Applications of QFT to Geometry

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Preliminary and **incomplete** draft

These are partial notes for a Fall 2017 course at UT Austin. They are extremely incomplete, unreliable, full of mistakes and omissions. Moreover, the course is now complete, but partway through the semester I stopped updating the typed notes. I hope to return to them, no later than the next time I teach the course.

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or as pull requests to the source repository hosted at

http://github.com/neitzke/qft-geometry

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1 Introductory motivation

1.1 Linear equations

Suppose we are interested in studying the topology of smooth manifolds X. One powerful tool for this purpose is to introduce an ingredient which at first seems alien to the problem: namely we fix a Riemannian metric g on X. This allows us to define the *form Laplacian* (A.32) and consider the space of *harmonic forms*

$$\mathcal{H}_k(X) = \{\omega \in \Omega^k(X) | \Delta\omega = 0\}. \tag{1.1}$$

The equation

$$\Delta\omega = 0 \tag{1.2}$$

is a *linear* equation over $\mathbb{R}$. Thus $\mathcal{H}_k(X)$ is a *vector space*. If X is compact, then $\mathcal{H}_k(X)$ is moreover finite-dimensional (a consequence of the fact that Δ is an *elliptic* operator). Thus we can define positive integers by

$$b_k(X) = \dim_{\mathbb{R}} \mathcal{H}_k(X). \tag{1.3}$$

There is a remarkable fact about these integers:

Fact 1.1. The integers b_k do not depend on the choice of Riemannian metric or smooth structure on X; instead they are invariants of the underlying topological manifold (the *Betti numbers*).

Exercise 1.1. Work out explicitly the spaces $\mathcal{H}^k(X)$ and Betti numbers $b_k(X)$ for some of the following: $X = S^1, T^2, S^2$. (In each case choose a convenient Riemannian metric; of course the b_k are independent of which metric you choose, though the $\mathcal{H}_k(X)$ naively are not.)

Fact 1.1 is a consequence of a stronger, "categorified" statement:

Fact 1.2. There is a canonical isomorphism

$$\mathcal{H}_k(X) \simeq H^k(X, \mathbb{R}) \tag{1.4}$$

(where H^k means de Rham or singular cohomology).

If X is an oriented Riemannian $4n$-manifold then there is a small refinement of the middle Betti number b_{2n}: we have the "Hodge star" operator

$$\star : \Omega^{2n}(X) \to \Omega^{2n}(X) \tag{1.5}$$

which has the crucial properties

- $\star^2 = 1$,
- $[\star, \Delta] = 0$.

The first property says we can decompose into the ±1-eigenspaces for $\star$:[1]

$$\Omega^{2n}(X) = \Omega^{2n,+}(X) \oplus \Omega^{2n,-}(X). \tag{1.6}$$

Combining this with the second property one sees that the harmonic forms also decompose:

$$\mathcal{H}_{2n}(X) = \mathcal{H}^+_{2n}(X) \oplus \mathcal{H}^-_{2n}(X), \qquad b_{2n}(X) = b^+_{2n}(X) + b^-_{2n}(X). \tag{1.7}$$

[1] concretely $\omega = \frac{1}{2}(1+\star)\omega + \frac{1}{2}(1-\star)\omega$

Exercise 1.2. Suppose instead that X is an oriented $(4n+2)$-manifold. Here the story is a bit different, because in dimension $k = 2n+1$ we have $\star^2 = -1$. Show that in this case the Betti number $b_{2n+1}(X)$ is even. (Is this still true if X is not orientable?)

1.2 Nonlinear equations

Now we will replace the linear equation (1.2) by a *nonlinear* equation.

Unlike linear equations — which in some sense behave uniformly in the dimension — nonlinear equations tend to behave very differently in different dimensions. With this in mind we now specialize to the case $\dim X = 4$. In this case, a new source of topological (or more precisely smooth) invariants was discovered by Donaldson in the 1980s. For an excellent reference see [1].

Fix a compact Lie group G and let P denote a principal G-bundle over X. Then we consider *connections* in P.[2] A connection in P may be locally represented by a 1-form[3]

$$A \in \Omega^1(\mathfrak{g}), \tag{1.8}$$

and has a curvature 2-form $F \in \Omega^2(\mathfrak{g}_P)$,[4] locally written[5]

$$F = \mathrm{d}A + A \wedge A \in \Omega^2(\mathfrak{g}). \tag{1.9}$$

Now, since F is a 2-form and we are in 4 dimensions, we can decompose F under $\star$ as

$$F = F^+ + F^-. \tag{1.10}$$

The *anti-self-dual Yang-Mills equation* is

$$F^+ = 0. \tag{1.11}$$

We view this as a condition on the connection. For G abelian (e.g. for $G = \mathrm{U}(1)$) the equation (1.11) is linear in A. For G nonabelian (e.g. for $G = SU(2)$) this equation is *nonlinear* in A, because of the quadratic part in (1.9).

Exercise 1.3. Write out (1.11) in detail in components, in two cases:

- For $G = \mathrm{U}(1)$: in this case you should get a system of 3 linear equations for 4 functions A_μ ($\mu \in \{1,2,3,4\}$).

[2] For background on connections in principal bundles I like [2] or [3], or for a briefer and to-the-point account [4].

[3] By "locally" here I mean "on a patch $U \subset X$ where we have chosen a trivialization of the bundle $P|_U$."

[4] $\mathfrak{g}_P$ means the *associated bundle* to P using the adjoint action of G on $\mathfrak{g}$, sometimes written $P \times_G \mathfrak{g}$; again see [2] for this notion. $\mathfrak{g}_P$ locally looks like $\mathfrak{g}$ but is globally twisted by the transition functions of P. Some people would call this bundle $\mathrm{ad}\, P$. $\mathfrak{g}_P$ is canonically globally trivial when $\mathfrak{g}$ is abelian, i.e. $\mathfrak{g}_P = X \times \mathfrak{g}$, so in that case we really have globally $F \in \Omega^2(\mathfrak{g})$.

[5] To spell out the notation here: suppose $A = A^a_\mu T_a \mathrm{d}x^\mu$, with T_a a basis for $\mathfrak{g}$, and define the structure constants f^c_{ab} by $[T_a, T_b] = f^c_{ab} T_c$; then $A \wedge A = \frac{1}{2} f^c_{ab} T_c A^a_\mu A^b_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Some people prefer to write this term as $\frac{1}{2}[A, A]$, which makes it more obviously sensible for arbitrary Lie groups as opposed to matrix groups. It vanishes when $\mathfrak{g}$ is abelian, so then we just have $F = \mathrm{d}A$.

- For $G = \mathrm{SU}(2)$: in this case you should get a system of 9 coupled nonlinear equations for 12 functions A^a_μ ($\mu \in \{1,2,3,4\}$, $a \in \{1,2,3\}$).

We consider the *instanton moduli space*

$$\mathcal{M} = \{\text{connections on } P \text{ obeying (1.11)}\}/\mathfrak{G} \tag{1.12}$$

where $\mathfrak{G}$ is the infinite-dimensional group of *gauge transformations* ie sections of $\mathrm{Aut}(P)$, acting on connections. Locally, a gauge transformation is represented by a map $g : U \to G$, and then this action is given by

$$A \to g^{-1}Ag + g^{-1}\mathrm{d}g. \tag{1.13}$$

Exercise 1.4. Write out the action of $\mathfrak{G}$ on connections explicitly in components, when $G = \mathrm{U}(1)$ or $G = \mathrm{SU}(2)$. (It may be convenient to consider the case where g is given as the exponential of a Lie algebra element, e.g. for $G = \mathrm{U}(1)$ the formulas will be simplest if you write $g = \exp(\chi T)$ with χ the generator of $\mathfrak{u}(1)$, and $\chi : U \to \mathbb{R}$ an ordinary function.)

Exercise 1.5. Show that when $G = \mathrm{U}(1)$, the structure of $\mathcal{M}$ depends on the image of $c_1(P)/2\pi \in H^2(X, \mathbb{Z})$ under the map $p : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$; namely, if $p(c_1(P)/2\pi) \in \mathcal{H}^2_-$ then $\mathcal{M}$ is $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$, and otherwise $\mathcal{M}$ is empty. [warning: needs Hodge theory] [warning: I got this wrong twice already, hopefully it's right now]

When G is nonabelian, the situation is much more difficult. Nevertheless $\mathcal{M}$ can be studied and moreover it turns out to have a reasonable geometric structure, as follows.

From now on let us specialize to the case $G = \mathrm{SU}(2)$. In this case P is classified by the integer

$$k = \int_X c_2(P). \tag{1.14}$$

Fact 1.3. If $k > 0$ and the metric g on X is *generic* (in a suitable sense), then $\mathcal{M}$ is a finite-dimensional manifold.

(For non-generic g, $\mathcal{M}$ is still close to being a manifold, but may develop singularities corresponding to reducible solutions of (1.11).)

1.3 Donaldson invariants

Donaldson's idea was to extract information about X from the study of $\mathcal{M}$. The direct nonlinear analogue of the Betti numbers is the *dimension* of $\mathcal{M}$: it turns out to be (for X connected)

$$\dim \mathcal{M} = 8k - 3(1 - b_1(X) + b_2^+(X)). \tag{1.15}$$

But because $\mathcal{M}$ is nonlinear there is more to it than just its dimension. Donaldson introduced an *orientation* on $\mathcal{M}$ and a family of canonically defined closed differential forms $\tau_\alpha \in \Omega^*(\mathcal{M})$, labeled by classes $\alpha \in H_*(X, \mathbb{Z})$. Then he defined new invariants $\langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_\ell} \rangle$ by, schematically,

$$\langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_\ell} \rangle = \int_{\mathcal{M}} \tau_{\alpha_1} \wedge \cdots \wedge \tau_{\alpha_\ell}, \tag{1.16}$$

and proved that they are independent of the Riemannian metric g (under the technical assumption $b_2^+(X) > 1$.)

These invariants proved very powerful: they could detect phenomena invisible to the standard differential-topology methods [explain something proved using them?] However, they were also very technically difficult to control, particularly because $\mathcal{M}$ is typically *noncompact*, so that integration over $\mathcal{M}$ is a delicate operation.

1.4 QFT and Donaldson invariants

In 1988, following some provocative suggestions of Atiyah, Witten found a remarkable new way of thinking about the Donaldson invariants [5]: he interpreted them in terms of a certain quantum field theory (QFT), *topologically twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory*. Very roughly, Witten imagined X to be the "spacetime" in some hypothetical universe, where the laws of physics are governed by topologically twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, and then imagined making some "experimental measurements" in that universe — captured in QFT language by *correlation functions*.

According to the rules of Lagrangian QFT[6] correlation functions are supposed to be integrals over an infinite-dimensional space $\mathcal{C}$, of the form

$$\langle \mathcal{O}_\alpha \rangle = \int_{\mathcal{C}} \mathrm{d}\mu \, \Phi_\alpha \mathrm{e}^{-S}, \tag{1.17}$$

where $S : \mathcal{C} \to \mathbb{R}$ is the "action" of the theory, $\mathrm{d}\mu$ is some measure of integration, and $\Phi_\alpha : \mathcal{C} \to \mathbb{R}$ are the "classical observables." $\mathcal{C}$ is sometimes called the "space of fields"; in a general Lagrangian QFT, it is something like the space of all[7] functions on X (or maybe differential forms on X, connections on bundles over X, sections of bundles over X, etc; different theories involve different notions of fields.)

In general, correlation functions (1.17) are difficult to calculate. In topologically twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, however, there is a remarkable *localization* phenomenon which reduces the desired integrals (1.17) to the simpler finite-dimensional integrals (1.16) above.

Impressive as this discovery was, it did not lead to an immediate breakthrough in Donaldson theory: the formulas one could directly derive from the QFT perspective were just the *same* formulas already written down by Donaldson. In 1994 Witten pushed forward somewhat further, using QFT to compute Donaldson invariants in the special case

[6] Lagrangian QFTs are an important and widely studied class of QFTs, which feature prominently (for good reason) in one's early QFT education. Nevertheless, not all QFTs are of this sort; for QFTs which are not Lagrangian, one needs other tools for computing the correlation functions. Many of the most interesting mathematical and physical applications of QFT involve non-Lagrangian QFTs.

[7] The meaning of "all" will hopefully become clearer as we go on.

where X is a Kähler manifold [6]. But the next major development had to wait for progress in physics: what was needed was a better understanding of the physics of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory.

1.5 The action

For aficionados, here is the standard way that a physicist would define $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, in Euclidean signature, on the spacetime $X = \mathbb{R}^4$.

First, we need to fix a compact Lie group G and two couplings: $g \in \mathbb{R}_+$ and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. The QFT we want to describe depends on these data.

We fix also an auxiliary 2-dimensional complex vector space R, carrying Hermitian structure $\delta : R \otimes \overline{R} \to \mathbb{C}$ and volume form $\epsilon \in \wedge^2(R)$. (Concretely, you may as well pick $R = \mathbb{C}^2$ with its standard Hermitian structure and volume form. The main reason for calling it R now is that later we will replace it by a rank 2 Hermitian vector bundle over X.)

Then we let $\mathcal{C}$ be the space of fields:

- (P, ∇) a principal G-bundle with connection (with curvature F),
- $\phi \in \Gamma(\mathfrak{g}_{\mathbb{C},P})$,
- $\lambda^\pm \in \Pi\Gamma(S^\pm \otimes \mathfrak{g}_{\mathbb{C},P} \otimes R)$,
- $D \in \Gamma(\mathfrak{g}_{\mathbb{C},P} \otimes \mathrm{Sym}^2 R)$,

where $S^\pm$ are the spin representations of $\mathrm{Spin}(4)$. The symbol Π here means "parity change" which means $\lambda^\pm$ are *Grassmann-odd* fields: we will explain this (or at least get used to it) later.

The *action* is: [explain notation v, w and inner product $\langle , \rangle$, and double-check factors]

$$S = \frac{1}{g^2}\int_X \mathrm{Tr}\left(-\frac{1}{4}F \wedge \star F + \nabla_\mu\bar{\phi}\nabla^\mu\phi - \mathrm{i}\delta^{vw}\langle\lambda_v^-, \not{\nabla}\lambda_w^+\rangle + \frac{1}{4}\delta^{vv'}\delta^{ww'}D_{vw}D_{v'w'} - \frac{1}{2}[\phi,\bar{\phi}]^2 \right.$$
$$\left. -\mathrm{i}\sqrt{2}\epsilon^{vw}\langle\lambda_v^-, [\bar{\phi}, \lambda_w^-]\rangle + \mathrm{i}\sqrt{2}\epsilon^{vw}\langle\lambda_v^+, [\phi, \lambda_w^+]\rangle\right)$$
$$+ \frac{\mathrm{i}\vartheta}{4\pi^2}\int_X \mathrm{Tr}(F \wedge F). \quad (1.18)$$

As described here, the theory only makes sense on $X = \mathbb{R}^4$. Later we will describe Witten's modification of the theory (topological twisting) which we will use when we put it on a general Riemannian 4-manifold X.

1.6 Effective field theory: an analogy

The real breakthrough came with the work of Seiberg and Witten in 1995 [7]. In this work Seiberg and Witten answered a fundamental question about $\mathcal{N} = 2$ supersymmetric Yang-Mills theory: *how does the theory behave at low energies?*

To understand how important this question is, let us make a quick analogy. Suppose that we want to study a pond full of water and how it will respond to, say, a gentle breeze, or a small toy boat. One approach to this problem which we could imagine would be to say to ourselves: well, the pond is made of about 10^{30} protons, neutrons and electrons; let's write down equations governing those objects, put them on the biggest supercomputer we can find, make a model for the perturbation we want to study, and then have the computer solve the equations and tell us what will happen. This (if it could be done) would be in some sense the most *direct* method. Of course it is also completely hopeless.

In practice, we know that the relevant physical laws governing a pond full of water are the Navier-Stokes equations. These describe the dynamics of new "effective" variables (velocity, pressure, density, viscosity), whose relation to the underlying 10^{30} particles would be complicated to describe directly. Nevertheless Navier-Stokes is really the description we want, for our practical purpose of studying boats interacting with a pond. (It is probably *not* the relevant description if we want to know what will happen if we shoot the pond with a high-intensity laser!)

The really hard and important problem is to go from the high-energy description (elementary particles) to the low-energy description (Navier-Stokes equations). Once this problem has been solved once, we can then use the low-energy description to answer the questions we care about.

1.7 Seiberg-Witten equations

Seiberg and Witten in [7] solved the analogous problem for $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Beginning with the action (1.18) (high-energy description) for $G = \mathrm{SU}(2)$, they completely determined the low-energy description. It turned out that this description is also in terms of gauge theory, but this time gauge theory for the group $G' = \mathrm{U}(1)$ (coupled to matter): the nonabelian group $\mathrm{SU}(2)$, with all its attendant nonlinearities, is gone!

Then one can try to compute the results of experiments, now using this effective low-energy description. As before, the answer turns out to localize on some simple equations; but now instead of (1.11) the equations are the *Seiberg-Witten equations*[8]

$$F^+ = q(\psi, \bar{\psi}), \tag{1.19a}$$
$$\not{D}\psi = 0, \tag{1.19b}$$

where the fields are:

- D is a connection in a $\mathrm{U}(1)$-bundle E (more precisely E is the determinant line of a Spin^c-structure on X),
- ψ is a section of S^+, with S^+ the spinor bundle attached to the Spin^c-structure,

[8]Some authors, including Witten in [8], call (1.19) the *monopole equations*. I think it is a good idea to avoid this name in order not to confuse (1.19) with the *Bogomolny equations* whose solutions are monopoles: the relation between the field ψ and the monopoles described by Bogomolny equations is rather subtle. We will explore it later in the course.

and q is a certain quadratic map $S^+ \otimes S^+ \to \wedge^2_+(T^*X)$. For an economical explanation of these equations see [9].

In our analogy to the physics of a pond, (1.19) is the moral analogue of the Navier-Stokes equations. What all this suggests is that (1.19) should be just as powerful in 4-manifold topology as was (1.11), but in some sense easier to work with, since in passing to (1.19) we have gotten rid of some irrelevant complexity. This point of view was advocated by Witten in the paper [8] and it turned out to be correct. This was the beginning of a revolution in 4-manifold topology which continues to the present day.

One preliminary indication that the effective (Seiberg-Witten) description may be more convenient than the high-energy (Donaldson) description is:

Fact 1.4. If the metric g on X is generic, then the moduli space

$$\widetilde{\mathcal{M}} = \{\text{pairs } (D, \psi) \text{ obeying } (1.19)\} / \mathfrak{G}' \tag{1.20}$$

is smooth and compact.

This is very different from the space $\mathcal{M}$ which is definitely *not* compact for $k > 0$.

1.8 Our goals

In this course we are going to explore various geometric applications of quantum field theory, emphasizing the two really nontrivial ingredients which have appeared above:

- *Localization*: the mechanism by which the formal integrals over infinite-dimensional spaces which appear in quantum field theory get related to finite-dimensional integrals which can be defined and computed. One derives (in the physicist's sense) nontrivial facts about the finite-dimensional integrals, using the infinite-dimensional integrals (i.e. the QFT) at some intermediate stages.
- *Effective field theory*: the reduction from a complicated "high-energy" description to a simple "low-energy" description of a physical system (say, a QFT).

Very roughly speaking, QFTs get more complicated as the dimension of spacetime increases. Dimension 0 and 1 are relatively tractable — even mathematically rigorous, with some effort. In dimension 2 there are still many rigorous things that can be said, but already we begin facing difficulties, and these become more serious in dimensions 3 and 4.

I expect that we will study dimension 0, dimension 1, maybe a short stop in dimension 2, then jump to dimension 4. The level of rigor will be inversely correlated with the dimension.

2 QFT in 0 dimensions

2.1 The partition function and expectation values

As we have explained above, Lagrangian QFT on a spacetime X generally involves performing integrals over some space $\mathcal{C}$ of (perhaps generalized) functions on X. Thus $\mathcal{C}$

is almost always infinite-dimensional, but there is one key exception: the case where X is 0-dimensional. Let's explore that case. We take X to be just a point, and $\mathcal{C}$ to be the space of real-valued functions on a point, i.e.

$$\mathcal{C} = \mathbb{R}. \tag{2.1}$$

Then, let's define the action

$$S : \mathcal{C} \to \mathbb{R} \tag{2.2}$$

by

$$S(x) = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4, \qquad \lambda \geq 0,\ m > 0. \tag{2.3}$$

Now we can define the *partition function*,[9]

$$Z = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-S(x)}. \tag{2.5}$$

More generally, let's define an *observable* to be any polynomial function $f : \mathcal{C} \to \mathbb{R}$, and then define its *(unnormalized) expectation value*

$$\langle f \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) \mathrm{e}^{-S(x)}. \tag{2.6}$$

Thus, we have

$$Z = \langle 1 \rangle. \tag{2.7}$$

Both (2.5) and (2.6) are functions of λ and m.

2.2 The perturbation series

Now, how do we *compute* these functions? Let's start with Z, given by (2.5). At $\lambda = 0$, the integral (2.5) is easy to do:

$$Z_0 = Z(m, \lambda = 0) = \sqrt{\frac{2\pi}{m}}. \tag{2.8}$$

For other λ, what to do? Computing for arbitrary λ looks hard, but since $\lambda = 0$ was easy, let's try to get the expansion around $\lambda = 0$. We begin by expanding the exponential under the integral sign:

$$Z(m, \lambda) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!} \right)^n \frac{x^{4n}}{n!} \mathrm{e}^{-\frac{m}{2}x^2} \tag{2.9}$$

[9] Incidentally, it turns out that in this particular theory $Z(m, \lambda)$ actually has a name: e.g. Mathematica gives it as

$$Z(m, \lambda) = \sqrt{\frac{3m}{\lambda}} \mathrm{e}^{3m^2/4\lambda} K_{\frac{1}{4}} \left(\frac{3m^2}{\lambda} \right). \tag{2.4}$$

This should increase your confidence that we are dealing here with an absolutely concrete and well-defined function.

