem 4.4. There is a moduli space $\mathcal{M}(q)$ parameterizing irregular Higgs bundles of type q up to equivalence. $\mathcal{M}(q)$ is a smooth manifold, of dimension $N-1$. The formula (2.6) defines a complete hyperkähler metric on $\mathcal{M}(q)$.

So this is a continuous family of hyperkähler spaces, parameterized by the choice of q. The space of polynomials q has dimension $N+1$, but we should mod out by automorphisms of $\mathbb{CP}^1$ preserving $z = \infty$ and by low-degree shifts, so the effective number of parameters in the family is $(N+1) - 2 - (\frac{N-3}{2} + 1) = \frac{N-1}{2}$.

The Hitchin base $\mathcal{B}(q)$ in this case is just the space of polynomials $P_2 = q + l$ appearing above, with q fixed and l varying. It has complex dimension $\frac{1}{2}(N-1)$ as it should.

Example 4.5. The case $N = 3$ is the first interesting one. Here we pick

$$q(z) = z^3 + \Lambda z \tag{4.4}$$

for $\Lambda \in \mathbb{C}$. Then $\mathcal{M}(q)$ is a hyperkähler space of complex dimension 2, depending on the choice of Λ. The Hitchin base is just $\mathcal{B}(q) = \mathbb{C}$. If $\Lambda = 0$ there is a single singular fiber, which is a cuspidal torus. If $\Lambda \neq 0$ there are two, which are both nodal tori.

5 The coordinates

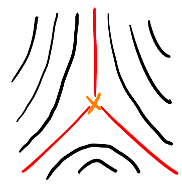
Now we describe the holomorphic coordinates $\mathcal{X}_\gamma(\zeta)$ on $(\mathcal{M}, I^\zeta)$ which we will use. We consider only the case of irregular Higgs bundles on $\mathbb{CP}^1$. (A very similar construction works in the case of Higgs bundles with regular singularities.)

5.1 Defining the coordinates

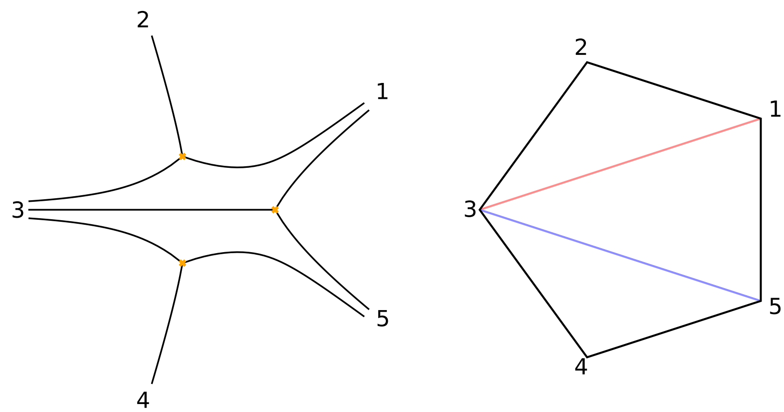
Fix $\phi_2 \in \mathcal{B}$ and $\zeta \in \mathbb{C}^\times$.

Definition 5.1 (ζ-trajectories of a quadratic differential). A *ζ-trajectory* of ϕ_2 is a path on C along which $\zeta^{-1}\sqrt{-\phi_2}$ (with either choice of sign for $\sqrt{-\phi_2}$) is a *real* and nowhere vanishing form.

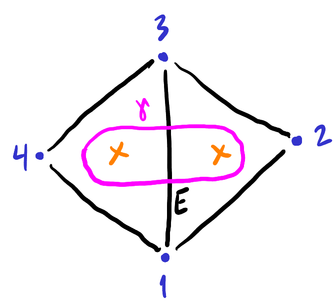
Proposition 5.2 (ζ-trajectories give a foliation). The ζ-trajectories are the leaves of a singular foliation of C, with singularities at the zeroes and poles of ϕ_2. At each zero of ϕ_2, the foliation by ζ-trajectories has a three-pronged singularity, as shown below.



Proposition 5.3 (Ideal triangulation determined by the ζ-trajectories). Suppose (ϕ_2, ζ) is generic, in the sense that $\zeta^{-1} Z_\gamma \notin \mathbb{R}$ for all $\gamma \in \Gamma_{\phi_2}$. Then there are $N+2$ rays r_i at infinity, such that any generic ζ-trajectory is asymptotic to one of the r_i. The ζ-trajectories determine a triangulation $T(\phi_2, \zeta)$ of an $(N+2)$-gon, as indicated below.