Next we make a dubious step: we exchange the orders of summation and integration.

$$Z(m,\lambda) \text{ ''$=$'' } \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \int_{-\infty}^{\infty} \frac{x^{4n}}{n!} \mathrm{e}^{-\frac{m}{2}x^2} \tag{2.10}$$

Next we use a fundamental integral identity:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2k} \mathrm{e}^{-\frac{m}{2}x^2} = \sqrt{\frac{2\pi}{m}} \frac{1}{m^k} \frac{(2k)!}{k! 2^k}. \tag{2.11}$$

Exercise 2.1. Prove the formula (2.11).

Using (2.11) the integrals in (2.10) can be done term by term, yielding

$$Z(m,\lambda) \text{ ''$=$'' } \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \left(-\frac{1}{96}\right)^n \frac{(4n)!}{n!(2n)!} \widetilde{\lambda}^n, \qquad \widetilde{\lambda} = \frac{\lambda}{m^2}, \tag{2.12}$$

$$\text{ ''$=$'' } \sqrt{\frac{2\pi}{m}} \left(1 - \frac{1}{8}\widetilde{\lambda} + \frac{35}{384}\widetilde{\lambda}^2 + \cdots + (1390.1\ldots)\widetilde{\lambda}^{10} + \cdots \right). \tag{2.13}$$

2.3 Meaning of the perturbation series

Looking at the coefficients, we see at once that (2.12) diverges for all $\lambda \neq 0$, *despite* the fact that the function $Z(m,\lambda)$, defined by the integral (2.5), really does exist whenever $\mathrm{Re}(\lambda) \geq 0$. In particular it follows that the interchange of summation and integration leading to (2.10) was not justified (if you try to justify it by the usual methods you will fail because of lack of uniform convergence [I think]).

Nevertheless, the series (2.12) is still useful:

Definition 2.1 (Asymptotic series). Given a function $f : \mathbb{R}_+ \to \mathbb{C}$, the formal series $\sum_{n=0}^{\infty} c_n t^n$ is an *asymptotic series for f as $t \to 0^+$* if, for all $N \geq 0$,

$$\lim_{t \to 0^+} t^{-N} \left| f(t) - \left(\sum_{n=0}^{N} c_n t^n \right) \right| = 0. \tag{2.14}$$

In this situation we write $f(t) \sim \sum_{n=0}^{\infty} c_n t^n$.

This means

$$\lim_{t \to 0^+} |f(t) - c_0| = 0, \tag{2.15}$$

$$\lim_{t \to 0^+} t^{-1} |f(t) - (c_0 + t c_1)| = 0, \tag{2.16}$$

and so on.

Proposition 2.2 (Perturbation series is an asymptotic series). The series (2.12) is an asymptotic series for $Z(m,\lambda)$ as $\lambda \to 0^+$ for fixed m.

Exercise 2.2. Prove Proposition 2.2. (This amounts to showing that the dubious step (2.10), while not justified at the level of convergent series, is justified at the level of asymptotic series. This is a very commonly-occurring situation.)

So the precise meaning of the "=" in (2.10) and (2.12) above is actually $\sim$.

One way to get a vivid illustration of what this asymptotic series expansion means is to do the next exercise:

Exercise 2.3. Make a plot of $Z(m = 1, \lambda)$ and the first few truncations of its asymptotic series around $\lambda \to 0^+$.

One might wonder whether there could be some *other* series expansion for $Z(m, \lambda)$. But this is impossible, as the next exercise shows.

Exercise 2.4. Do the following:

1. Show that if f has a convergent Taylor series expansion around $t = 0$ then this expansion is also an asymptotic expansion as $t \to 0^+$.

2. Show that any f can have at most one asymptotic series expansion.

In particular, since $Z(m, \lambda)$ has the divergent asymptotic expansion (2.12) it cannot also have a convergent one. For fun, we can diagnose a bit more precisely the problem with Z: it has an essential singularity at $\lambda = 0$, in the sense of the next exercise.

Exercise 2.5. Do the following things.

1. Show that $Z(m, \lambda) = \frac{1}{\sqrt{m}} f(\tilde{\lambda})$ for some f.

2. Show that $Z(m, \lambda)$ obeys the differential equation $(2\partial_m)^2 Z = -(4!\partial_\lambda) Z$.

3. Show that $f(\tilde{\lambda})$ obeys an ordinary differential equation in $\tilde{\lambda}$, with an irregular singularity at $\tilde{\lambda} = 0$.

4. Show that, for fixed m, $Z(m, \lambda)$ admits analytic continuation to a branched cover of $\mathbb{C} \setminus \{0\}$ and this continuation has an essential singularity at $\lambda = 0$.

5. Conclude (again) that $Z(m, \lambda)$ cannot have a convergent Taylor expansion around $\lambda = 0$.

What we have seen here is that even in 0-dimensional quantum field theory the perturbation series is "only" asymptotic. In higher-dimensional theories, we will meet very similar series, and there too we expect that these series are usually "only" asymptotic.

Exercise 2.6. Read the famous paper [10] of Freeman Dyson, in which he gives a heuristic physical argument that the perturbation series in quantum electrodynamics is only asymptotic.

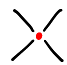
2.4 Feynman diagrams

Now we revisit (2.12) and rewrite it one more time, as

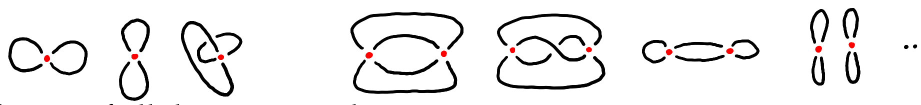
$$Z(m,\lambda) \sim \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!2^{2n}} \frac{(-\widetilde{\lambda})^n}{(4!)^n n!}, \qquad \widetilde{\lambda} = \frac{\lambda}{m^2}. \tag{2.17}$$

This formula has a neat combinatorial interpretation, in terms of *Feynman diagrams*, as follows.

Our basic object is a vertex with 4 half-edges attached.



To construct all Feynman diagrams with n vertices, we begin by fixing n vertices, and then considering all ways to pair up the $4n$ half-edges. All Feynman diagrams with one vertex, and some with two vertices, are shown below.



Let D_n be the set of all diagrams with n vertices.

Proposition 2.3 (Counting pairings of $2k$ objects). The number of ways to pair up $2k$ objects, i.e. to divide them into k 2-element subsets, is $\frac{(2k)!}{k!2^k}$.

Exercise 2.7. Prove Proposition 2.3.

Thus we have

$$|D_n| = \frac{(4n)!}{(2n)!2^{2n}}. \tag{2.18}$$

On the other hand, D_n is naturally acted on by the finite group

$$G_n = (S_4)^n \rtimes S_n \tag{2.19}$$

(permuting edges attached to a given vertex and also permuting vertices) which has

$$|G_n| = (4!)^n n!. \tag{2.20}$$

Thus (2.17) can be rewritten as

$$Z(m,\lambda) \sim \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \frac{|D_n|}{|G_n|}. \tag{2.21}$$

By the orbit-stabilizer theorem this becomes

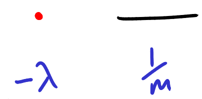
$$Z(m,\lambda) \sim \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \sum_{[\Gamma]\in D_n/G_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} \tag{2.22}$$

which we could also rewrite as

$$Z(m,\lambda) \sim \sqrt{\frac{2\pi}{m}} \sum_{[\Gamma]\in \sqcup_n D_n/G_n} \frac{(-\lambda)^{|\mathrm{vertices}(\Gamma)|}}{m^{|\mathrm{edges}(\Gamma)|}} \frac{1}{|\mathrm{Aut}\,\Gamma|}. \tag{2.23}$$

Thus we have proven a diagrammatic rule:

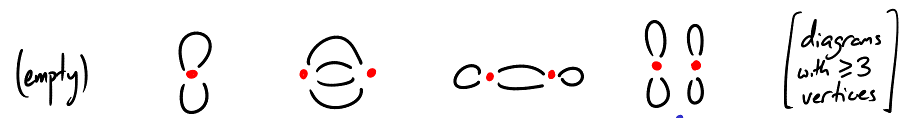
Proposition 2.4 (Feynman diagram expansion for the partition function (2.5)). To compute the perturbation expansion of the partition function (2.5) we can follow the following algorithm. Draw one representative Γ in each equivalence class. Define a weight w_Γ as a product of factors: one factor $(-\lambda)$ for each vertex, one factor $\frac{1}{m}$ for each edge,



and an overall "symmetry factor" $\frac{1}{|\mathrm{Aut}\,\Gamma|}$. Then

$$\frac{Z(m,\lambda)}{Z(m,0)} \sim \sum_{\Gamma} w_\Gamma. \tag{2.24}$$

The first few orders in the diagram expansion of $Z(m,\lambda)$ are:



$$\frac{Z}{Z(\lambda=0)} \sim 1 + \frac{-\lambda}{8m^2} + \frac{\lambda^2}{48m^4} + \frac{\lambda^2}{16m^4} + \frac{\lambda^2}{128m^4} + \mathcal{O}(\lambda^3)$$

This reproduces (2.12) as it should.

This basic mechanism can be extended in many ways:

1. In Proposition 2.4 we sum over both connected and disconnected Γ. But the contribution from disconnected diagrams is easily determined:

   Proposition 2.5 (Exponentiation of the connected diagrams). The sum over connected diagrams is related to the sum over all diagrams by:

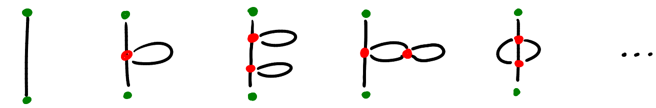
   $$\sum_{\Gamma} w_\Gamma = \exp\left(\sum_{\Gamma \text{ connected nonempty}} w_\Gamma \right). \tag{2.25}$$

   (e.g. look at the first disconnected term above to get an inkling of why.) Said otherwise,

   $$\log\left(\frac{Z(m,\lambda)}{Z(m,0)}\right) \sim \sum_{\Gamma \text{ connected nonempty}} w_\Gamma. \tag{2.26}$$

2. Suppose we want to compute the correlation function $\langle x^n \rangle$, as defined in (2.6) (generalizing Z which is the case $n = 0$). This is given by a similar sum over Feynman diagrams, except that now we introduce a new type of 1-valent vertex, and require that the diagram contains exactly n of these. The automorphisms of Γ are required to fix these vertices.

   To compute the *normalized* expectation value $\langle x^n \rangle / Z$, we compute similarly, with the additional rule that every connected component of each diagram must contain at least one of the 1-valent vertices.



$$\frac{\langle x^2\rangle}{Z} \sim \frac{1}{m} - \frac{\lambda}{2m^3} + \frac{\lambda^2}{4m^5} + \frac{\lambda^2}{4m^5} + \frac{\lambda^2}{6m^5} + \mathcal{O}(\lambda^3)$$

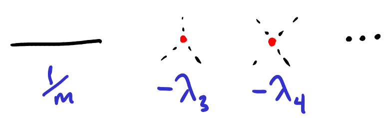
Exercise 2.8. Do the following:

- Compute the perturbative expansion of $\langle x^2\rangle/Z$ up to order λ^3, using Feynman diagrams. You should find $\langle x^2\rangle/Z = \frac{1}{m}(1 - \frac{1}{2}\tilde{\lambda} + \frac{2}{3}\tilde{\lambda}^2 - \frac{11}{8}\tilde{\lambda}^3 + \cdots)$. (I drew the diagrams up to order $\tilde{\lambda}^2$ above.)
- Compute the perturbative expansion of $\langle x^4\rangle/Z$ up to order λ^2, using Feynman diagrams. You should find $\langle x^4\rangle/Z = \frac{1}{m^2}(3 - 4\tilde{\lambda} + \frac{33}{4}\tilde{\lambda}^2 + \cdots)$.

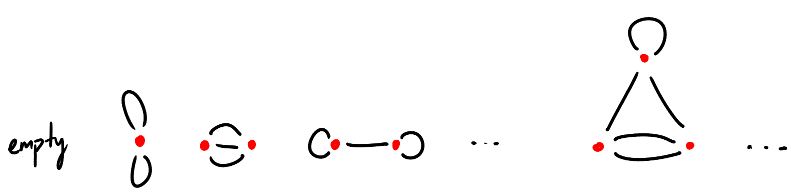
3. Instead of the action (2.3) we could take more generally

$$S(x) = \frac{m}{2}x^2 + \sum_{k=3}^{\infty}\frac{\lambda_k}{k!}x^k. \tag{2.27}$$

The Feynman diagram expansion then involves vertices of arbitrary valences, with each k-valent vertex contributing a factor $-\lambda_k$:



Here are some diagrams in the expansion of Z/Z_0:



$$\frac{Z}{Z_0} \sim 1 - \frac{\lambda_4}{8m^2} + \frac{\lambda_3^2}{12m^3} + \frac{\lambda_3^2}{8m^3} + \cdots - \frac{\lambda_3^2\lambda_4}{8} + \cdots$$

4. Generalizing in a different direction, we could take $\mathcal{C} = \mathbb{R}^N$ instead of $\mathbb{R}$ ("multiple fields"), with coordinates $x^1, \ldots, x^N$, and generalize the action to[10]

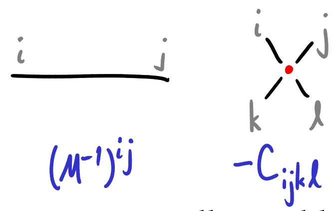
$$S = \frac{1}{2}x^i M_{ij} x^j + \frac{1}{4!}C_{ijkl}x^i x^j x^k x^l, \tag{2.28}$$

[10] Here and in many future equations we are using the "Einstein summation convention": any index which appears both up and down should be summed over. So the first term in (2.28) should be read $\frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N x^i M_{ij} x^j$ and similarly the second term involves four sums.

where M and C are both totally symmetric in their indices. Then we have

$$Z_0 = \int_{\mathcal{C}} \mathrm{d}\vec{x}\, \mathrm{e}^{-\frac{1}{2}x^i M_{ij} x^j} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}. \tag{2.29}$$

The diagrams Γ are just as before, with edges and quartic vertices. However, in the Feynman rules we attach additional labels $i \in \{1, \ldots, N\}$ on the half-edges:



To compute the weight w_Γ we sum over all possible labels for the half-edges (so for a diagram with k edges we sum N^{2k} terms), and divide by the usual symmetry factor $|\mathrm{Aut}\,\Gamma|$ for the unlabeled diagram Γ. (In higher-dimensional QFT, these sorts of labels would have an interpretation like labeling species of particle which could propagate along the edges.)

A basis-free description of this situation is as follows. Let $\mathcal{C}$ be a finite-dimensional real vector space V, with a density $\mathrm{d}\mu$, and two elements

$$M \in \mathrm{Sym}^2(V^*), \qquad C \in \mathrm{Sym}^4(V^*). \tag{2.30}$$

Then consider the action

$$S(x) = \frac{1}{2} M(x,x) + \frac{1}{4!} C(x,x,x,x). \tag{2.31}$$

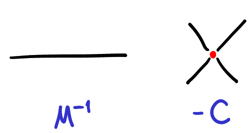
M determines a density $\sqrt{\det M}$ on V:

Exercise 2.9. Verify that a positive definite bilinear form $M \in \mathrm{Sym}^2(V^*)$ determines a density on V, which deserves to be called $\sqrt{\det M}$ in the sense that given a basis $\{e^1, \ldots, e^n\}$ on V, with $M(e^i, e^j) = M^{ij}$, $\sqrt{\det M} = \sqrt{\det(M^{ij})} |e_1^* \cdots e_n^*|$.

When $C = 0$ the partition function is

$$Z_0 = \int_V \mathrm{d}\mu\, \mathrm{e}^{-\frac{1}{2}M(x,x)} = (2\pi)^{\frac{1}{2}\dim V} \frac{\mathrm{d}\mu}{\sqrt{\det M}}. \tag{2.32}$$

The Feynman rules assign a vector in $(V^*)^{\otimes 4}$ to each vertex and $V^{\otimes 2}$ to each edge, contracted in the obvious way:

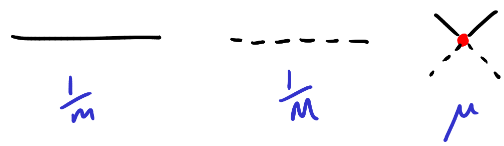


2.5 A coupled system

Now suppose $\mathcal{C} = \mathbb{R}^2$ and

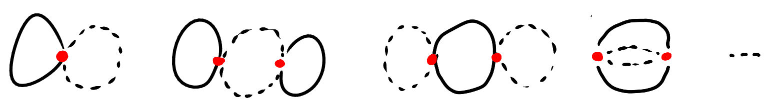
$$S(x,y) = \frac{m}{2}x^2 + \frac{M}{2}y^2 + \frac{\mu}{4}x^2y^2. \tag{2.33}$$

We think of this as two independent systems, one involving the field x and one involving the field y, which are "coupled" by the quartic interaction term $\frac{\mu}{4}x^2y^2$. You can see this point of view vividly in the Feynman rules for this theory, shown below:

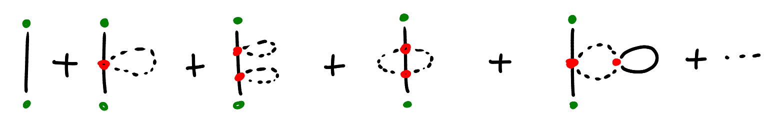


Exercise 2.10. Explain how these Feynman rules arise as a special case of the multiple-field rules given above.

A few sample computations are:



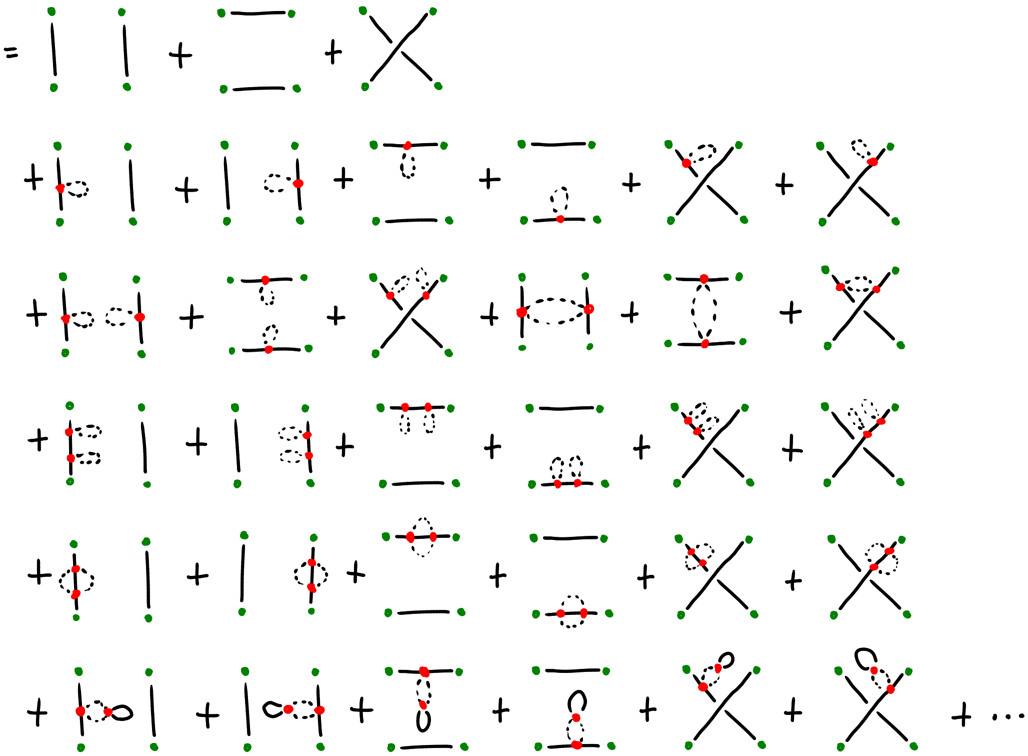
$$\log\left[\frac{Z}{Z_0}\right] \sim -\frac{\mu}{4mM} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{8m^2M^2} + \mathcal{O}(\mu^3)$$



$$\frac{\langle x^2\rangle}{Z} \sim \frac{1}{m} - \frac{\mu}{2m^2M} + \frac{\mu^2}{4m^3M^2} + \frac{\mu^2}{2m^3M^2} + \frac{\mu^2}{4m^3M^2} + \mathcal{O}(\mu^3)$$

But the job of computing by Feynman diagrams in this theory gets complicated fast. For example:

$$\frac{\langle x^4\rangle}{Z} =$$



$$= \frac{3}{m^2} - \frac{3\mu}{m^3 M} + \frac{33\mu^2}{4m^4M^2} + O(\mu^3)$$

How can we avoid this enormous profusion of diagrams every time we want to compute a correlation function? Suppose that we are only interested in computing correlations involving x. Then to simplify our task, we could use Fubini's theorem to integrate over y "once and for all": define $S_{\text{eff}}(x)$ by the equation

$$\int \mathrm{d}y\, \mathrm{e}^{-S(x,y)} = \mathrm{e}^{-S_{\text{eff}}(x)} \tag{2.34}$$

and then use $S_{\text{eff}}(x)$ as our action for subsequent computations.

In this particular theory we can compute the perturbation series of $S_{\text{eff}}(x)$ explicitly: it is of the form

$$S_{\text{eff}}(x) \sim \frac{m_{\text{eff}}}{2}x^2 + \sum_{k\geq 3} \frac{\lambda_k}{k!}x^k \tag{2.35}$$

with

$$m_{\text{eff}} = m + \frac{\mu}{2M}, \qquad \lambda_k = \begin{cases} 0 & \text{for } k \text{ odd,} \\ -\left(-\frac{\mu}{2M}\right)^{k/2}(k-1)! & \text{for } k \text{ even.} \end{cases} \tag{2.36}$$

Some qualitative remarks about $S_{\text{eff}}(x)$:

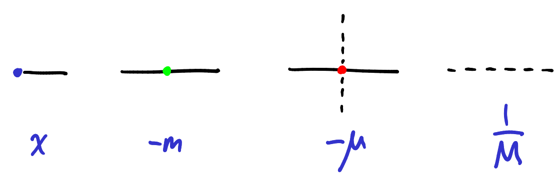
- Even though the original action $S(x,y)$ had only a quartic interaction, the effective action $S_{\text{eff}}(x)$ involves interactions of all even orders. Diagrammatically speaking, integrating out the field y to pass from S to S_{eff} amounts to "collapsing" parts of the Feynman diagrams involving only the dashed y lines; these parts get absorbed into the new effective vertices.

- The fact that in the effective theory we only get interaction vertices of even valence is related to the fact that the original action $S(x, y)$ has the symmetry $x \to -x$, which implies that $S_{\text{eff}}(x)$ must have the same symmetry.
- We could approximate $S_{\text{eff}}(x)$ by the simpler procedure of setting $y = 0$ in $S(x, y)$. This would not give the exact answer, because the fields are coupled: it would give $m_{\text{eff}} = m$ and all $\lambda_k = 0$. Thus the shift $m_{\text{eff}} - m$ and the nonzero values of the λ_k could be thought of as "quantum corrections" which are some vestige of the field y. Note these corrections go to zero in the limit $\mu \to 0$ (decoupling) and also go to zero in the limit $M \to \infty$.
- The series (2.35) for $S_{\text{eff}}(x)$ actually is convergent, not only asymptotic.

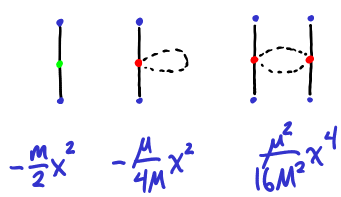
Exercise 2.11. Use the effective action $S_{\text{eff}}(x)$ to compute $\frac{\langle x^4 \rangle}{Z}$ up to order μ^2. Note that it is a lot easier than using the original action $S(x, y)$, but it indeed agrees with the result of the computation done above.

Exercise 2.12. Derive the formula (2.36).

There is also a Feynman-diagram expansion for the computation of the effective action S_{eff}. The rules here turn out to be as follows: we introduce a new 1-valent vertex and sum over connected diagrams where all x lines are required to terminate on a 1-valent vertex. In counting the symmetry factor we do *not* require that these 1-valent vertices are fixed by the automorphisms. The Feynman rules are:



Exercise 2.13. Use these Feynman rules to derive (2.36). (Hint: the first few diagrams contributing are shown below.)



2.6 Symmetries

Let us return to the original theory with action (2.3). In this theory we have

$$\langle x^n \rangle = 0 \quad \text{for } n \text{ odd} \tag{2.37}$$

One direct way of seeing this is to make the change of variables $x \to -x$ in the integral (2.5). Since $S(x) = S(-x)$ this change of variables gives

$$\langle x^n \rangle = \langle (-x)^n \rangle = (-1)^n \langle x^n \rangle \tag{2.38}$$

from which (2.37) follows.

The Feynman-diagrammatic expression of (2.37) is that there simply are no possible diagrams with an odd number of 1-valent vertices, since the number of half-edges would then be odd, no matter how many 4-valent vertices we add. The same wouldn't be true if we allow vertices of odd valence; of course such vertices arise only when there are odd-degree terms in S, which violate the symmetry $x \to -x$.

More generally,

Proposition 2.6 (Symmetries and correlation functions). Whenever $S : \mathcal{C} \to \mathbb{R}$ and the measure on $\mathcal{C}$ are both invariant under the action of a group G, then we have

$$\langle \mathcal{O}^g \rangle = \langle \mathcal{O} \rangle \tag{2.39}$$

where $\mathcal{O} : \mathcal{C} \to \mathbb{R}$ is any observable, and $\mathcal{O}^g = g^* \mathcal{O}$.

Exercise 2.14. Prove Proposition 2.6.

Exercise 2.15. Suppose the space Obs of all observables $\mathcal{O}$ is decomposed into isotypical components, $\mathrm{Obs} = \bigoplus_R \mathrm{Obs}_R$, where R runs over irreducible representations of G. Show that if $\mathcal{O} \in \mathrm{Obs}_R$ and R is nontrivial then $\langle \mathcal{O} \rangle = 0$.

If G is a Lie group, we can differentiate (2.39) to get

Proposition 2.7 (Infinitesimal symmetries and correlation functions). Whenever $S : \mathcal{C} \to \mathbb{R}$ and the measure on $\mathcal{C}$ are both invariant under the action of a Lie group G, then we have

$$\langle X \mathcal{O} \rangle = 0 \tag{2.40}$$

where $\mathcal{O} : \mathcal{C} \to \mathbb{R}$ is any observable, and $X \in \mathfrak{g}$.

Exercise 2.16. Take $\mathcal{C} = \mathbb{R}^2$ and $S(x,y) = \frac{1}{2} m x^2 + \frac{1}{2} m y^2$. Alternatively, letting $z = x + \mathrm{i}y$, we can say $\mathcal{C} = \mathbb{C}$ and $S(z) = \frac{1}{2} m |z|^2$. Consider the complex observables $\mathcal{O}_n : \mathcal{C} \to \mathbb{C}$ given by $\mathcal{O}_n(z) = z^n$. Use Proposition 2.6 to show that the correlation function $\langle \mathcal{O}_n \rangle$ vanishes for all $n \neq 0$. Is the same true if $S(z) = V(|z|)$ for more general V? What about if $S(z)$ is an arbitrary function of z?

2.7 Fermions

So far, the results of our QFT computations have been very far from "topological": Z and all the expectation values $\langle x^n \rangle$ are nontrivial functions of the parameters (m, λ) with no kind of deformation invariance in sight.

In the applications of QFT toward which we are headed, we will do things that are more deformation invariant. But to get there, we need one more key ingredient: *fermions*.

We will replace the field space $\mathcal{C}$, which so far has been a manifold (in fact a vector space), by a *supermanifold* (in fact super vector space).

Our treatment of supergeometry will be extremely superficial. Some references I have found useful are [11, 12, 13, 14].

Definition 2.8 (Super vector space). A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space,

$$V = V^0 \oplus V^1. \tag{2.41}$$

We let Π be the operation of *parity reversal*, i.e.

$$\Pi V = V^1 \oplus V^0. \tag{2.42}$$

Example 2.9. The basic example is the super vector space V with $V^0 = \mathbb{R}^p$, $V^1 = \mathbb{R}^q$; call this $\mathbb{R}^{p|q}$. Then $\Pi\mathbb{R}^{p|q} = \mathbb{R}^{q|p}$.

Definition 2.10 (Even and odd maps). An even map of super vector spaces $V \to W$ is a pair of maps $V^0 \to W^0$ and $V^1 \to W^1$. An odd map $V \to W$ is a pair of maps $V^0 \to W^1$ and $V^1 \to W^0$.

Definition 2.11 (Symmetric monoidal category of super vector spaces). The *symmetric monoidal category* of super vector spaces is the category of super vector spaces, with morphisms the even maps, and equipped with an unusual choice of symmetry isomorphism

$$s : V \otimes W \to W \otimes V, \tag{2.43}$$

namely, for homogeneous elements v, w of degrees $|v|$, $|w|$ we take

$$s(v \otimes w) = (-1)^{|v||w|} w \otimes v. \tag{2.44}$$

This turns out to be a very useful category!

Some standard constructions in linear algebra take on a different character when applied to super vector spaces. For example, if V is a super vector space then we define the symmetric algebra $\mathrm{Sym}^* V$ to be the quotient of T^*V by the two-sided ideal generated by $(v \otimes w - s(v \otimes w))$. If $V = V^0$ then $\mathrm{Sym}^* V$ is (forgetting its super structure) the usual $\mathrm{Sym}^* V^0$, but if $V = V^1$ then $\mathrm{Sym}^* V$ is the exterior algebra $\wedge^*(\Pi V^1)$.

Definition 2.12 (Polynomial functions on a super vector space). Given a super vector space V, we define the algebra of polynomial functions $\mathcal{O}(V)$ on V by

$$\mathcal{O}(V) = \mathrm{Sym}^*(V^*). \tag{2.45}$$

$\mathcal{O}(V)$ is itself a super vector space,

$$\mathcal{O}(V) = \mathcal{O}^0(V) \oplus \mathcal{O}^1(V), \tag{2.46}$$

and even a (super)commutative algebra.

In quantum field theory we want to consider a space $\mathcal{C}$ with an action S which is some kind of "function on $\mathcal{C}$". For $\mathcal{C}$ a super vector space, our model of "function on $\mathcal{C}$" will be an element

$$S \in \mathcal{O}^0(V). \tag{2.47}$$

2.8 A fermionic theory

The simplest example we can consider is to take

$$\mathcal{C} = \mathbb{R}^{0|2}. \tag{2.48}$$

$\mathcal{C}$ has two "coordinate functions"

$$\psi^1, \psi^2 \in \mathcal{O}^1(\mathcal{C}) = (V^1)^* \tag{2.49}$$

but these coordinates have "odd statistics"

$$\psi^1\psi^2 = -\psi^2\psi^1, \quad (\psi^1)^2 = 0, \quad (\psi^2)^2 = 0. \tag{2.50}$$

Note that $\psi^1\psi^2 \in \mathcal{O}^0(\mathcal{C})$ is a nice even function, but (2.50) implies it is nilpotent,

$$(\psi^1\psi^2)^2 = 0. \tag{2.51}$$

(In fact $\dim \mathcal{O}(C) = 2|2$ as we expect from the identification with the exterior algebra on $\mathbb{R}^2$; the even part $\mathcal{O}^0(\mathcal{C})$ has basis $\{1, \psi^1\psi^2\}$, the odd part $\mathcal{O}^1(\mathcal{C})$ has basis $\{\psi^1, \psi^2\}$.)

Now let us take the action functional

$$S = \frac{1}{2}M\psi^1\psi^2. \tag{2.52}$$

(Unlike in the bosonic case, here this is all we can do — there is no way of introducing an interaction term!)

We would like to make sense of the partition function in this setting,

$$Z = \int_{\mathcal{C}} \mathrm{d}\mu \, \mathrm{e}^{-S}. \tag{2.53}$$

Expanding the exponential, the fact that $S^2 = 0$ means it *truncates* to a polynomial:

$$Z = \int_{\mathcal{C}} \mathrm{d}\mu \left(1 - \frac{1}{2}M\psi^1\psi^2\right). \tag{2.54}$$

What will we mean by such an integral? Integration over a purely odd vector space is defined to mean taking the "top order part" of the function. More exactly:

Definition 2.13. A *translation invariant measure* $\mathrm{d}\mu$ on a purely odd super vector space $V = V^1$ is

$$\mathrm{d}\mu \in \wedge^{\mathrm{top}}(\Pi V^1). \tag{2.55}$$

For any $f \in \mathcal{O}(V) \simeq \wedge^*((\Pi V^1)^*)$, let $f^{\mathrm{top}} \in \wedge^{\mathrm{top}}((\Pi V^1)^*)$ be the top component of f; then we define the integral by

$$\int_V \mathrm{d}\mu \, f = \mathrm{d}\mu \cdot f^{\mathrm{top}}. \tag{2.56}$$

Exercise 2.17. Suppose $V = \mathbb{R}^{0|1}$ with odd coordinate ψ. Show that there exists a translation invariant measure $\mathrm{d}\mu$ on V with the property that

$$\int_V \mathrm{d}\mu\, (a\psi + b) = a. \tag{2.57}$$

We call this measure $\mathrm{d}\psi$.

Exercise 2.18. Again take $V = \mathbb{R}^{0|1}$, and $c \in \mathbb{R}$. Show that there exist translation invariant measures $c\mathrm{d}\psi$ and $\mathrm{d}(c\psi)$ on V, obeying the formulas

$$\int_V (c\mathrm{d}\psi) f(\psi) = c \int_V \mathrm{d}\psi f(\psi), \qquad \int_V \mathrm{d}(c\psi) f(c\psi) = \int_V \mathrm{d}\psi f(\psi), \tag{2.58}$$

where $f(c\psi)$ is defined in the obvious way — explicitly, if $f(\psi) = a\psi + b$ then $f(c\psi) = ac\psi + b$. Then prove the change of variables formula in one odd variable:

$$\mathrm{d}(c\psi) = \frac{1}{c}\mathrm{d}\psi. \tag{2.59}$$

Similarly on $\mathbb{R}^{0|q}$ we choose once and for all the measure $\mathrm{d}\mu = \mathrm{d}\vec{\psi}$, characterized by

$$\int \mathrm{d}\vec{\psi}\, \psi^q \psi^{q-1} \cdots \psi^1 = 1. \tag{2.60}$$

Now we can evaluate the odd Gaussian integral (2.54), obtaining

$$Z = \frac{1}{2} M. \tag{2.61}$$

Note a key difference between this and the usual even Gaussian integrals: here the M appears in the *numerator*, not the *denominator* as we had in (2.8).

2.9 More fermions

More generally suppose we take $\mathcal{C}$ to be any purely odd super vector space, $\mathcal{C} = V = V^1$, equipped with a measure in the sense of (2.55), and elements

$$M \in \mathrm{Sym}^2(V^*) = \wedge^2((\Pi V^1)^*), \qquad C \in \mathrm{Sym}^4(V^*) = \wedge^4((\Pi V^1)^*). \tag{2.62}$$

Then we can take for the action

$$S = \frac{1}{2} M + \frac{1}{4!} C \in \mathcal{O}(\mathcal{C}), \tag{2.63}$$

which we also write in parallel with (2.31) as

$$S(\psi) = \frac{1}{2} M(\psi, \psi) + \frac{1}{4!} C(\psi, \psi, \psi, \psi), \tag{2.64}$$

or after choosing a basis for ΠV^1,

$$S(\psi) = \frac{1}{2} M_{IJ} \psi^I \psi^J + \frac{1}{4!} C_{IJKL} \psi^I \psi^J \psi^K \psi^L \tag{2.65}$$

with M and C totally antisymmetric in their indices.

The partition function is an odd integral

$$Z = \int_{\mathcal{C}} \mathrm{d}\mu \, \mathrm{e}^{-S} \tag{2.66}$$

which can again be evaluated in a purely algebraic fashion, by expanding out the exponential.

Exercise 2.19. Suppose $\mathcal{C} = \mathbb{R}^{0|4}$ and $S = m\psi^1\psi^2 + m\psi^3\psi^4 + \lambda\psi^1\psi^2\psi^3\psi^4$. Show that $Z = m^2 - \lambda$.

2.10 Perturbation theory with fermions

Integrals over finite-dimensional odd vector spaces always give *polynomials* in the couplings; thus a development of "perturbation theory" for them might seem unnecessary. Nevertheless, with an eye toward the future, it is interesting to develop a Feynman diagram expansion for these integrals.

First let us see what happens when $C = 0$. Then the odd Gaussian integral gives the *Pfaffian* of M, generalizing (2.61) (compare (2.29)):

$$Z_0 = \int_{\mathcal{C}} \mathrm{d}\vec{\psi} \, \mathrm{e}^{-\frac{1}{2}\psi^I M_{IJ} \psi^J} = \mathrm{Pf}(M). \tag{2.67}$$

(Recall that the Pfaffian is a polynomial in the entries of M, defined only for skew-symmetric M, with the property that $(\mathrm{Pf}\, M)^2 = \det M$. For example, $\mathrm{Pf} \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$. Note that for a general 2×2 matrix the determinant is *not* the square of any polynomial.)

Exercise 2.20. Prove (2.67). One way that presumably works is to expand directly and compare with the combinatorial expression for the Pfaffian, namely if $\operatorname{rank} M = 2n$ then

$$\mathrm{Pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} M_{\sigma(2i-1),\sigma(2i)}. \tag{2.68}$$

The coordinate-independent version of this is:

Exercise 2.21. Suppose $V = V^1$ is an odd vector space with a quadratic element $M \in \mathrm{Sym}^2(V)$. Show that there is a canonical element $\mathrm{Pf}(M) \in \wedge^{\mathrm{top}}(\Pi(V^1)^*)$, and that if we choose an element $\mathrm{d}\mu \in \wedge^{\mathrm{top}}(\Pi V^1)$, then

$$\int_V \mathrm{d}\mu \, \mathrm{e}^{-M} = \mathrm{d}\mu \cdot \mathrm{Pf}(M). \tag{2.69}$$

When $C \neq 0$ we compute Z/Z_0 by Feynman diagrams. The rules are just as they were in the even case, except that we have to be careful about signs. [...]

Exercise 2.22. Suppose again $\mathcal{C} = \mathbb{R}^{0|4}$ and $S = m\psi^1\psi^2 + m\psi^3\psi^4 + \lambda\psi^1\psi^2\psi^3\psi^4$. Show that $Z = m^2 - \lambda$ by the Feynman diagram expansion. (There is only one nonempty diagram which contributes.)

2.11 Bosons and fermions together

Now suppose $\mathcal{C} = V = V^0 \oplus V^1$ is a super vector space which has nontrivial odd *and* even parts. We want to extend our integration theory to this situation. The strategy will be to integrate first over the odd directions, then over the even directions.

To get anything convergent, we need to extend the class of functions we consider: we let

$$C^\infty(V) = C^\infty(V^0) \otimes \mathcal{O}(V^1). \tag{2.70}$$

To define measures we need the super analogue of the determinant line:

Definition 2.14 (Berezinian line). The *Berezinian line* of a super vector space V is

$$\operatorname{Ber} V = \wedge^{\mathrm{top}} V^0 \otimes \wedge^{\mathrm{top}} (\Pi V^1)^*. \tag{2.71}$$

An element $\mathrm{d}\mu \in \operatorname{Ber} V^*$ plays the role of a volume measure on V. By an *orientation* of V we mean an orientation of V^0. Then:

Definition 2.15 (Integration over super vector space). If V is an oriented super vector space, $\mathrm{d}\mu = \omega^0 \otimes \omega^1 \in \operatorname{Ber} V^*$, and $f = f^0 \otimes f^1 \in C^\infty(V)$, then

$$\int_V \mathrm{d}\mu\, f = \int_{V^0} \left(\omega^0 f^0\right) \int_{V^1} \left(\omega^1 f^1\right). \tag{2.72}$$

On $\mathbb{R}^{p|q}$ we have the canonical element

$$\mathrm{d}\mu = \mathrm{d}\vec{x}\,\mathrm{d}\vec{\psi} = (\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p) \otimes (\mathrm{d}\vec{\psi}) \in \operatorname{Ber} V^*. \tag{2.73}$$

2.12 A supersymmetric example

Let us now consider the example $\mathcal{C} = \mathbb{R}^{1|2}$, and write an action of the form

$$S(x, \psi^1, \psi^2) = S_1(x) + S_2(x)\psi^1\psi^2. \tag{2.74}$$

The partition function is

$$Z = \int \mathrm{d}x\mathrm{d}\vec{\psi}\, \mathrm{e}^{-S} \tag{2.75}$$

$$= \int \mathrm{d}x\, S_2(x)\mathrm{e}^{-S_1(x)}. \tag{2.76}$$

For generic $S_1(x)$ and $S_2(x)$ we could compute this in perturbation theory: indeed it just reduces to a computation of the correlation function $\langle S_2(x)\rangle$ in the pure bosonic theory with action $S_1(x)$. The answer has no particularly good property.

There is a special case where we can do much better. (This example is more or less lifted from [15].) This is the case where for some $h : \mathbb{R} \to \mathbb{R}$, with $|h(x)| \to \infty$ as $|x| \to \infty$, we have

$$S_1(x) = \frac{1}{2}h(x)^2, \qquad S_2(x) = h'(x), \tag{2.77}$$

so that the action (2.74) becomes

$$S(x, \psi^1, \psi^2) = \frac{1}{2}h(x)^2 + h'(x)\psi^1\psi^2. \tag{2.78}$$

The basic reason why this case is better is that the action (2.78) is invariant under a certain *odd vector field* or *supersymmetry*.

Definition 2.16 (Graded derivations). If $\mathcal{A}$ is a commutative superalgebra, then a map $D : \mathcal{A} \to \mathcal{A}$ (either even or odd) is a *derivation* of $\mathcal{A}$ if it obeys

$$D(aa') = (Da)a' + (-1)^{|a||D|}a(Da'). \tag{2.79}$$

Definition 2.17 (Super vector fields on a super vector space). If V is a super vector space, let $\mathrm{Vect}(V)$ denote the space of all graded derivations on $C^\infty(V)$.