Definition 5.4 (Fock-Goncharov coordinate attached to an edge). Fix an interior edge $E \in T(\phi_2, \zeta)$. E determines a class $\gamma \in \Gamma_{\phi_2}$, shown below:[2]



To define $\mathcal{X}_\gamma(\zeta)$, we consider the connection $\nabla(\zeta)$ restricted to the quadrilateral shown. Its space of flat sections is a 2-dimensional vector space V, equipped with 4 distinguished lines $\ell_i \subset V$: ℓ_i consists of the flat sections which have exponentially *decaying* norm as we go to infinity along a leaf of $T(\phi_2, \zeta)$ in the i-th direction. Said otherwise, the ℓ_i give 4 points of $\mathbb{CP}^1$. We define $\mathcal{X}_\gamma(\zeta)$ to be the $\mathrm{SL}(2, \mathbb{C})$-invariant cross-ratio of these 4 points:

$$\mathcal{X}_\gamma(\zeta) = -\frac{(\ell_1 \wedge \ell_2)(\ell_3 \wedge \ell_4)}{(\ell_2 \wedge \ell_3)(\ell_4 \wedge \ell_1)}. \tag{5.1}$$

This definition appears in Fock-Goncharov [8]; it is a complexification of the notion of shear coordinate.

[2] More precisely, the picture shows only the projection of γ to C, and does not show the orientation. The ambiguity can be fixed as follows: the intersection $\langle \gamma, \hat{E} \rangle$ should be positive, where $\hat{E}$ denotes one of the lifts of E to Σ, oriented so that λ is a positive 1-form along $\hat{E}$.

Applying Definition 5.4 for all edges E of $T(\phi_2, \vartheta)$ gives functions $\mathcal{X}_\gamma(\zeta)$ with γ running over a basis for Γ. We extend to arbitrary γ by requiring $\mathcal{X}_\gamma \mathcal{X}_\mu = \mathcal{X}_{\gamma+\mu}$. These are local Darboux coordinates:

$$\Omega^\zeta = \langle\langle \mathrm{d}\log \mathcal{X}(\zeta), \mathrm{d}\log \mathcal{X}(\zeta) \rangle\rangle. \tag{5.2}$$

5.2 Asymptotic behavior of the coordinates

The main asymptotic property of the coordinates $\mathcal{X}_\gamma(\zeta)$ is:

Conjecture 5.5. Fix a point of $\mathcal{M}$. Then, as $\zeta \to 0$ along any ray,

$$\mathcal{X}_\gamma(\zeta) \sim \exp\left(\zeta^{-1} Z_\gamma + \mathrm{i}\theta_\gamma + c_\gamma\right) \tag{5.3}$$

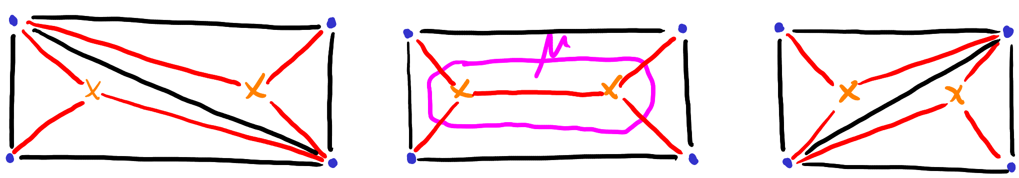
where the constants $c_\gamma = \int_\gamma \alpha$ for some $\alpha \in \Omega^{1,1}(\Sigma)$. Moreover, if all $\theta_\gamma = 0$, then all $c_\gamma = 0$, so in that case

$$\mathcal{X}_\gamma(\zeta) \sim \exp\left(\zeta^{-1} Z_\gamma\right). \tag{5.4}$$

(The idea: it would follow from the exact WKB method applied to the connections $\nabla(\zeta) = \zeta^{-1}\varphi + \cdots$.)