In particular, on $\mathbb{R}^{p|q}$, in addition to the usual even vector fields ∂_{x^i} we also have odd vector fields ∂_{ψ^I}, defined by

$$\partial_{\psi^I}x^i = 0, \quad \partial_{\psi^I}\psi^J = \delta^J_I. \tag{2.80}$$

Together with the derivation property this implies e.g.

$$\partial_{\psi^1}\psi^1\psi^2 = \psi^2, \qquad \partial_{\psi^2}\psi^1\psi^2 = -\psi^1. \tag{2.81}$$

The ∂_{x^i} and ∂_{ψ^I} together generate $\mathrm{Vect}(V)$ as a module over $C^\infty(V)$.

Definition 2.18 (Super Lie algebra). A *super Lie algebra* is a super vector space $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ with a bracket obeying

$$[X, Y] = -(-1)^{|X||Y|}[Y, X] \tag{2.82}$$

and

$$[X, [Y, Z]] + (-1)^{|X|(|Y|+|Z|)}[Y, [Z, X]] + (-1)^{|Z|(|X|+|Y|)}[Z, [X, Y]] = 0. \tag{2.83}$$

Exercise 2.23. Verify that $\mathrm{Vect}(V)$ is a super Lie algebra under supercommutator: $[X, Y]f = X(Yf) - (-1)^{|X||Y|}Y(Xf)$.

Now in our example of $\mathbb{R}^{1|2}$ we consider the odd vector fields

$$Q_1 = \psi^1\partial_x + h(x)\partial_{\psi^2}, \tag{2.84}$$

$$Q_2 = \psi^2\partial_x - h(x)\partial_{\psi^1}. \tag{2.85}$$

These are both symmetries of the action (2.78), in the sense that

$$Q_1 S = Q_2 S = 0. \tag{2.86}$$

Exercise 2.24. $[Q_1, Q_1]$ is a nontrivial even vector field on V. Compute it, and verify directly that it is a symmetry of S.

The vector fields Q_1, Q_2 are also divergence-free, i.e. they preserve the integration measure $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}\vec{\psi}$, in the following sense. For any super vector space F we can consider the super vector space

$$C^\infty(V, F) = C^\infty(V) \otimes_{\mathbb{R}} F \tag{2.87}$$

which we could think of as a space of "sections of the trivial super vector bundle with fiber F over the super vector space V." Then,

Definition 2.19 (Lie derivative of section of $\operatorname{Ber} V^*$ along super vector field). If X is a vector field on V, the *Lie derivative*[11]

$$\mathcal{L}_X : C^\infty(V, \operatorname{Ber} V^*) \to C^\infty(V, \operatorname{Ber} V^*) \tag{2.88}$$

is characterized by

$$\mathcal{L}_X(f\mathrm{d}\mu) = (Xf)\mathrm{d}\mu + (-1)^{|f||X|} f \mathcal{L}_X \mathrm{d}\mu \tag{2.89}$$

and if $X = h^i \partial_{x^i} + g^I \partial_{\psi^I}$ relative to a basis of V, then

$$\mathcal{L}_X(\mathrm{d}\vec{x}\,\mathrm{d}\vec{\psi}) = \left(\partial_{x^i} h^i + (-1)^{|Q|}\partial_{\psi^I} g^I\right)\mathrm{d}\mu. \tag{2.90}$$

Exercise 2.25. Verify that $\mathcal{L}_{Q_1}\mathrm{d}\mu = 0$ and similarly for Q_2.

The existence of the supersymmetries Q_1 and Q_2 will give us a powerful tool for analyzing the partition function. First we need a preliminary:

Lemma 2.20. Suppose Q is a divergence-free vector field (even or odd) on a super vector space V with translation invariant measure $\mathrm{d}\mu$, and $f \in C_c^\infty(V)$. Then

$$\int_V \mathrm{d}\mu\, Qf = 0. \tag{2.91}$$

Proof. By direct computation: if $Q = h^i \partial_{x^i} + g^I \partial_{\psi^I}$ and $\mathrm{d}\mu = \mathrm{d}\vec{x}\mathrm{d}\vec{\psi}$, then

$$\int_V \mathrm{d}\mu\, Qf = \int \mathrm{d}\vec{x}\,(Qf)^{\mathrm{top}} \tag{2.92}$$

$$= \int \mathrm{d}\vec{x}\,(h^i\partial_{x^i} f + g^I \partial_{\psi^I} f)^{\mathrm{top}} \tag{2.93}$$

$$= \int \mathrm{d}\vec{x}\,(h^i\partial_{x^i} f + (-1)^{|g|}\partial_{\psi^I} g^I) f)^{\mathrm{top}} \tag{2.94}$$

$$= \int \mathrm{d}\vec{x}\,((-\partial_{x^i} h^i + (-1)^{|g|}\partial_{\psi^I} g^I) f)^{\mathrm{top}} \tag{2.95}$$

$$= 0 \tag{2.96}$$

[11] For a coordinate-free description of $\mathcal{L}_X$, more in keeping with the notion of the infinitesimal variation along a flow which one has in ordinary geometry, see e.g. [14].

where in the second line we used the fact that $(\partial_{\psi^I}(g^I f))^{\mathrm{top}} = 0$, and in the third line we integrated by parts, using the compact support of f. $\square$

This permits us to make the following fundamental construction:

Proposition 2.21 (Invariance of partition function under Q-exact deformations). Suppose V is a super vector space with measure $\mathrm{d}\mu$. Let $\{S_t\}$ be a family of actions, invariant under a family of divergence-free odd symmetries Q_t, i.e. $Q_t S_t = 0$. Finally, suppose

$$\partial_t S_t = Q_t \Psi_t \tag{2.97}$$

with $\Psi_t \in C_c^\infty(\mathcal{C})$. Then the partition function Z_t computed with action S_t is independent of t.

Proof. After this deformation the partition function is

$$Z_t = \int_{\mathcal{C}} \mathrm{d}\mu\, \mathrm{e}^{-S_t} \tag{2.98}$$

and thus

$$\begin{aligned} \partial_t Z_t &= -\int_{\mathcal{C}} \mathrm{d}\mu\, (Q_t \Psi_t) \mathrm{e}^{-S_t} && (2.99) \\ &= -\int_{\mathcal{C}} \mathrm{d}\mu\, Q_t \left(\Psi_t \mathrm{e}^{-S_t} \right) && (2.100) \\ &= 0. && (2.101) \end{aligned}$$

At the first step, to justify the differentiation under the integral sign, we use the fact that Ψ_t is compactly supported. $\square$

One often considers the special case where $Q_t = Q$ is independent of t and has $Q^2 = 0$, in which case it is natural to consider the *homology* of Q acting on $C^\infty(\mathcal{C})$; in particular S is a Q-closed element, and Proposition 2.21 says roughly that Z only depends on the *homology class* of S.

We will apply Proposition 2.21 in our example. We consider deforming the action S in (2.78) to a family of actions S_t, by deforming $h(x)$ to a family $h_t(x)$. Using dots for derivatives with respect to t, we have

$$\dot{S} = h(x)\dot{h}(x) + \dot{h}'(x)\psi^1\psi^2. \tag{2.102}$$

But this is actually Q_1-exact: indeed

$$\dot{S} = Q_1 \Psi, \qquad \Psi = \dot{h}(x)\psi^2. \tag{2.103}$$

Thus, applying Proposition 2.21, we conclude:

Proposition 2.22 (Weak deformation invariance of partition function). For the action (2.78), Z is unchanged by compactly supported variations of h.

Making deformations with larger and larger compact support $[L, -L]$ and using an *a priori* estimate of the contribution to Z from the region $|x| > L$, we can bootstrap to something slightly stronger:

Proposition 2.23 (Strong deformation invariance of partition function). For the action (2.78), Z depends only on the asymptotic signs ϵ_+, ϵ_- of $h(x)$, defined by $\lim_{x \to \pm\infty} h(x) = \epsilon_\pm \infty$.

Exercise 2.26. Prove Proposition 2.23.

As we will see below, one cannot improve this further: Z really *does* depend on the asymptotic signs $\epsilon_\pm$. Crudely speaking, the point is that so long as $h(x)^2 \to \infty$ for large $|x|$, it suppresses the contribution to Z from large $|x|$, making the field space $\mathcal{C}$ effectively compact. If we try to change the sign ϵ_+ (say) by interpolating through a family of functions $h_t(x)$, there will be some critical t where we lose this compactness, and so we lose the deformation invariance. This is a prototype for "wall-crossing" phenomena which recur frequently in topological QFT. In particular, the failure of Donaldson invariants on X to be fully independent of the metric on X when $b_2^+(X) = 1$ has a similar origin.

2.13 Localization

We continue with the 0-dimensional supersymmetric theory of the last section. We want to use the deformation invariance to compute Z.

The strategy we follow is roughly to deform $h(x) \to \lambda h(x)$ and take $\lambda \to \infty$. In this limit, the bosonic term $h(x)^2$ in the action becomes extremely large away from the zeroes of $h(x)$; thus one might expect that the contribution to Z becomes concentrated near the zeroes, and can be computed locally there.

Let us recall how the analogous phenomenon plays out for a purely bosonic integral in one dimension: consider a family of actions $S_\lambda(x) = \lambda S(x)$, with all critical points nondegenerate. Then the $\lambda \to \infty$ asymptotics are governed by the method of *steepest descent*. See [16] for a very clear account of this method.

Proposition 2.24 (Steepest descent in one dimension). As $\lambda \to \infty$ we have

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\lambda S(x)} \sim \sum_{x_c : S'(x_c) = 0} \sqrt{\frac{2\pi}{\lambda S''(x_c)}} \mathrm{e}^{-\lambda S(x_c)} \tag{2.104}$$

(Of course this expansion is dominated by the critical point(s) where $S(x_c)$ takes its minimum value: the others are exponentially suppressed, and could be dropped.) The proof goes by replacing the original integral by

$$\sum_{x_c : S'(x_c) = 0} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\lambda (S(x_c) + \frac{1}{2} S''(x_c)(x - x_c)^2)}, \tag{2.105}$$

a replacement which (maybe surprisingly) involves only an exponentially small error.

Making the same kind of replacement in our supersymmetric theory leads to

$$Z(\lambda) \sim \sum_{x_c : h(x_c) = 0} \int \mathrm{d}x \mathrm{d}\vec{\psi}\, \mathrm{e}^{-\frac{1}{2}\lambda h'(x_c)^2 (x - x_c)^2 - \lambda h'(x_c) \psi^1 \psi^2} \tag{2.106}$$

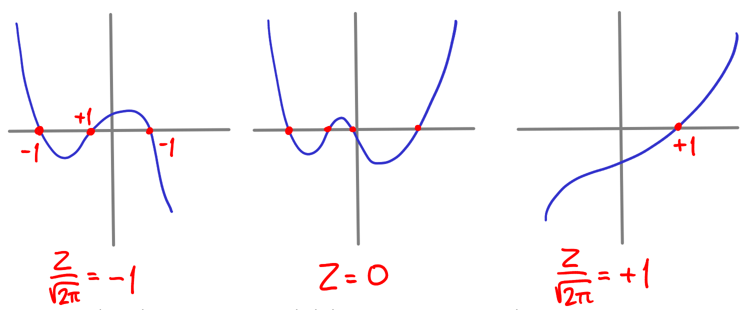
and now performing the Gaussian integrals over fermions and bosons gives

$$Z(\lambda) \sim \sqrt{2\pi} \sum_{x_c : h(x_c)=0} \frac{h'(x_c)}{|h'(x_c)|} = \sqrt{2\pi} \sum_{x_c : h(x_c)=0} \operatorname{sgn}(h'(x_c)) \tag{2.107}$$

Note the cancellation between the fermions which contribute in the numerator and the bosons in the denominator. Since we already know $Z(\lambda)$ is independent of λ, this proves:

Proposition 2.25. In our supersymmetric theory with action (2.78), the partition function can be evaluated exactly:

$$\frac{Z(\lambda)}{\sqrt{2\pi}} = \sum_{x_c : h(x_c)=0} \operatorname{sgn}(h'(x_c)) = \frac{1}{2}(\epsilon_+ - \epsilon_-). \tag{2.108}$$



This is such a toy example that we could have gotten this answer more directly:

Exercise 2.27. Prove Proposition 2.25 directly by integrating out the fermionic directions and then making the formal change of variables $y = h(x)$.

[study the cancellation in perturbation theory?]

In contrast, expectation values like $\langle f(x) \rangle$ in this theory cannot be evaluated exactly by localization: the reason is that functions $f(x)$ are not Q-invariant.

2.14 Localization in a zero-dimensional sigma model

Now we consider a less trivial and more geometric example of supersymmetric localization. Suppose (M, ω) is a compact symplectic manifold with $\dim M = 2n$, with a $U(1)$ action generated by a Hamiltonian vector field

$$Y = \omega^{-1}(\mathrm{d}H). \tag{2.109}$$

Suppose moreover that all fixed points of Y are isolated. Fix some $\alpha \in \mathbb{R}$. Our interest is in the integral

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H}. \tag{2.110}$$

Example 2.26. The fundamental example is the case $M = S^2$ with ω the standard volume form $\omega = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi$, and the $U(1)$ action rotating φ. This action is generated by the function

$$H = z = \cos\theta \tag{2.111}$$

and thus

$$\int_M \frac{\omega^n}{n!} \mathrm{e}^{\mathrm{i}\alpha H} = 2\pi \int_0^\pi \mathrm{e}^{\mathrm{i}\alpha\cos\theta} \sin\theta\, \mathrm{d}\theta \tag{2.112}$$

$$= -2\pi \int_1^{-1} \mathrm{e}^{\mathrm{i}\alpha z} \mathrm{d}z \tag{2.113}$$

$$= \frac{2\pi}{\mathrm{i}\alpha}(\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha}) \tag{2.114}$$

$$= 4\pi \frac{\sin\alpha}{\alpha}. \tag{2.115}$$

This answer exhibits another localization phenomenon: it is a sum of contributions

$$(\pm)\frac{2\pi}{\mathrm{i}\alpha} \mathrm{e}^{\mathrm{i}\alpha H(x_c)} \tag{2.116}$$

from the two fixed points x_c of the $U(1)$ action.

We would like to explain this localization as an instance of the supersymmetric localization we have been discussing. For this, we need a generalization of what we have done so far: we take our field space $\mathcal{C}$ to be a *supermanifold*

$$\mathcal{C} = \Pi T M \tag{2.117}$$

i.e. the total space of the tangent bundle to M, with the parity of the fibers reversed. Everything we have done for super vector spaces has a supermanifold version, obtained by appropriate patching. I will be vague about this, again referring to the references [11, 12, 13, 14] for a more detailed treatment.

$\mathcal{C}$ has local charts induced from the charts on M. In each such chart, we identify $\mathcal{C}$ with a patch of $\mathbb{R}^{2n|2n}$, with base coordinates x^i (even) and fiber coordinates ψ^i (odd). The latter generate an exterior algebra. Thinking of ψ^i as $\mathrm{d}x^i$, this suggests that globally we should have

$$C^\infty(\mathcal{C}) = \Omega^*(M) \tag{2.118}$$

and this is indeed true. (More generally, for any $\mathcal{C} = \Pi E$ with $E \to M$ some vector bundle we would have $C^\infty(\mathcal{C}) = C^\infty(M, \wedge^* E^*)$.)

Now we take for our action the function

$$S = -\mathrm{i}\alpha(H + \omega) = -\mathrm{i}\alpha(H + \omega_{ij}\psi^i\psi^j). \tag{2.119}$$

The partition function is

$$Z = \int_{\mathcal{C}} \mathrm{d}\vec{x}\mathrm{d}\vec{\psi}\, \mathrm{e}^{-S}. \tag{2.120}$$

Here $\mathrm{d}\vec{x}\mathrm{d}\vec{\psi}$ denotes the canonical measure on $\mathcal{C}$; this measure exists roughly because the $\mathrm{d}x^i$ and $\mathrm{d}\psi^i$ transform oppositely under change of coordinates; more precisely, we have an extension

$$0 \to \Pi\pi^* TM \to T\mathcal{C} \to TM \to 0 \tag{2.121}$$

from which it follows that $\operatorname{Ber} T\mathcal{C} = \operatorname{Ber} TM \otimes \operatorname{Ber} \Pi TM$ which is canonically trivial. (The existence of this measure is the parity-changed analogue of the fact that T^*M has a canonical volume form.) Thus, provided that M itself is oriented, we get a canonical measure of integration on $\mathcal{C}$.

Since our supermanifold $\mathcal{C}$ comes to us as an odd vector bundle, the rule for integration is a straightforward generalization of what we have done before: namely we first integrate over the odd directions fiber by fiber, thus reducing to an integral over the base,

$$Z = (\mathrm{i}\alpha)^n \int_M \frac{\omega^n}{n!} \mathrm{e}^{\mathrm{i}\alpha H}. \tag{2.122}$$

Exercise 2.28. Verify (2.122), by computing in a local coordinate patch. (It might be easiest to use local Darboux coordinates for ω.)

Now we want to compute Z by localization. Begin by noting that the action S is invariant under the odd vector field

$$Q = \mathrm{d} + \iota_Y = \psi^i \partial_{x^i} + Y^j \partial_{\psi^j} \tag{2.123}$$

which has (Cartan's formula)

$$\frac{1}{2}[Q, Q] = \mathcal{L}_Y = \psi^i \partial_{x^i} Y^j \partial_{\psi^j} + Y^i \partial_{x^i}. \tag{2.124}$$

As before, we want to get a localization to some small subset of $\mathcal{C}$ by making a perturbation $S \to S + Q\Psi$. For this we fix a $\mathrm{U}(1)$-invariant metric g on M and then take the odd function

$$\Psi = g(\psi, Y) = g_{ij}\psi^i Y^j = \psi^i Y_i \tag{2.125}$$

where in the last line we defined $Y_i = g_{ij} Y^j$. Then we have

$$Q\Psi = g(Y, Y) + \mathrm{d}(gY) = Y^i Y_i - \psi^i \psi^k \partial_{x^k} Y_i \tag{2.126}$$

and

$$Q^2 \Psi = 0 \tag{2.127}$$

(using the fact that g is $\mathrm{U}(1)$-invariant).

Exercise 2.29. Verify (2.127) by direct computation using the coordinate expressions of Q and Ψ.

Now we make the deformation $S \to S + \lambda Q\Psi$. Because of (2.127), the deformed action is still Q-invariant, for all values of λ. Then, by Proposition 2.21, Z is independent of the deformation parameter λ. Taking λ very large, we can as usual reduce to a neighborhood of the zero locus of $g(Y,Y)$, i.e. to the fixed locus of Y. Then the steepest-descent method gives

$$Z \sim \sum_{x_c \in M : Y(x_c)=0} \mathrm{e}^{\mathrm{i}\alpha H(x_c)} \times (2\pi)^n \frac{(\mathrm{d}(gY)(x_c))^n/n!}{\sqrt{\det(g(Y,Y))''(x_c)}} \tag{2.128}$$

Here both the numerator $(\mathrm{d}(gY)(x_c))^n/n!$ and the denominator $\sqrt{\det(g(Y,Y))''(x_c)}$ are valued in $\wedge^{\mathrm{top}} T^*_{x_c} M$. [it would be nicer to be careful about twists by orientation bundle here; we always use the standard orientation to avoid having to worry about it] What remains is to compute their ratio.

One convenient way to do this is to consider a local model: $M = \mathbb{R}^2$ with its standard metric and symplectic form,

$$g = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2, \quad \omega = r\,\mathrm{d}r \wedge \mathrm{d}\theta, \tag{2.129}$$

with $U(1)$ acting in the charge-k representation, generated by

$$Y = k\partial_\theta, \qquad H = \frac{1}{2} k r^2. \tag{2.130}$$

Then we compute

$$\mathrm{d}(gY) = 2kr\,\mathrm{d}r \wedge \mathrm{d}\theta, \quad \sqrt{\det g(Y,Y)''(0)} = 2k^2 r\,\mathrm{d}r \wedge \mathrm{d}\theta \tag{2.131}$$

and thus the ratio comes to

$$\frac{(\mathrm{d}(gY)(x_c))}{\sqrt{\det(g(Y,Y))''(x_c)}} = \frac{1}{k}. \tag{2.132}$$

For an isolated fixed point we can decompose $T_{x_c}M$ as a $\mathrm{U}(1)$ representation into a direct sum of n 2-dimensional pieces, with $\mathrm{U}(1)$ weights $k_1, \dots, k_n$, and put each piece in standard form as above. Thus (2.128) becomes

$$Z = (2\pi)^n \sum_{x_c \in M : Y(x_c)=0} \frac{\mathrm{e}^{\mathrm{i}\alpha H(x_c)}}{\prod_{i=1}^n k_i(x_c)}. \tag{2.133}$$

Comparing this with (2.122) proves:

Theorem 2.27 (Duistermaat-Heckman localization formula [17]).

$$\int_M \frac{\omega^n}{n!} \mathrm{e}^{\mathrm{i}\alpha H} = \left(\frac{2\pi}{\mathrm{i}\alpha}\right)^n \sum_{x_c} \frac{\mathrm{e}^{\mathrm{i}\alpha H(x_c)}}{\prod_{i=1}^n k_i(x_c)} \tag{2.134}$$

where the $k_i(x_c) \in \mathbb{Z}$ are the weights of the $U(1)$ action on the normal bundle to x_c.

In the case of S^2 which we considered above, the weights at the two fixed points were $k = +1$ and $k = -1$, the sign determined by whether the local orientation induced by the $U(1)$ action agreed or disagreed with the orientation induced by ω.