5.3 Piecewise analytic behavior of the coordinates

As we vary (ϕ_2, ζ), the function $\mathcal{X}_\gamma(\zeta)$ is only piecewise smooth: it suffers a jump whenever the triangulation $T(\phi_2, \zeta)$ changes. The simplest kind of jump is shown below:



This jump is associated with the "saddle connection" connecting two zeroes of ϕ_2, appearing in the middle of the figure. Such a saddle connection can only appear when $\zeta^{-1} Z_\mu \in \mathbb{R}_-$. The coordinates on the two sides of the jump are related by:

$$\mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 + \mathcal{X}_\mu)^{\langle \mu, \gamma \rangle}. \tag{5.5}$$

A similar (but more intricate) phenomenon occurs when we cross a (ϕ_2, ζ) for which an annulus of closed trajectories appears: then the $\mathcal{X}_\gamma$ undergo a jump of the form

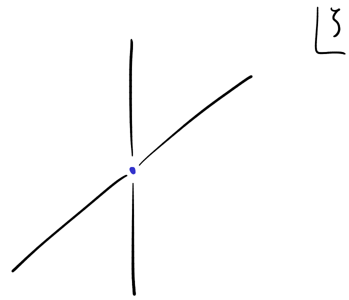
$$\mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 - \mathcal{X}_\mu)^{-2\langle \mu, \gamma \rangle}. \tag{5.6}$$

Both of these are instances of the following general structure:

$$\mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 - \sigma(\mu) \mathcal{X}_\mu)^{\mathrm{DT}(\mu) \langle \mu, \gamma \rangle} \tag{5.7}$$

where for a saddle connection we have $\mathrm{DT}(\mu) = +1$ and $\sigma(\mu) = -1$, while for a closed loop we have $\mathrm{DT}(\mu) = -2$ and $\sigma(\mu) = +1$.

Fix ϕ_2 and just let ζ vary. Then $\mathcal{X}_\gamma(\zeta)$ depends on ζ in a piecewise-analytic way: the collection $\{\mathcal{X}_\gamma(\zeta)\}_{\gamma\in\Gamma}$ jumps at various rays ℓ in the ζ-plane.



At each such ray, the jump is a product of transformations of the form (5.7), where the μ in (5.7) can be any $\mu \in \Gamma$ such that $Z_\mu/\zeta \in \mathbb{R}_-$ along ℓ.

5.4 The integral equation

We are building up an elaborate structure, but it is only going to be useful if it allows us to say something concrete about $\mathcal{X}_\gamma(\zeta)$. Here is one approach:

Conjecture 5.6 (Integral equation for $\theta_\gamma = 0$). When all $\theta_\gamma = 0$,

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\zeta) \exp\left[\frac{1}{4\pi\mathrm{i}} \sum_{\mu\in\Gamma} \mathrm{DT}(\mu)\langle\gamma,\mu\rangle \int_{Z_\mu \mathbb{R}_-} \frac{\mathrm{d}\zeta'}{\zeta'} \frac{\zeta'+\zeta}{\zeta'-\zeta} \log(1-\sigma(\mu)\mathcal{X}_\mu(\zeta'))\right] \tag{5.8}$$

where

$$\mathcal{X}_\gamma^{\mathrm{sf}}(\zeta) = \exp\left(\zeta^{-1} Z_\gamma + \zeta \overline{Z}_\gamma\right). \tag{5.9}$$

The signs $\sigma(\mu) = \pm 1$ obey the relation

$$\sigma(\mu)\sigma(\mu') = (-1)^{\langle\mu,\mu'\rangle}\sigma(\mu+\mu'). \tag{5.10}$$

The functions $\mathcal{X}_\gamma(\zeta)$ appear on both sides of (5.8). Thus (5.8) is an integral equation, which needs to be solved for the whole collection $\{\mathcal{X}_\gamma(\zeta)\}_{\gamma\in\Gamma}$ at once, rather than an integral formula.

- **Q**: Where does this equation come from? **A**: It would lead to $\mathcal{X}_\gamma(\zeta)$ with the right analytic properties in the ζ-plane: asymptotics as $\zeta \to 0, \infty$ and jumps at the rays $Z_\mu/\zeta \in \mathbb{R}_-$ with $\mathrm{DT}(\mu) \neq 0$. The optimistic hope is that these analytic properties are strong enough to *determine* $\mathcal{X}_\gamma(\zeta)$.

- **Q**: How do you actually solve it? **A**: By iteration: pick $\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\zeta)$ as initial guess, and then iterate.

- **Q**: Why would you think that that iteration would converge? **A1**: If all $|Z_\gamma|$ are large enough, and $\mathrm{DT}(\mu)$ doesn't grow too fast as a function of μ (e.g. if only finitely many are nonzero) saddle-point estimates show the iteration defines a contraction mapping, so it must converge to a (unique) fixed point. **A2**: Actually, experimentally it seems that it *always* converges! This is strange, and deserves an explanation.