A fancy way of interpreting the factor $\prod_{i=1}^n k_i(x_c)$ is:

Exercise 2.30. Suppose V is a vector space with $\mathrm{SO}(2n)$-structure. Then define a canonical 1-dimensional vector space $\mathrm{Pf}\, V$ (the "Pfaffian line") with the property $(\mathrm{Pf}\, V)^2 = \det V$. If $\mathrm{U}(1)$ acts on V preserving the $\mathrm{SO}(2n)$-structure, show that the induced action on $\mathrm{Pf}\, V$ is multiplication by $\prod_{i=1}^n k_i$ where k_i are the weights of the $\mathrm{U}(1)$ action.

The next simplest example of Theorem 2.27 beyond $M = S^2$ is:

Exercise 2.31. Work out the concrete statement of Theorem 2.27 in the case where $M = \mathbb{CP}^2$, with ω the Fubini-Study Kähler form, and the $U(1)$ action given in homogeneous coordinates by $(z_1, z_2, z_3) \to (\mathrm{e}^{\mathrm{i}b_1\theta} z_1, \mathrm{e}^{\mathrm{i}b_2\theta} z_2, \mathrm{e}^{\mathrm{i}b_3\theta} z_3)$ for $b_1, b_2, b_3 \in \mathbb{Z}$, with all b_i distinct.

Incidentally, there is another way of thinking about the localization formula Theorem 2.27. The contributions from the fixed points match what one would get by making a quadratic approximation to the *original* integrand $\omega^n \mathrm{e}^{\mathrm{i}\alpha H}$ around the critical points x_c of H. Such an approximation generally gives the leading $\alpha \to \infty$ asymptotics of the integral (this is called the *stationary phase* method, similar to the steepest descent method which we have been using). So, Theorem 2.27 can be rephrased as the statement that, for this particular integral, *the stationary phase approximation is exact*.

Since there are a few tricky points here, let us explain in a bit more detail how one computes the local contributions in the stationary phase method. (See also the very nice reference [18].) A key difference between the stationary phase method and the steepest-descent expansion is that in the stationary phase method we have the i in the exponent, hence no exponential suppression. One deals with this by rotating the contour of integration over each real variable by an angle $\pm\pi/4$ into complex space, so that the integrand becomes exponentially suppressed again. This produces a factor

$$\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}(n_+ - n_-)} \tag{2.135}$$

where (n_+, n_-) is the signature of the bilinear form $H''(x_c)$. Then, making the quadratic approximation and performing the Gaussian integral leads to the local contribution

$$\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}(n_+ - n_-)} \left(\frac{2\pi}{\alpha}\right)^n \frac{\omega^n / n!}{\sqrt{|\det H''(x_c)|}} \mathrm{e}^{\mathrm{i}\alpha H(x_c)}. \tag{2.136}$$

Looking again at the local models one sees that this is

$$\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}(n_+ - n_-)} \left(\frac{2\pi}{\alpha}\right)^n \frac{\mathrm{e}^{\mathrm{i}\alpha H(x_c)}}{\prod_{i=1}^n |k_i(x_c)|}. \tag{2.137}$$

and since $\frac{1}{2} n_-$ is the number of $k_i(x_c)$ which are negative, this matches the contribution we found in Theorem 2.27.

I do not really know the significance of the fact that the stationary phase approximation to $\int \omega^n \mathrm{e}^{\mathrm{i}\alpha H}$ is exact. As we have seen above, this is *not* the same method one uses in the supersymmetric localization proof of Theorem 2.27, despite the obvious resemblance.

2.15 Some generalizations

The proof of Theorem 2.27 which we have given naturally generalizes: we could have replaced the action S by any Q-invariant $S \in C^\infty(\mathcal{C})$. Under the identification $C^\infty(\mathcal{C}) \simeq$

$\Omega^*(M)$, such an S would be called an *equivariantly closed form* in the Cartan model for the $U(1)$-equivariant cohomology of M, where Q becomes the *equivariant differential* $\mathrm{d} + \mathcal{L}_Y$. Then Theorem 2.27 generalizes to

Theorem 2.28 (Atiyah-Bott-Berline-Vergne localization for isolated fixed points [18, 19]). Suppose M is a compact manifold with a $\mathrm{U}(1)$-action, with isolated fixed points. Also suppose $\beta \in \Omega^*(M)$ is an equivariantly closed form. Then

$$\int_M \mathrm{e}^\beta = (-2\pi \mathrm{i})^n \sum_{x_c} \frac{\mathrm{e}^{\beta^{\mathrm{bot}}(x_c)}}{\prod_{i=1}^n k_i(x_c)} \tag{2.138}$$

where β^{bot} means the bottom (0-form) component of β.

Exercise 2.32. Generalize the proof of Theorem 2.27 to a proof of Theorem 2.28.

Finally we can consider a further generalization: let us imagine trying to repeat the proof in the case where the fixed points are not isolated. In this situation, the leading term in the asymptotics produced by the steepest-descent method is an integral over the fixed locus $F \subset M$. Thus the supersymmetric localization must reduce $\int_M \mathrm{e}^\beta$ to an integral over F. The integrand turns out to be determined by the local structure around F, as follows.

The codimension of F is always even, say $2n$, since it has a nontrivial $\mathrm{U}(1)$-action. Then the normal bundle NF has an $\mathrm{SO}(2n)$-structure and connection induced by our choice of metric g on M, and a compatible $\mathrm{U}(1)$-action induced by the $\mathrm{U}(1)$-action on M. [worry about orientations]

Definition 2.29 (Equivariant Euler form). Suppose X is a manifold, with a $\mathrm{U}(1)$-equivariant $\mathrm{SO}(2n)$-bundle E, carrying a $\mathrm{U}(1)$-equivariant connection. The *equivariant Euler form* of E is [check signs]

$$\mathrm{Euler}(E) = \mathrm{Pf}\left(\frac{1}{2\pi}(Y + F)\right) \in \Omega^*(X) \tag{2.139}$$

where $Y \in \Omega^0(\mathfrak{so}(E))$ is the generator of the $\mathrm{U}(1)$ action and $F \in \Omega^2(\mathfrak{so}(E))$ is the curvature of the equivariant connection on E.

So $\mathrm{Euler}(E)$ is a form concentrated in even degrees. For example, in the case $n = 1$, using the standard trivialization of $\mathfrak{so}(2)$, we have simply

$$\mathrm{Euler}(E) = \frac{1}{2\pi}(\mathrm{i}k + F), \tag{2.140}$$

the sum of a 0-form and a 2-form. More generally, the bottom component of $\mathrm{Euler}(E)$ is $\prod_{i=1}^n \frac{\mathrm{i}k_i}{2\pi}$ where the k_i are the weights of the $\mathrm{U}(1)$-action. In particular, if no $k_i = 0$ then $\mathrm{Euler}(E)$ has nowhere-vanishing bottom component. This means $\mathrm{Euler}(E)$ is invertible in $\Omega^*(X)$, i.e. there is a form

$$\frac{1}{\mathrm{Euler}(E)} \in \Omega^*(X). \tag{2.141}$$

Exercise 2.33. Write the form $\frac{1}{\mathrm{Euler}(E)}$ explicitly in the case $n = 1$.

We use this in the statement of the next theorem:

Theorem 2.30 (Atiyah-Bott-Berline-Vergne localization in general [18, 19]). Suppose M is a compact manifold with a $\mathrm{U}(1)$-action, with fixed locus F. Also suppose $\beta \in \Omega^*(M)$ is an equivariantly closed form. Choose a $\mathrm{U}(1)$-invariant metric g on M. Then

$$\int_M \mathrm{e}^\beta = \int_F \frac{\mathrm{e}^\beta}{\mathrm{Euler}(NF)}. \tag{2.142}$$

Exercise 2.34. Show that Theorem 2.30 reduces to Theorem 2.28 when the fixed points of the $\mathrm{U}(1)$-action are isolated.

Exercise 2.35. Use Theorem 2.30 to generalize Exercise 2.31 to the case where the b_i need not be distinct.

Exercise 2.36. Generalize the proof of Theorem 2.28 to a proof of Theorem 2.30. [hard? at least needs more delicacy with the steepest descent expansion]

3 QFT in 1 dimension

Now we move to 1-dimensional quantum field theory. This involves a choice of Riemannian 1-manifold (X, η). We will take X to be compact: either

$$X = [0, T] \quad \text{or} \quad X = S^1(T) \tag{3.1}$$

(where by $S^1(T)$ we mean the circle with circumference T). We parameterize X by t which we sometimes think of as "time."

Our configuration space $\mathcal{C}_X$, over which we want to integrate, will now be some kind of space of "generalized functions on X."

3.1 The 1-dimensional sigma model

Fix the data:

- A Riemannian manifold (Y, g),
- A function $V : Y \to R$ ("potential").

We are going to define a 1-dimensional quantum field theory which will be equivalent to the usual nonrelativistic *quantum mechanics* describing a single particle propagating on Y.

In this quantum field theory, when $X = S^1$, the field space is the space of continuous maps

$$\mathcal{C}_{S^1} = \{\phi : S^1 \to Y\}. \tag{3.2}$$

When X is the interval, we will include a bit more data: use the notation $X = [0,T]_{y_0}^{y_1}$ to mean the interval decorated by boundary conditions at the two ends, and define

$$\mathcal{C}_{[0,T]_{y_0}^{y_1}} = \{\phi : [0,T] \to Y \,|\, \phi(0) = y_0, \phi(T) = y_1\}. \tag{3.3}$$

The action is, for differentiable ϕ,

$$S(\phi) = \int_X \mathrm{dvol}_X \left(\frac{1}{2}g(\dot\phi,\dot\phi) + V(\phi) - \frac{1}{6}R(\phi)\right) \tag{3.4}$$

$$= \int_X \mathrm{d}t\sqrt{\eta_{tt}}\left(\frac{1}{2}g_{ij}(\phi)\dot\phi^i\dot\phi^j\eta^{tt} + V(\phi) - \frac{1}{6}R(\phi)\right) \tag{3.5}$$

where $R : Y \to \mathbb{R}$ is the scalar curvature.

If we choose the parameter t on X to be the arc-length, then the action reduces to

$$S(\phi) = \int_X \mathrm{d}t\,\left(\frac{1}{2}g_{ij}(\phi)\dot\phi^i\dot\phi^j + V(\phi) - \frac{1}{6}R(\phi)\right). \tag{3.6}$$

We can think of various $\phi \in \mathcal{C}_X$ as various possible trajectories that the particle could take on Y.

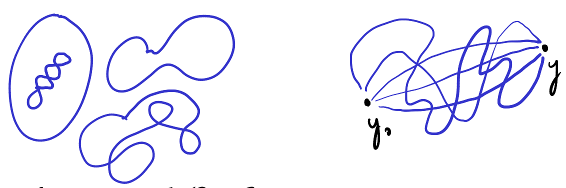
Formally we would like to consider the partition function on $X = S^1$,

$$Z_{S^1(T)} \text{ ''='' } \int_{\mathcal{C}_{S^1(T)}} \mathrm{d}\phi\,\mathrm{e}^{-S}. \tag{3.7}$$

or the analogue on the interval,

$$Z_{[0,T]_{y_0}^{y_1}} \text{ ''='' } \int_{\mathcal{C}_{[0,T]_{y_0}^{y_1}}} \mathrm{d}\phi\,\mathrm{e}^{-S} \tag{3.8}$$

So (3.7) is a sum over all closed trajectories in Y, and (3.8) is a sum over all possible paths a particle can take from y_0 to y_1.



More generally, given any functional $\mathcal{O} : \mathcal{C}_X \to \mathbb{R}$ we can contemplate the (unnormalized) expectation value

$$\langle\mathcal{O}\rangle \text{ ''='' } \int_{\mathcal{C}_X} \mathrm{d}\phi\,\mathcal{O}(\phi)\,\mathrm{e}^{-S}. \tag{3.9}$$

How are we to make sense of (3.9)? As it stands, it has (at least) two difficulties. First, there is no reasonable notion of translation-invariant Lebesgue measure on an infinite-dimensional space,[12] so it is not clear what we could mean by $\mathrm{d}\phi$. Second, the action S is not defined for arbitrary continuous paths, only for differentiable ones. For non-differentiable paths it looks like we should have $S(\phi) = \infty$ in some sense. As we will discuss momentarily, these difficulties actually "cancel" one another in some sense.

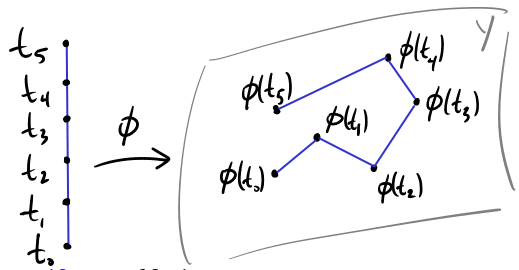
[12]The basic problem is that in an infinite-dimensional Banach space a ball of radius 1 contains infinitely many disjoint balls of radius ϵ.

3.2 Discretization

Let us start with the case $X = [0,T]_{y_0}^{y_1}$. One natural approach to defining (3.8) is to try *discretization* i.e. replacing X by a lattice of $N+1$ points, with spacing $\Delta t = \frac{T}{N}$. Then we define a discretized field space

$$\mathcal{C}_{X;N}, \tag{3.10}$$

the space of *geodesic polygons* i.e. piecewise-smooth paths, with each segment the unique minimizing geodesic between its endpoints.



$\mathcal{C}_{X;N}$ is an open dense subset[13] in Y^{N+1}. Restricting S to these paths gives $S : \mathcal{C}_{X;N} \to \mathbb{R}$. $\mathcal{C}_{X;N}$ also has a natural measure induced from the volume measure on Y, up to a tricky constant:

$$\mathrm{d}\mu_N = \frac{1}{(2\pi\Delta t)^{\frac{\dim Y}{2}\times N}} \prod_{n=1}^{N-1} \mathrm{dvol}_Y(\phi(t_n)). \tag{3.11}$$

Thus we can define the discretized partition function

$$Z_N = \int_{\mathcal{C}_{X;N}} \mathrm{d}\mu_N \, \mathrm{e}^{-S} \tag{3.12}$$

and then try to make sense of the *limit* as $N \to \infty$. The limit does indeed exist (in "good" situations, e.g. Y compact or $Y = \mathbb{R}^n$), which I think has been known for a long time when $Y = \mathbb{R}^n$, but is apparently much more recent for more general Y, e.g. see [20, 21] for the case of Y compact. To describe what the limit is, we need a digression on the heat kernel.

3.3 Heat kernel

We continue with a Riemannian manifold Y and a function $V : Y \to \mathbb{R}$. Then we have an operator acting on $C^\infty(Y)$,

$$H = -\frac{1}{2}\Delta + V. \tag{3.13}$$

The *heat equation* defined by these data is an equation for functions of two variables, $f : \mathbb{R} \times Y \to \mathbb{R}$. We view such an f as a family of functions $f_t : Y \to \mathbb{R}$; then the heat equation is

$$\partial_t f_t(x) + H f_t(x) = 0. \tag{3.14}$$

[13] The restriction to an open dense subset is a technicality, brought on by the fact that some pairs of points are connected by more than one length-minimizing geodesic.

Proposition 3.1 (Heat kernel). Suppose Y is compact or $Y = \mathbb{R}$.[14] For any $t \in \mathbb{R}_+$, the *heat kernel* (deformed by V) is a smooth function k_t on $Y \times Y$, obeying the heat equation in the first variable,[15]

$$\partial_t k_t(x,y) + H_x k_t(x,y) = 0 \tag{3.15}$$

and such that as $t \to 0^+$ this solution "concentrates" at the point y,

$$\lim_{t \to 0^+} k_t(x,y) = \delta(x,y). \tag{3.16}$$

These properties characterize k_t.

Exercise 3.1. Show that on $Y = \mathbb{R}^n$, when $V = 0$, the heat kernel is[16]

$$k_t(x,y) = \left(\frac{1}{2\pi t}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2t}\|x-y\|^2\right). \tag{3.17}$$

The heat kernel is really a kernel in the following sense. Given a smooth function $f : Y \to \mathbb{R}$, one can extend it to a family of smooth functions $f_t : Y \to \mathbb{R}$ for $t \in \mathbb{R}_{\geq 0}$ by solving the heat equation (3.14). In particular, this defines a map $U_t : C^\infty(Y) \to C^\infty(Y)$, taking $f \mapsto f_t$, which we might call "evolving the heat equation for time t."

A convenient notation is to write

$$U_t = \mathrm{e}^{-tH}. \tag{3.18}$$

The justification for this notation is that U_t obeys the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t = -HU_t \tag{3.19}$$

which is really just a rephrasing of the heat equation (3.14), and $U_0 = 1$.

Exercise 3.2. Prove (3.19).

Proposition 3.2. $k_t(x,y)$ is the integral kernel for U_t, in the sense that

$$(U_t f)(x) = \int_M \mathrm{dvol}_y \, k_t(x,y) f(y). \tag{3.20}$$

U_t is an extremely nice operator, with the "smoothing" property: for any $t > 0$ it maps distributions to C^∞ functions. In particular it gives a linear operator on $L^2(Y)$. This is *not* a unitary operator.[17]

[14] I don't know the precise class of manifolds where the heat kernel is known to exist.

[15] Our convention is that Δ is the usual Laplacian, i.e. $\Delta = \sum \partial_i^2$ on $\mathbb{R}^n$. This has the inconvenient consequence that Δ is a *negative* definite operator on $L^2(Y)$. Thus we will often find ourselves considering $-\Delta$.

[16] The usual formulas for the heat kernel differ from this by the replacement $t \to 2t$, because they omit the factor $\frac{1}{2}$ in the definition of H. The same replacement is needed in comparing the formulas we write below on a Riemannian manifold to those in the literature, e.g. [22].

[17] One could have instead considered the analytic continuation $t \to it$ which would have been the time evolution for the Schrödinger equation; in that case it really would be unitary.

3.4 Path integral and heat kernel

Now we can state the main fact about the 1-dimensional sigma model: its path integral actually exists and the result is the heat kernel!

Proposition 3.3 (Lattice approximations to 1-dimensional sigma model converge to the heat kernel). If $X = [0,T]_{y_0}^{y_1}$, then the discretized path integrals converge, so we can define

$$Z_{[0,T]_{y_0}^{y_1}} = \lim_{N\to\infty} Z_N. \tag{3.21}$$

Moreover,

$$Z_{[0,T]_{y_0}^{y_1}} = k_T(y_0, y_1). \tag{3.22}$$

In the stochastic-process community this would be rephrased as the statement that, in the case $V = 0$, $\lim_{N\to\infty} \mathrm{d}\mu_N \mathrm{e}^{-S}$ is the *(conditional) Wiener measure*. Indeed the defining property of Wiener measure is its relation to the heat kernel. Note that although $\mathrm{d}\mu_N \mathrm{e}^{-S}$ has a well defined limit, the measure $\mathrm{d}\mu_N$ by itself does not.

If we take $X = S^1(T)$ then we can make a similar discretization of the path integral: the only difference is that we require y_0 and y_N to be equal and then we integrate over their common value. Thus for the partition function we get

$$Z_{S^1(T)} = \int_Y \mathrm{dvol}_Y \, k_t(y,y). \tag{3.23}$$

This integral has another interpretation: it is the trace of the integral operator U_T acting on $L^2(Y)$. Stated formally,

Proposition 3.4 (Lattice approximations to 1-dimensional sigma model on S^1 converge to the trace of the heat flow). If $X = S^1(T)$, then the discretized path integrals converge,

$$Z_{S^1(T)} = \lim_{N\to\infty} Z_N, \tag{3.24}$$

and then

$$Z_{S^1(T)} = \mathrm{Tr}_{L^2(Y)} \, \mathrm{e}^{-TH}. \tag{3.25}$$

3.5 A discretization computation

Here is a rough computation which gives some motivation for Proposition 3.3. (See [21] for an actual proof along these lines.) We consider the special case $V = 0$. First, the property $U_T = U_{\Delta t}^N$ gives a relation between the integral kernels,

$$k_T(y_N, y_0) = \int_{Y^{N-1}} \prod_{n=1}^{N-1} \mathrm{dvol}_{y_n} \prod_{n=0}^{N-1} k_{\Delta t}(y_{n+1}, y_n). \tag{3.26}$$

Next we use the short time asymptotics of the heat kernel. As $\Delta t \to 0$, there is a sort of complicated expansion described in [23]. Fortunately we need only the leading behavior of $k_t(x,y)$ as $\Delta t \to 0$ and $x \to y$, plus the first-order correction in Δt, plus the first-order correction in $d(x,y)$; all the higher-order terms will not contribute in (3.26) after we take the large N limit.