- **Q**: How does this lead to the weak conjecture, Conjecture 2.15, from the previous lecture? **A**: If we substitute $\mathcal{X} = \mathcal{X}^{\mathrm{sf}}$, the log in the integrand is bounded above by $\mathrm{e}^{-2|Z_\mu|}$; thus we expect that the first step of the iteration is already suppressed by e^{-2M} where M is the minimum $|Z_\mu|$ for which $\mathrm{DT}(\mu) \neq 0$, and later steps should be further exponentially suppressed. That suggests that just truncating to the zeroth iteration (i.e. taking g^{sf}) would already give a result exponentially close to the true metric, and the accuracy will improve with each iteration we take. In particular we can truncate to the *first* iteration. Working this out leads to

$$g = g^{\mathrm{sf}} - \frac{2}{\pi} \sum_{\mu \in \Gamma} \mathrm{DT}(\mu) K_0\left(2|Z_\mu|\right) \mathrm{d}|Z_\mu|^2 + \cdots \tag{5.11}$$

  where K_0 is the modified Bessel function. Note that $K_0(x) \sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}$, so $g - g^{\mathrm{sf}}$ is exponentially suppressed as e^{-2M}. The omitted terms $\cdots$ should be of order e^{-4M}.

6 Numerical tests

The strongest evidence supporting Conjecture 5.6 is numerical, given in [21].

6.1 The Hitchin section

Definition 6.1 (Hitchin section). Let $\mathcal{O}(\alpha)$ denote the trivial meromorphic bundle over $\mathbb{CP}^1$, with a valuation ν_∞ given by the usual pole order at $z = \infty$ shifted by $-\alpha$. Then given $\phi_2 = P_2(z)\mathrm{d}z^2 \in \mathcal{B}(q)$ we consider the Higgs bundle (E, φ):

$$E = \mathcal{O}\left(\frac{N}{4}\right) \oplus \mathcal{O}\left(-\frac{N}{4}\right), \qquad \varphi = \begin{pmatrix} 0 & -P_2 \\ 1 & 0 \end{pmatrix}. \tag{6.1}$$

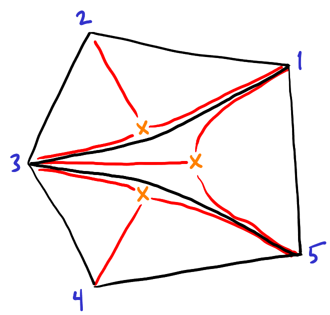
This gives a section of the Hitchin map for $\mathcal{M}(q)$.

6.2 The $\mathcal{X}_\gamma$ in an example

For example, when $N = 3$, we have 5 asymptotic rays and $\dim \mathcal{M} = 2$. The $\mathcal{X}_\gamma(\zeta)$ are monomials in 2 out of the 5 possible cross-ratios. Precisely *which* cross-ratios we take depends on P_2 and ζ, as we explained. We take the concrete example

$$P_2(z) = R^2(z^3 - 1), \qquad R \in \mathbb{R}_+. \tag{6.2}$$

In this case the triangulation $T(\phi_2, \zeta = 1)$ looks like:



From this picture we can read off that the relevant cross-ratios are r_{1235} and r_{1345}.

6.3 Numerical results

In **[DNexp]** we computed the $X_\gamma = \mathcal{X}_\gamma(\zeta = 1)$ numerically in this example, in two different ways:

- by directly solving Hitchin's equation i.e. finding the harmonic maps,
- by solving the integral equations of §5.4.

The github repository `neitzke/stokes-numerics` contains the code we used. Some sample output:

```
comparisons.compareClusters("A1A2", R = 0.07, scratch = True, pde_nmesh = 511)
```

```
{'xarcluster': [-0.5108779665615462, -1.0],
 'fdcluster': [-0.510880773551951, -1.0000000000000009],
 'sfcluster': [-0.7023314112631698, -1.0],
 'absdiff': [2.8069904048910743e-06, 8.881784197001252e-16],
 'logdiff': [-5.4944289580305394e-06, -8.881784197001248e-16],
 'phasediff': [0.0, 0.0],
 'reldiff': [5.494428958012677e-06, 8.881784197001248e-16],
 'frames': <framedata.framedata at 0x7f1a79c6a9e0>,
 'errest': {'absode': [2.7422339558380702e-14, 6.735318507305554e-14],
  'relode': [5.3676924495678885e-14, 6.739053759474921e-14]}}
```

So e.g. this says that the quantity $X_1 = r_{1235}$ at $R = 0.07$ is approximately 0.51088, and the integral equation computation agrees with the PDE computation to this precision.

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