The leading behavior as $\Delta t \to 0$ and $x \to y$ is not hard to guess: it is the same as on $\mathbb{R}^n$, namely

$$k_{\Delta t}(x,y) \sim \left(\frac{1}{2\pi \Delta t}\right)^{\frac{\dim Y}{2}} \exp\left(-\frac{1}{2\Delta t} d(x,y)^2\right). \tag{3.27}$$

The first-order correction in $d(x,y)$ can be extracted from [23]: it is of the form

$$k_{\Delta t}(x,y) \sim \left(\frac{1}{2\pi \Delta t}\right)^{\frac{\dim Y}{2}} \exp\left(-\frac{1}{2\Delta t} d(x,y)^2\right) \left(1 + \frac{1}{12} \mathrm{Ric}_x(x-y, x-y) + \cdots\right). \tag{3.28}$$

To get the first-order correction in Δt we can restrict to the diagonal, and then look in [22] which says that as $\Delta t \to 0$ we have

$$k_{\Delta t}(x,x) \sim \left(\frac{1}{2\pi \Delta t}\right)^{\frac{\dim Y}{2}} \exp\left(-\frac{1}{2\Delta t} d(x,x)^2\right) \left(1 + \frac{1}{12} R(x)\Delta t + \cdots\right) \tag{3.29}$$

Substituting these in (3.26) we get

$$\begin{aligned} k_T(y_N, y_0) \sim \int_{Y^{N-1}} \prod_{n=1}^{N-1} \mathrm{dvol}_{y_n} \prod_{n=0}^{N-1} \left(\frac{1}{2\pi\Delta t}\right)^{\frac{\dim Y}{2}} \times \\ \exp\left(-\frac{1}{2\Delta t} d(y_{n+1}, y_n)^2\right) \left(1 + \frac{1}{12} R(y_n)\Delta t + \frac{1}{12} \mathrm{Ric}_{y_n}(y_{n+1} - y_n, y_{n+1} - y_n) + \cdots\right) \end{aligned} \tag{3.30}$$

The last term is a bit tricky to deal with, but under the integral it can be replaced by $\frac{1}{12} R(y_n)\Delta t$. [explain how] Thus altogether we get

$$k_T(y_N, y_0) \sim \int_{Y^{N-1}} \mathrm{d}\mu_N \exp\left[\sum_{n=0}^{N-1} \Delta t \left(-\frac{1}{2}\left(\frac{d(y_{n+1}, y_n)}{\Delta t}\right)^2 + \frac{1}{6} R(y_n)\right) + \cdots\right]. \tag{3.31}$$

Finally, this last expression agrees with the restriction of S to polygonal paths, up to terms which vanish in the limit $\Delta t \to 0$. This completes our formal "proof" of Proposition 3.3 when $V = 0$.

When $V \neq 0$ we can make a similar formal argument to motivate Proposition 3.3. One technical point is that we need to be able to separate the pieces involving Δ from the pieces involving V. This is done using the following, applied to $A = -\frac{1}{2} T\Delta$, $B = TV$:

Proposition 3.5 (Trotter product formula). For operators A, B obeying appropriate functional-analytic hypotheses,

$$\mathrm{e}^{A+B} = \lim_{N\to\infty} \left(\mathrm{e}^{\frac{A}{N}} \mathrm{e}^{\frac{B}{N}}\right)^N. \tag{3.32}$$

Exercise 3.3. Use Proposition 3.5 to extend the formal argument we gave for Proposition 3.3 to the case $V \neq 0$.

3.6 Local observables

There is a similar interpretation for the path integral with a "local observable" inserted. By this we mean an observable $\mathcal{O} : \mathcal{C}_X \to \mathbb{R}$ whose value only depends on some finite-order jet of $\phi \in \mathcal{C}_X$ at some fixed t: the "locality" is along X, in the t coordinate. The simplest example would be to take some function $F : Y \to \mathbb{R}$ and then define

$$\mathcal{O}_F(t) : \mathcal{C}_X \to \mathbb{R} \tag{3.33}$$

by

$$(\mathcal{O}_F(t))(\phi) = F(\phi(t)). \tag{3.34}$$

Then we can define correlation functions

$$\langle \mathcal{O}_F(t) \rangle_X \text{ "=" } \int_{\mathcal{C}_X} \mathrm{d}\phi \, F(\phi(t)) \, \mathrm{e}^{-S} \tag{3.35}$$

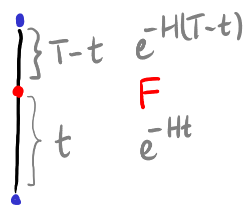
by the same kind of discretization we discussed above (except that we should take subintervals of irregular lengths, so that t can be on the boundary between two of them). The result is [ref?]

Proposition 3.6. $\langle \mathcal{O}_F(t) \rangle_{[0,T]_{y_0}^{y_1}}$ is the integral kernel representing the operator

$$\mathrm{e}^{-H(T-t)} \hat{F} \mathrm{e}^{-Ht} \tag{3.36}$$

on $L^2(Y)$, where $\hat{F}$ means the operator of multiplication by F.

One pictures this as an instruction: "propagate for time t, then do F, then propagate for another time $T - t$."



Exercise 3.4. Bootstrap Proposition 3.6 into a similar formula for $\langle \mathcal{O}_{F_1}(t_1) \mathcal{O}_{F_2}(t_2) \cdots \mathcal{O}_{F_k}(t_k) \rangle$.

Exercise 3.5. Give a heuristic proof of Proposition 3.6 along the lines of what we did above for Proposition 3.3.

Proposition 3.6 says that the path integral converts the observable $\mathcal{O}_F$ associated to the function $F : Y \to \mathbb{R}$ into an operator $\hat{F}$ on $L^2(Y)$, albeit a rather obvious one. This process is sometimes called *path-integral quantization* of the function F.

We could also consider observables depending on, say, the 1-jet of ϕ instead of the 0-jet. This would amount to considering a function $F : TY \to \mathbb{R}$, defining

$$(\mathcal{O}_F(t))(\phi) = F(\phi'(t)), \tag{3.37}$$

and constructing $\langle \mathcal{O}_F(t) \rangle$ again by discretization. Then it is an interesting question to identify the corresponding operator on $L^2(Y)$.

At least when $Y = \mathbb{R}$ we can answer this question: [though strictly speaking $Y = \mathbb{R}$ was not allowed in our previous discussion]

Proposition 3.7. When $Y = \mathbb{R}$, use the standard metric to identify $TY = T\mathbb{R}$ with $T^*Y = T^*\mathbb{R}$ coordinatized by (x, p). Then the function $F = p$ induces (in the above sense) the operator

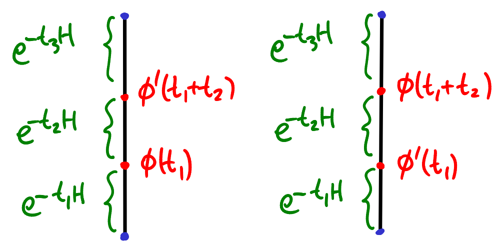
$$\hat{p} = -\partial_x \tag{3.38}$$

on $L^2(\mathbb{R})$.

Exercise 3.6. Give a heuristic proof of Proposition 3.7.

3.7 Noncommutativity and discretization

Thus the path-integral quantization converts the functions p, q on $T^*\mathbb{R}$ to operators $\hat{p}$, $\hat{q}$ on $L^2(\mathbb{R})$. Of course $\hat{p}$ and $\hat{q}$ do not commute with one another, which might at first seem puzzling: where does the noncommutativity come from? Consider the decorated intervals below:



Naively we would have (where $\leftrightarrow$ means "is the heat kernel for," and $T = t_1 + t_2 + t_3$)

$$\mathrm{e}^{-t_3 H}\hat{p}\mathrm{e}^{-t_2 H}\hat{x}\mathrm{e}^{-t_1 H} \leftrightarrow \int_{\mathcal{C}_{[0,T]}{}_{y_0}^{y_1}} \mathrm{d}\phi\, \phi'(t_1+t_2)\phi(t_1)\mathrm{e}^{-S(\phi)}, \tag{3.39}$$

$$\mathrm{e}^{-t_3 H}\hat{x}\mathrm{e}^{-t_2 H}\hat{p}\mathrm{e}^{-t_1 H} \leftrightarrow \int_{\mathcal{C}_{[0,T]}{}_{y_0}^{y_1}} \mathrm{d}\phi\, \phi(t_1+t_2)\phi'(t_1)\mathrm{e}^{-S(\phi)}. \tag{3.40}$$

In the limit $t_2 \to 0$ these two path integrals both formally limit to the same object,

$$\int_{\mathcal{C}_{[0,T]}{}_{y_0}^{y_1}} \mathrm{d}\phi\, \phi(t_1)\phi'(t_1)\mathrm{e}^{-S(\phi)} \tag{3.41}$$

but we want to say that actually the two limits are not quite the same: one is $\mathrm{e}^{-t_3 H}\hat{p}\hat{x}\mathrm{e}^{-t_1 H}$ and the other is $\mathrm{e}^{-t_3 H}\hat{x}\hat{p}\mathrm{e}^{-t_1 H}$. How can this be?

To understand this issue let us think a bit more carefully about how we defined the path integral by discretization. When we discretize the interval $[0, T]$ we will put one of the discretization points at t_1, in order to have a clean definition of the operator $\phi(t_1)$. But what do we do about $\phi'(t_1)$? Let us set

$$y_1 = \phi(t_1 - \Delta t), \qquad y_2 = \phi(t_1), \qquad y_3 = \phi(t_1 + \Delta t). \tag{3.42}$$

The two limits we are considering would lead to the two definitions

$$\phi'(t_1) = (x_2 - x_1)/\Delta t, \qquad \phi'(t_1) = (x_3 - x_2)/\Delta t. \tag{3.43}$$

In the discretized path integrals with these two insertions the x_2-dependent part is respectively

$$\frac{1}{\Delta t}\int \mathrm{d}x_2\,(x_2-x_1)x_2\,\mathrm{e}^{-F(x)}, \qquad \frac{1}{\Delta t}\int \mathrm{d}x_2\,(x_3-x_2)x_2\,\mathrm{e}^{-F(x)} \tag{3.44}$$

where

$$F(x) = [(x_1-x_2)^2 + (x_3-x_2)^2]/(2\Delta t). \tag{3.45}$$

These two integrals are not quite the same: indeed, since $\int \mathrm{d}x_2\,\partial_{x_2}(x_2\mathrm{e}^{-F(x)}) = 0$ we get for their difference

$$\frac{1}{\Delta t}\int \mathrm{d}x_2\,x_2((x_2-x_1)-(x_3-x_2))\,\mathrm{e}^{-F(x)} = -\int \mathrm{d}x_2\,\mathrm{e}^{-F(x)}. \tag{3.46}$$

Note this difference survives in the limit $\Delta t \to 0$! Now, (3.46) says the difference $\hat{x}\hat{p} - \hat{p}\hat{x}$ behaves the same in the path integral as the operator -1. Thus it is what we expected from the commutation relation

$$[\hat{p},\hat{x}] = -1. \tag{3.47}$$

3.8 Symmetries

Recall from the 0-dimensional case that symmetries of S — i.e. vector fields on $\mathcal{C}$ which annihilate S — lead to constraints on correlation functions.

In the 1-dimensional theory we are considering here, with action (3.4), we have two different sources of symmetries:

- isometries of X,
- isometries of Y.

For $X = S^1(T)$, the group of isometries of X is $\mathrm{Isom}(X) \simeq U(1)$, acting by shifts $t \to t + c$. The immediate consequence of this $U(1)$ symmetry is that for any c we have

$$\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\rangle_{S^1(T)} = \langle \mathcal{O}_1(t_1+c)\mathcal{O}_2(t_2+c)\cdots\mathcal{O}_n(t_n+c)\rangle_{S^1(T)}. \tag{3.48}$$

Similarly for any $g \in G = \mathrm{Isom}(Y)$ we have

$$\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\rangle_{S^1(T)} = \langle \mathcal{O}_1^g(t_1)\mathcal{O}_2^g(t_2)\cdots\mathcal{O}_n^g(t_n)\rangle_{S^1(T)}. \tag{3.49}$$

The facts (3.48), (3.49) can also be understood from the Hamiltonian point of view, i.e. using the fact that (if $T > t_n > t_{n-1} > \cdots > t_1 > 0$)

$$\langle \mathcal{O}_n(t_n)\mathcal{O}_{n-1}(t_{n-1})\cdots\mathcal{O}_1(t_1)\rangle_{S^1(T)} = \mathrm{Tr}_{\mathcal{H}}\,\mathrm{e}^{-(T-t_n)H}\hat{\mathcal{O}}_n\mathrm{e}^{-(t_n-t_{n-1})H}\hat{\mathcal{O}}_{n-1}\cdots\hat{\mathcal{O}}_1\mathrm{e}^{-t_1 H}. \tag{3.50}$$

Exercise 3.7. Prove (3.48) and (3.49) using (3.50).

In the proof of (3.49), we use the fact that G acts on $\mathcal{H} = L^2(Y)$, and $\hat{\mathcal{O}}^g = g\hat{\mathcal{O}}g^{-1}$.
[the conserved charge]
[coupling to "background field": connection on the circle]

Finally let us introduce a piece of notation which will be useful later. Each of the symmetry groups we considered above is a connected Lie group, so we can describe its action on $\mathcal{C}$ by giving the action of the Lie algebra. For example, the infinitesimal shift $t \to t + \epsilon$ acts by $\phi(t) \to \phi(t) + \epsilon\dot{\phi}(t)$, which we represent by writing

$$\delta\phi = \epsilon\dot{\phi}, \tag{3.51}$$

or relative to any coordinate system on Y,

$$\delta\phi^I = \epsilon\dot{\phi}^I. \tag{3.52}$$

Similarly an element $X \in \mathfrak{g} = \mathrm{Lie}(\mathrm{Isom}(Y))$ acts by

$$\delta\phi = \epsilon X(\phi), \tag{3.53}$$

or in coordinates

$$\delta\phi^I = \epsilon X^I(\phi), \tag{3.54}$$

where $X^I\partial_I$ is the coordinate expression of the vector field by which X acts on Y.

3.9 Simple examples

Example 3.8 (Harmonic oscillator). A fundamental example is the case

$$Y = \mathbb{R}, \qquad V = \frac{1}{2}\omega^2 x^2, \qquad \omega \in \mathbb{R}. \tag{3.55}$$

Then we have

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2\omega^2 \tag{3.56}$$

acting on $\mathcal{H} = L^2(\mathbb{R})$. There is a nice explicit basis of eigenfunctions,

$$\psi_0 = \mathrm{e}^{-\omega x^2/2}, \tag{3.57}$$

$$\psi_1 = x\mathrm{e}^{-\omega x^2/2}, \tag{3.58}$$

$$\psi_2 = \left(x^2 - \frac{1}{2\omega}\right)\mathrm{e}^{-\omega x^2/2}, \tag{3.59}$$

$$\psi_n = H_n(x\sqrt{\omega})\mathrm{e}^{-\omega x^2/2} \tag{3.60}$$

where H_n is the n-th Hermite polynomial. They obey

$$H\psi_n = \left(\frac{1}{2} + n\right)\omega\psi_n, \tag{3.61}$$

so the eigenvalues of H are

$$\left(\frac{1}{2}+n\right)\omega \text{ for } n \geq 0. \tag{3.62}$$

Thus the partition function on $X = S^1(T)$ is

$$Z_{S^1(T)} = \operatorname{Tr} e^{-TH} = \sum_{n=0}^{\infty} \exp\left(-\omega T\left(n+\frac{1}{2}\right)\right) = \frac{1}{2\sinh(\omega T/2)}. \tag{3.63}$$

Example 3.9 (Sigma model into S^1). Another basic example is

$$Y = S^1_R, \qquad V = 0. \tag{3.64}$$

Here we have

$$H = -\frac{1}{2}\partial_x^2 \tag{3.65}$$

acting on $\mathcal{H} = L^2(S^1_R)$. A simple basis of eigenfunctions is

$$\psi_0(x) = 1, \qquad \psi_{2n-1}(x) = \sin\left(\frac{2\pi n x}{R}\right), \qquad \psi_{2n}(x) = \cos\left(\frac{2\pi n x}{R}\right) \tag{3.66}$$

and so the eigenvalues are

$$0, \frac{2\pi^2 n^2}{R^2} \text{ (with multiplicity 2) for } n > 0. \tag{3.67}$$

Thus the partition function on $X = S^1(T)$ is

$$Z_{S^1(T)} = \operatorname{Tr} e^{-TH} = 1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{2\pi^2 n^2 T}{R^2}\right) = \sum_{n=-\infty}^{\infty} \exp\left(-\frac{2\pi^2 n^2 T}{R^2}\right), \tag{3.68}$$

also known as the Jacobi theta function,

$$Z_{S^1(T)} = \vartheta\left(\tau = \frac{2\pi i T}{R^2}, z = 0\right). \tag{3.69}$$

Exercise 3.8. For $\alpha \in \mathbb{R}$, let $S_\alpha : \mathcal{H} \to \mathcal{H}$ be the operation of shifting by α, $\psi(x) \mapsto \psi(x+\alpha)$. Compute $\operatorname{Tr}_{\mathcal{H}} e^{-TH} S_\alpha$. (You should find a theta function with the argument z nonzero, generalizing (3.69).)

3.10 Infinite-dimensional determinants

In the examples of subsection 3.9 the discretized path integrals involve only Gaussian integrals, albeit Gaussian integrals over unbounded numbers of variables. Recall from (2.32) that for finite-dimensional Gaussian integrals we have

$$(2\pi c)^{-\frac{1}{2}\dim V} \int_V d\mu\, e^{-\frac{1}{2}M(x,x)} = \frac{d\mu}{\sqrt{\det(cM)}}. \tag{3.70}$$

The discretized versions of the path integrals in subsection 3.9 have the form of the LHS, if we take $c = \Delta t = T/N$. Now, we have seen that the $N \to \infty$ limit of the discretized path integral exists. Thus the limit of the RHS also exists. We would like to interpret it as some kind of infinite-dimensional determinant.

One obstacle is that in the infinite-dimensional case the measure $\mathrm{d}\mu$ does not exist. So, we need to get rid of that. For this note that in the finite-dimensional case, we can identify the bilinear pairing $M\Delta t$ with a linear operator A on V, by choosing an identification $V \simeq V^*$ coming from a metric on V. If we choose our metric on V such that $\|\mathrm{d}\mu\| = 1$ then we have the simple relation

$$\frac{\mathrm{d}\mu}{\sqrt{\det M\Delta t}} = \frac{1}{\sqrt{\det A}}. \tag{3.71}$$

Exercise 3.9. Show that the $N \times N$ matrix A which appears in the discretization of the path integral of Example 3.8 is

$$A_{ij} = \left(2 + \frac{\omega^2 T^2}{N^2}\right) \delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}. \tag{3.72}$$

Check (at least numerically) that $1/\sqrt{\det A}$ indeed approaches the value (3.63) as $N \to \infty$. (Explicitly the eigenvalues of A are: $\frac{\omega^2 T^2}{N^2}$, $4\sin^2(\pi n/N) + \frac{\omega^2 T^2}{N^2}$ (with multiplicity 2) for $1 \leq n \leq \frac{N}{2} - 1$, and $4 + \frac{\omega^2 T^2}{N^2}$ if N is even.)

Exercise 3.10. Do similarly for Example 3.9.

We would like to make sense of the limit of the operator A, and its determinant, as $N \to \infty$. One interpretation is as follows. Let V denote the space of loops $S^1 \to \mathbb{R}$ for which the total energy is finite. This is an infinite-dimensional vector space (even a Hilbert space) carrying the quadratic function

$$S(\phi) = \frac{1}{2}\int_0^T \mathrm{d}t \left(\|\dot{\phi}(t)\|^2 + \omega^2 \phi(t)^2\right). \tag{3.73}$$

We expand

$$\phi(t) = c\sqrt{T} + \sum_{n=1}^{\infty} \frac{\sqrt{2T}}{2\pi n}\left(a_n \sin\left(\frac{2\pi n}{T}t\right) + b_n \cos\left(\frac{2\pi n}{T}t\right)\right). \tag{3.74}$$

Now, we choose the norm

$$\|\phi\|^2 = c^2 + \sum_n a_n^2 + b_n^2. \tag{3.75}$$

Then we have

$$S(\phi) = \frac{1}{2}\left(\omega^2 T^2 c^2 + \sum_{n=1}^{\infty}\left(1 + \frac{\omega^2 T^2}{4\pi^2 n^2}\right)(a_n^2 + b_n^2)\right) \tag{3.76}$$

Thus the eigenvalues of the operator A corresponding to the quadratic function $M = 2S$ are

$$\omega^2 T^2, \qquad 1 + \frac{\omega^2 T^2}{4\pi^2 n^2} \text{ (with multiplicity 2) for all } n > 0, \tag{3.77}$$

which gives[18]

$$\sqrt{\det A} = (\omega T) \prod_{n=1}^{\infty} \left(1 + \frac{\omega^2 T^2}{4\pi^2 n^2}\right) = 2\sinh(\omega T/2) \tag{3.79}$$

as desired.

What is puzzling about this is that it is not clear *a priori* why the norm (3.75) is the right one to choose. One encouraging sign: the paper [24] proves a similar statement in the case of an arbitrary Riemannian Y with $V = 0$ (under some nondegeneracy conditions); there it says the correct norm to take is a Sobolev H^1 norm, which is indeed what we did above. [we can verify more directly that the two correspond by making the appropriate rescalings in the discretization]

Exercise 3.11. Carry out a similar analysis for Example 3.9, relating the partition function $Z_{S^1(T)}$ we found there to determinants of operators on infinite-dimensional spaces. (In this case the field space $\mathcal{C}_{S^1(T)}$ is actually disconnected, because maps $S^1 \to S^1$ have a discrete invariant, the winding number. Nevertheless each connected component is an infinite-dimensional vector space, which we can analyze as above. In each component you meet the *same* infinite-dimensional determinant, multiplied by a prefactor depending on the component; so even without computing this determinant, you can get the answer up to a single undetermined constant. To see that it agrees with (3.69) you will need to use the Poisson summation formula, aka the modular property of the theta function.)

3.11 Perturbation theory in quantum mechanics

Now how about path integrals for actions which are not quadratic but are close to quadratic, in the same sense as subsection 2.4 above?

For example, let's consider the *quartic oscillator*, which is the case $Y = \mathbb{R}$,

$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{4!}x^4. \tag{3.80}$$

We will treat the case $X = S^1(T)$. Following the same pattern as subsection 2.4, we can write down Feynman rules for the asymptotic expansion of the partition function, or correlation functions, in powers of λ. In fact our action is an infinite-dimensional version of (2.31), and so we use exactly the same rules we used in that case, just adapted to the case where $\mathcal{C}$ is an infinite-dimensional vector space.

[18] We use the product formula

$$\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 n^2}\right). \tag{3.78}$$

Thus to compute Z/Z_0 we sum over "bubble" diagrams with only 4-valent vertices. The role of $M^{-1} \in V^{\otimes 2}$ is now played by the *Green's function* for the quadratic part of the action. Namely, let

$$M(x,x) = \int_0^T \mathrm{d}t \left(\frac{1}{2}\|\dot{x}(t)\|^2 + \frac{1}{2}\omega^2 x(t)^2 \right) \tag{3.81}$$

Then the function

$$G(t,t') = G(t-t') = \frac{1}{2\omega} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\omega|(t-t')+nT|} \tag{3.82}$$

inverts M in the sense that it obeys[19]

$$M\left(G(t,t'), f(t)\right) = f(t'). \tag{3.83}$$

Then, in parallel with the story of the 0-dimensional theory with quartic potential in subsection 2.4, the Feynman diagrams with ≤ 2 vertices give

$$\log \frac{Z}{Z_0} \sim -\frac{\lambda}{8} \int \mathrm{d}t\, G(0)^2 + \frac{\lambda^2}{48} \int \mathrm{d}t\, \mathrm{d}t'\, G(t-t')^4 + \frac{\lambda^2}{16} \int \mathrm{d}t\, \mathrm{d}t'\, G(t-t')^2 G(0)^2 + O(\lambda^3) \tag{3.84}$$

$$= -\frac{\lambda T}{32\omega^2} \left(\coth\left(\frac{\omega T}{2} \right) \right)^2 + O(\lambda^2). \tag{3.85}$$

Here is one concrete consequence of this formula. In the limit of large T, $\omega T \gg 1$, we have $\log Z \sim -TE$ where E is the ground state energy, i.e. the smallest eigenvalue of H. Then (3.85) gives the first-order correction to E as

$$E - E_0 \sim \frac{\lambda}{32\omega^2} + O(\lambda^2). \tag{3.86}$$

More generally, (3.85) contains the first-order perturbations of all the energy eigenvalues, not only the lowest one.

Exercise 3.12. Verify that (3.86) agrees with the usual result of first-order perturbation theory, which says that if we perturb the Hamiltonian by $H \to H + \delta H$, the resulting perturbation of the energy is $\delta E = \langle \psi_0, (\delta H)\psi_0 \rangle$, where ψ_0 is the lowest eigenstate of the unperturbed Hamiltonian H, normalized by $\langle \psi_0, \psi_0 \rangle = 1$.

Exercise 3.13. Evaluate explicitly the $O(\lambda^2)$ terms in (3.84). Use them to compute the $O(\lambda^2)$ correction to the ground state energy.

Exercise 3.14. Work out the prediction of first-order perturbation theory for the "sextic oscillator" with potential

$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{6!} x^6. \tag{3.87}$$

[19] When V is a finite-dimensional vector space, for a map $M : V \otimes V \to \mathbb{R}$, $M^{-1} : V^* \otimes V^* \to \mathbb{R}$ can be characterized by the equation $M(M^{-1}(\eta, \cdot), v) = \eta(v)$. (3.83) is an infinite-dimensional version of that equation, where we take V to be the space of functions on S^1.

Exercise 3.15. Extract from (3.85) a prediction for the first-order perturbation of the *second*-lowest energy eigenvalue of the quartic oscillator.

[perturbation theory for heat kernel?]

3.12 Quantum mechanics coupled to a vector bundle

Suppose Y is a Riemannian manifold, carrying a metric vector bundle E with connection ∇, and a Laplace-type operator $\Delta : C^\infty(Y, E) \to C^\infty(Y, E)$,

$$\Delta = \nabla^* \nabla + V. \tag{3.88}$$

Then we can define a heat equation coupled to E, just like (3.14) except that we use the operator $H = -\frac{1}{2}\Delta$ on $L^2(Y, E)$. The heat kernel $k_t(x, y)$ in this case is a section of $E \boxtimes E^*$ over $Y \times Y$: thus integrating it against a section of E gives another section of E.

Much like (3.17), this heat kernel arises from a 1-dimensional quantum field theory, with one new coupling added giving the *parallel transport* in E. Namely we write path integrals in the form

$$Z_X = \int \mathrm{d}\phi \; \mathrm{Hol}_\phi \, \mathrm{e}^{-S(\phi)} \tag{3.89}$$

where

- $S(\phi)$ is defined just as in (3.6).
- In case $X = S^1$, Hol_ϕ means the *trace* of the holonomy of $\phi^* E$.
- In case $X = [0, 1]_{y_0}^{y_1}$, Hol_ϕ means the holonomy itself, so that Z_X is not a number but a map $E_{y_0} \to E_{y_1}$.

Then we have:

Proposition 3.10 (Partition function of 1-d QFT coupled to a vector bundle on the target). $Z_{S^1(T)} = \mathrm{Tr}_{L^2(E)} \, \mathrm{e}^{-TH}$.

3.13 Supersymmetric quantum mechanics

We've seen that given a Riemannian manifold Y there is a corresponding 1-dimensional QFT of maps $X \to Y$, formally involving integrals over the path space

$$\mathcal{C}_X = \mathrm{Map}(X, Y), \tag{3.90}$$

which has to do with the heat flow on Y. In particular, its partition function on $X = S^1(T)$ is

$$Z_{S^1(T)} = \mathrm{Tr}\; \mathrm{e}^{-HT} \tag{3.91}$$

which contains the whole spectrum of $H = -\Delta$. This is very far from being deformation invariant: morally it depends on every little detail of the metric on Y.

To get something more topological we repeat the strategy we used in subsection 2.12 and subsection 2.14: add fermions! Concretely, we will make a new QFT whose configuration space is formally

$$\mathcal{C}_X = \Pi T\,\mathrm{Map}(X,Y). \tag{3.92}$$

One immediate observation is that if $\mathrm{Map}(X,Y)$ is oriented (whatever this means!) then we expect to have (at least formally) a *canonical* measure of integration $\mathrm{d}\vec{x}\mathrm{d}\vec{\psi}$, like what we had in subsection 2.14 where we took $\mathcal{C} = \Pi TM$. Thus we may expect that the constructions will be more canonical than they were in the non-supersymmetric case, where our choice of measure was somewhat arbitrary.

A point of $\mathcal{C}_X = \Pi T\,\mathrm{Map}(X,Y)$ concretely means a pair of:

- $\phi : X \to Y$,
- $\psi \in \Pi\Gamma(\phi^* TY)$.

Now we write the action $S \in C^\infty(\mathcal{C}_X)$:

$$S(\varphi,\psi) = \int \mathrm{d}t\,\frac{1}{2}(g(\dot{\phi},\dot{\phi}) + g(\psi,\nabla_t\psi)) \tag{3.93}$$

where ∇_t is the pullback of the Levi-Civita connection from Y, acting on sections of TY. The fermion bilinear term $g(\psi,\nabla_t\psi)$ should as usual be understood in a purely algebraic way: it defines an element of $\wedge^2(\Gamma(\phi^*TY))$, which by definition gives a function on $\Pi\Gamma(\phi^*TY)$ as needed.

Exercise 3.16. Check that $\int \mathrm{d}t\, g(\cdot,\nabla_t\cdot)$ is indeed a *skew-symmetric* pairing on $\Gamma(\phi^*TY)$. (Thus it would vanish if we inserted an ordinary even field in both slots; this is why we don't write such first-order terms in the ordinary bosonic 1-d QFT.)

As before, this action is invariant under the symmetry of time translations, which we represent by its infinitesimal action $H \in \mathrm{Vect}^0(\mathcal{C}_X)$,[20]

$$\delta\phi = \epsilon\dot{\phi}, \qquad \delta\psi = \epsilon\dot{\psi}. \tag{3.94}$$

But now it is also invariant under an odd symmetry $Q \in \mathrm{Vect}^1(\mathcal{C}_X)$, whose infinitesimal action is

$$\delta\phi = \epsilon\psi, \qquad \delta\psi = -\epsilon\dot{\phi}. \tag{3.95}$$

Let us verify this invariance in the special case $Y = \mathbb{R}^n$: then we have

$$S = \int \mathrm{d}t\frac{1}{2}(\dot{\phi}^I\dot{\phi}^I + \psi^I\dot{\psi}^I) \tag{3.96}$$

and thus

$$\delta S = \int \mathrm{d}t\,\frac{1}{2}(2\epsilon\dot{\psi}^I\dot{\phi}^I - \epsilon\dot{\phi}^I\dot{\psi}^I - \psi^I\epsilon\ddot{\phi}^I) \tag{3.97}$$

[20] It's possible to get confused about the invariant meaning of this formula, so let's spell it out: it means that relative to any local coordinate system on Y, we have $\delta\phi^I(t) = \epsilon\dot{\phi}^I(t)$ and $\delta\psi^I(t) = \epsilon\dot{\psi}^I(t)$.

which indeed vanishes (using integration by parts in the last term, and keeping track of the fact that ϵ and ψ^I anticommute to get the signs right.)

For a more general Riemannian manifold Y the verification is somewhat trickier, but the outcome is the same:

Exercise 3.17. Verify that S is invariant under Q.

The two symmetries Q and H are not unrelated:

Exercise 3.18. Verify that Q and H obey the 1-*dimensional supersymmetry algebra*

$$\frac{1}{2}[Q,Q] = -H. \tag{3.98}$$

Thus, in this theory Q provides a "square root of time translations."

Moreover, variations of the metric g on Y are Q-exact:

Exercise 3.19. Show that under a variation $g \to g + \delta g$, the action S changes by $S \to S + Q\Psi$, where $\Psi = -\frac{1}{2}\int_0^T \mathrm{d}t\,(\delta g)_{IJ}\dot{\phi}^I\psi^J$. [check]

Thus, if we were dealing with finite-dimensional integrals, Proposition 2.21 would show that the partition function in this theory is independent of the metric on Y. In our infinite-dimensional setting we will see that this metric-independence still holds. The infinite-dimensional setting does bring a surprising twist, discussed in subsection 3.14: in order for $Z_{S^1(T)}$ to be well defined, Y needs to be a *spinnable* manifold.

3.14 Integrating out fermions

[warning, doing this by discretization seems to be subtle, because of fermion doubling problem! need to give some explicit way of understanding the result here]

Suppose Y is even-dimensional, say $\dim Y = 2n$.

For fixed $\phi : S^1(T) \to Y$, the integral over the fermions is formally

$$W(\phi) = \int_{\Pi\Gamma(\phi^*TY)} \mathrm{d}\vec{\psi}\, \mathrm{e}^{-\frac{1}{2}\int_0^T g(\psi,\nabla_t\psi)} \tag{3.99}$$

We claim that here an important topological subtlety appears: in order to regulate the theory, i.e. to actually define this integral, we will need to fix an orthonormal trivialization F_ϕ of ϕ^*TY, and the path integral $W(\phi)$ will actually depend on this choice (so it is more accurately written $W(\phi, F_\phi)$.) For each fixed ϕ, the space of possible F_ϕ is a torsor for the loop group $\mathcal{L}\,\mathrm{SO}(2n)$. In particular, it divides up into connected components, which are a torsor for $\pi_1(\mathrm{SO}(2n))$; this group is $\mathbb{Z}$ for $n = 1$ and $\mathbb{Z}/2\mathbb{Z}$ for $n > 1$. Let τ denote the generator. Naively $W(\phi, F_\phi)$ would be independent of F_ϕ, but instead we have the following:

Proposition 3.11. $W(\phi, F_\phi)$ is invariant under small deformations of F_ϕ, but $W(\phi, F_\phi) = -W(\phi, \tau F_\phi)$.

For general Y this sign ambiguity would prevent us from defining the path integral, at least in the ordinary sense. To get rid of the problem we assume Y is spinnable, and then fix a spin structure on Y. This picks out a restricted class of orthonormal trivializations F_ϕ, namely those which lift to trivializations of the $\mathrm{Spin}(2n)$-bundle covering $\phi^* TY$. If we use only these trivializations in defining our discretized fermion path integral, then the troubling sign ambiguity does not occur.

Now let us see how to study $W(\phi, F_\phi)$ explicitly, roughly following a calculation given in [25, 26].

Recall the discussion in subsection 3.10: given a bilinear pairing and a metric on a vector space V, we can extract a number, which in finite dimensions is naturally the result of a Gaussian integral, and which we use to *define* Gaussian integrals in infinite dimensions.

Proposition 3.12. Relative to the Sobolev H^1 norm in $\Gamma(\phi^* TY)$, the determinant of the skew bilinear pairing $\langle \psi, \nabla_t \psi \rangle$ is (up to an overall constant) $\det(1 - \mathrm{Hol}_\phi)$, where Hol_ϕ is the holonomy of $\phi^* TY$.

Proof. The eigenvalues of the operator ∇_t are $\frac{2\pi \mathrm{i}}{T}(k \pm \frac{\alpha_i}{2\pi})$, $k \in \mathbb{Z}$, where $e^{\pm \mathrm{i}\alpha_i}$ are the eigenvalues of Hol_ϕ. (To see this just diagonalize ∇_t acting on complex sections: then its diagonal entries look like $\partial_t \pm \mathrm{i}\alpha_i / T$, with eigenfunctions $e^{2\pi \mathrm{i} k t / T}$.) Thus formally the determinant of this operator would be the infinite product

$$\prod_{k \in \mathbb{Z}} \prod_{i=1}^{n} -\frac{4\pi^2}{T^2} \left(k + \frac{\alpha_i}{2\pi} \right) \left(k - \frac{\alpha_i}{2\pi} \right). \tag{3.100}$$

This infinite product has a double zero at $\alpha_i = 2\pi k$, for any k.

Now, we take the Sobolev H^1 norm in $\Gamma(\phi^* TY)$; then the eigenfunctions are orthogonal but not orthonormal: for large k, the k-th eigenfunction has squared-norm $\sim k^2 T^2$. Thus the determinant relative to this norm will be a product of factors which for large k go like

$$\prod_{k} \prod_{i=1}^{n} \left(1 + \frac{\alpha_i}{2\pi k} \right) \left(1 - \frac{\alpha_i}{2\pi k} \right). \tag{3.101}$$

In particular it is now convergent, and must converge to an honest function of the α_i, with zeroes at $\alpha_i = 2\pi k$, which moreover is periodic as $\alpha_i \to \alpha_i + 2\pi$. This [plus some condition on asymptotic growth] determines it up to a constant as $\prod_{i=1}^{n} (2 \sin \frac{\alpha_i}{2})^2$. The next lemma then finishes the proof. □

Lemma 3.13. If $A \in \mathrm{SO}(2n)$ has eigenvalues $e^{\pm \mathrm{i}\alpha_i}$ then

$$\det(1 - A) = \prod_{i=1}^{n} \left(2 \sin \frac{\alpha_i}{2} \right)^2. \tag{3.102}$$

[comment on zeta reg?]

What we are really after is the Pfaffian of the pairing $\langle \psi, \nabla_t \psi \rangle$, not the determinant; so we need to take a square root of $\det(1 - \mathrm{Hol}_\phi)$. Since (3.102) represents this quantity as a square, it gives a natural-looking choice of square root, namely $\prod (2 \sin \frac{\alpha_i}{2})$. We claim that this is indeed the correct square root to take. Note that this requires us to lift Hol_ϕ to $\mathrm{Spin}(2n)$, since it changes by a sign under $\alpha_i \to \alpha_i + 2\pi$.[21]

[21] We might also have decided to just always take the *positive* square root. Then there would be no sign

For later purposes it will be convenient to have another interpretation of this answer which brings in the spin representation:

Proposition 3.14. If $A \in \mathrm{Spin}(2n)$ then

$$\det_V(1 - A) = (-1)^n (\mathrm{STr}_S A)^2, \tag{3.103}$$

where V denotes the vector representation.

Here STr means the *supertrace*, i.e.

$$\mathrm{STr}_S A = \mathrm{Tr}_{S^0} A - \mathrm{Tr}_{S^1} A. \tag{3.104}$$

Proof. When $n = 1$, from our description of S in subsection A.1, we see immediately that $\mathrm{STr}_S A$ is simply $(e^{i\alpha/2} - e^{-i\alpha/2}) = 2i \sin \frac{\alpha}{2}$. More generally we can block-diagonalize to reduce to the case where $A \in \mathrm{SO}(2)^n \subset \mathrm{SO}(2n)$ and use the fact that, as a representation of $\mathrm{SO}(2)^n \subset \mathrm{SO}(2n)$, S is the tensor product of 2-dimensional representations, and the grading on S is the induced grading. □

Motivated by Proposition 3.12 and Proposition 3.14, we propose that the integral over the fermions should be interpreted as [watch the $(-1)^n$]

$$W(\phi, F_\phi) = \mathrm{STr}_S \mathrm{Hol}_\phi \,. \tag{3.105}$$

3.15 Supersymmetric path integral and heat kernel

What we have just seen is that integrating out the fermions in our supersymmetric quantum mechanics has the effect of creating an insertion of $\mathrm{STr}_S \mathrm{Hol}_\phi$. This means that the effective theory we get is an ordinary bosonic quantum mechanics, but one coupled to the vector bundle $S = S^0 \oplus S^1$, with an unconventional sign rule; said differently, the partition function of the supersymmetric theory is a *difference*

$$Z_{S^1(T)} = Z_{S^1(T);S^0} - Z_{S^1(T);S^1} \tag{3.106}$$

where $Z_{S^1(T);E}$ means the partition function of the bosonic theory coupled to the vector bundle E, in the sense of subsection 3.12. Using Proposition 3.10, we then have

Proposition 3.15 (Partition function of supersymmetric quantum mechanics is the supertrace of spinor heat flow).

$$Z_{S^1(T)} = \mathrm{STr}_{\mathcal{H}} e^{-TH}, \tag{3.107}$$

where $H = -\frac{1}{2}\Delta$ acting on $\mathcal{H} = L^2(Y, SY)$. [should be careful about which Laplacian we use here: scalar curvature terms could intervene, but somehow they don't]

This function is remarkably simple to analyze:

Proposition 3.16 (Supertrace of spinor heat flow is the index). $\mathrm{STr}\, e^{-TH} = \mathrm{ind}\, \not{\partial}^0$.

ambiguity, but some lack of smoothness as a function of ϕ; it's not clear that this is fatal, but it does seem a little off-putting, at least.

Proof. This just uses the fact that $\mathcal{H} = L^2(Y, SY)$ is a unitary graded representation of the $1|1$-dimensional superalgebra spanned by $\not{\partial}$ and Δ, with $\not{\partial}^\dagger = \not{\partial}$ and $\Delta^\dagger = \Delta$.

First we show that $\ker \Delta = \ker \not{\partial}$. Indeed, obviously $\ker \not{\partial} \subset \ker \Delta$. Conversely, if $\psi \in \ker \Delta$ then we have

$$\|\not{\partial}\psi\|^2 = \langle \not{\partial}\psi, \not{\partial}\psi \rangle = \langle \psi, \not{\partial}^2\psi \rangle = \langle \psi, -\Delta\psi \rangle = 0 \tag{3.108}$$

and thus $\psi \in \ker \not{\partial}$.

Since $[\not{\partial}, \Delta] = 0$, Δ is central, and thus it acts as a multiple of the identity in each graded irreducible representation. So, consider a graded irreducible representation V in which $-\Delta = E$. Then it is straightforward to see there are only three possibilities:

- $\dim V = 1|1$, $E > 0$,
- $\dim V = 1|0$, $E = 0$,
- $\dim V = 0|1$, $E = 0$.

Each representation of dimension $1|1$ contributes $\mathrm{e}^{-TE} - \mathrm{e}^{-TE} = 0$ to $\mathrm{STr}_{\mathcal{H}}\, \mathrm{e}^{-TH}$. The representations of dimension $1|0$ contribute in total $\dim \ker \not{\partial}^0$, and those of dimension $0|1$ give $-\dim \ker \not{\partial}^1$. □

The upshot of this section is that supersymmetric quantum mechanics computes the index of the Dirac operator:

$$Z_{S^1(T)} = \mathrm{ind}\, \not{\partial}^0. \tag{3.109}$$

3.16 Quantization of local operators

The next proposition makes it a bit clearer why spinors are involved in the story. Fix a section γ of TY. Then for any $t \in X$ there is a corresponding odd function $\mathcal{O}_\gamma(t) \in C^\infty(\mathcal{C})$ which gives a local observable. Just as we "quantized" the local observables in [...], relating them to operators on $L^2(Y)$, now we can ask what is the quantization of these odd functions. The answer is that they map to operators $\hat{\gamma}$, which just act by the Clifford action of γ on $L^2(Y, S)$:

Proposition 3.17. $\langle \mathcal{O}_{\gamma_1}(t_1)\mathcal{O}_{\gamma_2}(t_2)\rangle_{S^1(T)} = \mathrm{STr}(\mathrm{e}^{-t_1 H}\hat{\gamma}_1 \mathrm{e}^{-t_2 H}\hat{\gamma}_2 \mathrm{e}^{-(T-t_1-t_2)H})$, and similarly for products of more $\mathcal{O}_\gamma(t)$.

[give some sketch proof? discretization?]
[supercurrent which maps to the Dirac operator]

3.17 Localization and the index theorem

In this section we are going to "prove" the index theorem, Theorem A.22, using our supersymmetric quantum mechanics. Of course, because we have been rather cavalier about infinite-dimensional determinants, this is not a rigorous proof as it stands. The original references for this proof are [27, 26, 25].

Now we want to compute the partition function $Z_{S^1(T)}$ by localization, similar to what we did in subsection 2.14. Indeed, the path integral we want to do is an infinite-dimensional version of the one given there, where:

- M is the loop space $M = \mathcal{L}Y$,
- the $U(1)$ action on M is by rotation of the loops,
- H is the bosonic part of the action (3.93), $\frac{1}{2}\int g(\dot{\phi}, \dot{\phi})$,
- ω is the fermionic part of (3.93), $\frac{1}{2}\int g(\psi, \nabla_t \psi)$.

Exercise 3.20. Verify that H is indeed a generating function for the $U(1)$ action, i.e. that $\iota_Y \omega = \mathrm{d}H$. [watch out for factor of 2 here]

Exercise 3.21. Verify that the odd symmetry Q corresponds to $\mathrm{d} + \iota_Y$.

[notation problem: used Y for the vector field earlier, now using it for target]

We are as usual free to choose any $U(1)$-invariant metric on M; we choose the one induced from the metric on Y.

Now, formally applying Theorem 2.30 would lead to

$$Z_{S^1(T)} = \int_F \frac{\mathrm{e}^{H+\omega}}{\mathrm{Euler}(NF)} \tag{3.110}$$

where $F \subset \mathcal{L}Y$ is the fixed locus of the $U(1)$ action. But this is just the space of *constant* loops, which make up a copy of Y inside $\mathcal{L}Y$, i.e. $F \simeq Y$. Restricted to F, we have $H = \omega = 0$. It just remains to evaluate the form $\mathrm{Euler}(NF)$.

First, what is NF? A tangent vector to the constant loop $\phi(t) = y$ is just a map $S^1 \to T_yY$. Fourier expanding, the space of these maps is $T_yY \oplus \bigoplus_{k>0}(T_yY \otimes \mathbb{R}^2)$; using our metric on $\mathcal{L}Y$ to project off the tangent bundle TF, we get $NF = \bigoplus_{k>0}(T_yY \otimes \mathbb{R}^2)$. The induced metric on NF is orthogonal and, in each summand, restricts to that of TY; in particular its curvature is $R \otimes \mathbf{1}$, with $R \in \mathfrak{so}(TY)$ the Riemann curvature 2-form, which we block-diagonalize with weight 2-forms R_i, $R = \bigoplus_{i=1}^n R_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Meanwhile the $U(1)$ generator acts as $\mathbf{1} \otimes \bigoplus_{k>0} k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So altogether

$$\mathrm{Euler}(NF) = \mathrm{Pf}\left(\left[\mathbf{1} \otimes \bigoplus_{k>0} k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] + \left[\bigoplus_{i=1}^{n} R_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1}\right]\right). \tag{3.111}$$

Now, we have

Exercise 3.22. Show that $\mathrm{Pf}\left(a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1} + b\mathbf{1} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = a^2 - b^2$. [check signs]

From this we compute the equivariant Euler form, formally, as an infinite product:

$$\mathrm{Euler}(NF) = \prod_{i=1}^{n} \prod_{k>0} (k + R_i)(k - R_i). \tag{3.112}$$

Our arguments have been too heuristic to determine how this product is to be regulated: in principle such a prescription should be part of the infinite-dimensional version of Theorem 2.30. Still, we notice that this product is just the same as one we encountered earlier [...] so we try regulating it the same way:

$$\mathrm{Euler}(NF) = \prod_{i=1}^{n} \frac{\sin \pi R_i}{\pi R_i} \tag{3.113}$$

This almost matches $\hat{A}(Y)^{-1}$ defined in Definition A.21, but not quite: its k-form part differs by a rescaling by $(4\pi^2)^k$. A more precise treatment of the regularization should allow us to fix this constant (in particular I think zeta function regularization works). Alternatively we could fix it by computing one example in each dimension.

Thus the formal (3.110) becomes

$$Z_{S^1(T)} = \int_Y \hat{A}(Y), \tag{3.114}$$

and combining this with (3.109) we obtain the index theorem Theorem A.22. Needless to say, this is not a rigorous proof of Theorem A.22 as it stands!

3.18 The twisted case

One extension of the physical proof of the index theorem is worth mentioning. It was given first in [28].

Suppose we fix a vector bundle E over Y with metric and compatible connection.[22] Then we can define a variant of our 1-d supersymmetric QFT, similar to what we did in the bosonic version in subsection 3.12: we define path integrals in the form

$$Z_X = \int \mathrm{d}\phi \mathrm{d}\psi \, \widetilde{\mathrm{Hol}}_\phi \, \mathrm{e}^{-S(\phi,\psi)} \tag{3.115}$$

One important caveat is that in order for the full theory to be Q-invariant we need to use a "supersymmetrized" holonomy $\widetilde{\mathrm{Hol}}_\phi$, which includes some dependence on the fermions and is engineered to have $Q(\widetilde{\mathrm{Hol}}_\phi) = 0$.

Let us specialize to the case where E has rank 1 and $X = S^1(T)$. Then it is easy enough to write the supersymmetrized holonomy explicitly: in components, it is

$$\widetilde{\mathrm{Hol}}_\phi = \exp \oint_{S^1(T)} A^I(\phi)\dot{\phi}_I + \frac{1}{2} F^{IJ}(\phi)\psi_I\psi_J \tag{3.116}$$

where F is the curvature of the connection in E.

[22] E can even be a super vector bundle [...]

Exercise 3.23. Verify that (3.116) is Q-invariant.

Exercise 3.24. Describe the generalization of (3.116) to bundles E of higher rank. [hard?]

After integrating out the fermions we obtain the bosonic theory coupled to the bundle $S \otimes E$, and in parallel to (3.109) we get

$$Z_{S^1(T)} = \operatorname{ind} \not{\partial}_E^0 \tag{3.117}$$

where now $\not{\partial}_E^0$ is the Dirac operator coupled to E as in subsection A.4.

The localization computation of $Z_{S^1(T)}$ in this case goes just as in subsection 3.17, except that now on the fixed locus we no longer have $\omega = 0$; rather, because of the fermionic part in (3.116), we get $\omega = F$, with F the curvature of the connection in E. Thus in the localization formula we obtain an extra factor of e^F. The result is the index theorem for twisted Dirac operators, Theorem A.25.

[case of higher rank E]

The index theorem for the Dirac operator is part of a much broader story. I do not know how much of this story can be understood in the supersymmetric-quantum-mechanics framework we consider here. (For example, what should we say about the *families index theorem*? Can we think of it as a formula for the Berry connection which one normally has on the space of ground states of a family of quantum-mechanical systems? One relevant reference here appears to be [29].)

4 Some QFT generalities

[Hilbert spaces associated to spatial slices] [canonical quantization]

5 QFT in 2 dimensions

Now we move on to two-dimensional theories, i.e. theories defined on a Riemannian 2-manifold X.

5.1 The free boson

A simple case to start with is the theory of a single scalar field. This theory has

$$\mathcal{C}_X = \operatorname{Map}(X, \mathbb{R}) \tag{5.1}$$

and the action

$$S(\phi) = \frac{1}{2}\int_X \|\mathrm{d}\phi\|^2 \mathrm{dvol}_X = \frac{1}{2}\int_X \mathrm{d}\phi \wedge \star\mathrm{d}\phi. \tag{5.2}$$

Exercise 5.1. Verify that $S(\phi)$ is *conformally invariant*, i.e. it depends only on the conformal class of the metric on X.

In principle this theory should be defined by some kind of discretization, cutoff or the like. For a while, let us try to work formally, without being careful about precisely how the theory is defined.

We consider the special case $X = S^1(T) \times S^1(L)$. In this case we will see that the theory essentially decomposes into an infinite number of copies of the 1-dimensional "harmonic oscillator" we considered in Example 3.8. Indeed, we can decompose the field[23]

$$\phi(x,t) = \frac{1}{\sqrt{2L}} \sum_n a_n(t) \mathrm{e}^{2\pi \mathrm{i} n x/L} \tag{5.3}$$

where $a_n(t) = \overline{a_{-n}(t)}$, and then the action (5.2) is

$$S(\phi) = \frac{1}{4} \sum_{n \in \mathbb{Z}} \int_0^T \mathrm{d}t \, \frac{4\pi^2 n^2}{L^2} |a_n(t)|^2 + |\dot{a}_n(t)|^2. \tag{5.4}$$

Alternatively, writing $a_n(t) = b_n(t) + \mathrm{i} c_n(t)$ with $b_n(t)$ and $c_n(t)$ real, this is

$$S(\phi) = \sum_{n>0} \left[\int_0^T \mathrm{d}t \, \frac{2\pi^2 n^2}{L^2} (b_n(t)^2 + c_n(t)^2) + \frac{1}{2} (\dot{b}_n(t)^2 + \dot{c}_n(t)^2) \right] + \frac{1}{4} \int_0^T \mathrm{d}t \, \dot{a}_0(t)^2. \tag{5.5}$$

Thus the action is a sum of terms, each involving just one of the fields $a_0(t), b_n(t), c_n(t)$. Each of the $b_n(t)$ and $c_n(t)$ has an action like the field $x(t)$ of Example 3.8, with the frequency

$$\omega_n = 2\pi \frac{n}{L}. \tag{5.6}$$

We have already computed the contribution to the partition function from such a field: it is given by (3.63). Thus formally the fields $b_n(t)$ and $c_n(t)$ contribute to the partition function a factor

$$\prod_{n=1}^{\infty} \frac{1}{(2 \sinh(\pi n T/L))^2} = \prod_{n=1}^{\infty} \left(\frac{q^{n/2}}{1-q^n} \right)^2, \qquad q = \mathrm{e}^{-2\pi T/L}. \tag{5.7}$$

As it stands, this product diverges to 0 for any q, because of the factor $\prod_{n=1}^{\infty} q^n$. So our strategy of working formally has not worked out well. On the other hand, if we define the path integral by discretization, the situation should be better. [explain regulation of the zero point energy] So the regulated version of (5.7) is

$$q^{-1/12} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \tag{5.8}$$

This function has another name: it is $\eta(q)^{-2}$, where $\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ is the Dedekind eta function.

We also have the remaining field $a_0(t)$, which has to be treated differently. [it's an IR divergence, regulate it] Finally we get

$$Z_X = \frac{V}{2\pi\sqrt{T/L}} \eta(q)^{-2} \tag{5.9}$$

[23] We use x as the coordinate on "space" $S^1(L)$ and t on "time" $S^1(T)$.

More generally, if we consider a *tilted* torus $X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ with τ in the upper half-plane, then we get

$$Z_X = \frac{V}{2\pi\sqrt{\operatorname{Im}\tau}} |\eta(\tau)|^{-2}. \tag{5.10}$$

Exercise 5.2. Check directly that Z_X given in Equation 5.10 is modular invariant, i.e. invariant under $\tau \to \tau + 1$ and $\tau \to -1/\tau$. [comment on this]

[comment on Polyakov formula: if we transform the metric from $\rho_1 |\mathrm{d}z|^2$ to $\rho_2 |\mathrm{d}z|^2$, $\det \Delta$ changes by $\exp\left[\frac{1}{3\pi} \int_X \log \frac{\rho_2}{\rho_1} \partial^2_{z\bar{z}} \log(\rho_2\rho_1) |\mathrm{d}z|\right]$ (cite Fay)]

5.2 Compactified free boson

Now let us generalize slightly: instead of the target $\mathbb{R}$ let us take $Y = S^1(R)$, with the same action (5.2). The previous analysis now has to be modified in two ways. First, the expansion (5.3) gets replaced by

$$\phi(x,t) = \frac{1}{\sqrt{2L}} \sum_n a_n(t) \mathrm{e}^{2\pi \mathrm{i} n x/L} + wxR/L \tag{5.11}$$

where $w \in \mathbb{Z}$ is an integer, the *winding number*. Second, the field $a_0(t)$ appearing here is now a map to $S^1(R)$ rather than to $\mathbb{R}$. (All the other fields $a_n(t)$ are still maps to $\mathbb{R}$.)

The spectrum is correspondingly modified. First, the winding term contributes [...]

[T-duality]

6 Four-dimensional field theory

6.1 Abelian gauge theory

[abelian gauge theory]

[summing over bundles]

[electric/magnetic duality]

[exercise: modular property of partition function on a Riemannian manifold, ref Witten]

[coupling to background currents: Wilson line]

[coupling to dynamical currents]

[nonperturbative sectors]

6.2 Interactions

Now we come to an important point. The action [...] is not purely quadratic: it involves cubic and quartic interaction terms. Thus, for any computation we do in this theory, we will need to deal with the interactions. We have done a few such computations before, in 0-dimensional and 1-dimensional theories [...], without any major difficulties arising.

Now let's consider an analogous computation in a 4-dimensional theory: we take a scalar field theory,

$$\mathcal{C}_X = \mathrm{Map}(X, \mathbb{R}), \tag{6.1}$$

with the action

$$S = \int_X \frac{1}{2}\|\mathrm{d}\phi\|^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{6.2}$$

This is completely parallel to the anharmonic oscillator which we considered in [...], except for the crucial difference that X has dimension 4 instead of 1.

Suppose we try to compute the simplest nontrivial observable in this theory, the 2-point function. Following the rules of perturbation theory as we have developed them to this point, we would get the formal expression

$$\frac{\langle \phi(x_1)\phi(x_2)\rangle}{Z} = D(x_1, x_2) + \frac{\lambda}{2}\int_X \mathrm{d}x'\, D(x_1, x')D(x_2, x')D(x', x') + \cdots \tag{6.3}$$

where $D(x, y)$ is the Green's function for the deformed Laplace operator, obeying

$$(-\Delta_x + m^2)D(x, y) = \delta(x, y) \tag{6.4}$$

and $D(x, y) \to 0$ as $\|x - y\| \to \infty$. Recall [...] that for $X = \mathbb{R}$ and $m = 0$ this would be $D(x, y) = \frac{1}{2}|x - y|$. For $X = \mathbb{R}^4$ and $m = 0$ on the other hand it is $D(x, y) = \frac{1}{\|x-y\|^2}$. [constant?] In particular, there is an evident difficulty with (6.3): the quantity $D(x', x')$ appearing there is infinite.

This infinity might seem like a disaster, but it isn't: it is just a symptom of the fact that we have not been particularly careful about how the theory is really defined. Were we more careful, we might try to define the theory by first defining a lattice version of it and then taking the lattice spacing to zero, as we did in [...]. One important effect of such a discretization is to provide a "UV cutoff": roughly speaking, the Fourier transform $\hat{\phi}$ cannot be supported at values larger than the inverse lattice spacing. In the case $X = \mathbb{R}^4$, we can model this effect by defining a new field space,

$$\mathcal{C}_\Lambda = \{\phi : \mathbb{R}^4 \to \mathbb{R}, \quad \hat{\phi}(p) = 0 \text{ for } \|p\| > \Lambda\}. \tag{6.5}$$

Now, suppose that we perform our path integrals over the cutoff field space $\mathcal{C}_\Lambda$ instead of the original $\mathcal{C}_{\mathbb{R}^4}$. Repeating the formal development of perturbation theory, we would find that the only change [check] is that $D(x, y)$ gets replaced by a cutoff version $D_\Lambda(x, y)$, defined as follows.

Recall that the original $D(x, y)$ can be written conveniently in Fourier space, as [factors of 2π?]

$$D(x, y) = \int_{(\mathbb{R}^4)^*} \mathrm{d}p \frac{e^{\mathrm{i}p\cdot(x-y)}}{\|p\|^2 + m^2}. \tag{6.6}$$

Exercise 6.1. Check (6.6). (Hint: the Fourier transform of $-\Delta + m^2$ is the operator of multiplication by $\|p\|^2 + m^2$, and the Fourier transform of the delta-function $\delta(0)$ is a constant function.)

In particular this gives

$$D(x,x) = \int_{(\mathbb{R}^4)^*} \mathrm{d}p \frac{1}{\|p\|^2 + m^2} \tag{6.7}$$

which makes it obvious that it indeed diverges, with the divergence coming from the region of large $\|p\|$, i.e. from the high-frequency modes of the field ϕ. In the cutoff theory we get instead

$$D_\Lambda(x,y) = \int_{\|p\|<\Lambda} \mathrm{d}p \frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{\|p\|^2 + m^2} \tag{6.8}$$

and in particular $D_\Lambda(x,x)$ is now finite. So, by introducing the cutoff, we have at least made the individual terms in our perturbation expansion (6.3) well defined. However, as a practical tool, this expansion still suffers from some drawbacks. In particular, let us consider the behavior for large Λ. Then we can expand [fix the constants]

$$D_\Lambda(x,y) \sim \begin{cases} \Lambda^2 + \cdots & \text{if } x = y, \\ \frac{1}{\|x-y\|^2} + \cdots & \text{if } x \neq y \end{cases} \tag{6.9}$$

Using this expansion we can study the large Λ behavior of the individual terms in the expansion (6.3). The leading term behaves as $\|x_1 - x_2\|^{-2}$ while the next term behaves as $\lambda\Lambda^2$. In particular, if $\Lambda \gg \|x_1 - x_2\|^{-1}$, then this expansion seems to be rather ill behaved. [...]

A Background

Here I collect a little background material for convenience.

A.1 Spinors

Good references for this are [30, 9].

Definition A.1 (Free graded tensor algebra). Fix a real vector space V. The *free graded tensor algebra* $T(V)$ is

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} \tag{A.1}$$

equipped with the concatenation product

$$(v_1 \otimes \cdots \otimes v_n)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m \tag{A.2}$$

and the obvious $\mathbb{Z}$-grading.

Definition A.2 (Clifford algebra). Fix a real vector space V with a positive definite quadratic form $\langle \cdot, \cdot \rangle$. The *Clifford algebra* $\mathrm{Cliff}(V)$ is the quotient of $T(V)$ by the two-sided ideal generated by the relation

$$v \otimes v = -\langle v, v \rangle, \qquad v \in V. \tag{A.3}$$

More informally we could say $\mathrm{Cliff}(V)$ is the free algebra on V subject to the single relation (A.3). This relation violates the $\mathbb{Z}$-grading but preserves a residual $\mathbb{Z}/2\mathbb{Z}$. Thus $\mathrm{Cliff}(V)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra:

$$\mathrm{Cliff}(V) = \mathrm{Cliff}^0(V) \oplus \mathrm{Cliff}^1(V). \tag{A.4}$$

By polarization (A.3) implies that for any $v, w \in V$ we have

$$[v, w] = vw - wv = -2\langle v, w\rangle. \tag{A.5}$$

If we allow $\langle \cdot, \cdot \rangle$ to be the zero quadratic form then $\mathrm{Cliff}(V)$ becomes the exterior algebra $\wedge^*(V)$; more generally $\mathrm{Cliff}(V)$ is a *deformation* of $\wedge^*(V)$, or, it is a $\mathbb{Z}$*-filtered* algebra whose associated $\mathbb{Z}$-graded algebra is $\wedge^*(V)$.

Exercise A.1. Show that $\dim_{\mathbb{R}} \mathrm{Cliff}(V) = 2^{\dim V}$.

Definition A.3 (Pin group). $\mathrm{Pin}(V)$ is the group of all elements

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathrm{Cliff}(V) \tag{A.6}$$

where all v_i have $\langle v_i, v_i\rangle = 1$.

Exercise A.2. Show that $\mathrm{Pin}(V)$ is indeed a group.

Definition A.4 (Spin group). $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cliff}^0(V)$.

Now any $v \in V \subset \mathrm{Pin}(V)$ acts on $V \subset \mathrm{Cliff}(V)$ by conjugation:

Exercise A.3. Show that this action is just the reflection in v.

More generally,

Exercise A.4. Show that $\mathrm{Pin}(V)$ acts on $V \subset \mathrm{Cliff}(V)$ by conjugation, and that this action gives a map $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ which is a double cover.

As a convenient notation we let $\mathrm{Cliff}(n)$ mean the Clifford algebra on the vector space $\mathbb{R}^n$ with its standard positive definite quadratic form. $\mathrm{Cliff}(n)$ is generated by odd elements e_i, $i = 1, \dots, n$, with

$$\frac{1}{2}[e_i, e_j] = -\delta_{ij}. \tag{A.7}$$

Similarly we define $\mathrm{Spin}(n)$, $\mathrm{Pin}(n)$ etc.

When $n > 2$, $\mathrm{Spin}(n)$ is simply connected, and the double-covering map above identifies $\mathrm{Spin}(n)$ as the universal covering of $\mathrm{SO}(n)$. When $n = 2$, both $\mathrm{Spin}(2)$ and $\mathrm{SO}(2)$ are the circle group, and this map is just the double-covering map $\theta \to 2\theta$.

Exercise A.5. Show that:

- $\mathrm{Cliff}(1) \simeq \mathbb{C}$, $\mathrm{Cliff}^0(1) \simeq \mathbb{R}$, $\mathrm{Spin}(1) \simeq \mathbb{Z}/2\mathbb{Z}$.
- $\mathrm{Cliff}(2) \simeq \mathbb{H}$, $\mathrm{Cliff}^0(2) \simeq \mathbb{C}$, $\mathrm{Spin}(2) \simeq \mathrm{U}(1)$.

- $\mathrm{Cliff}(3) \simeq \mathbb{H} \oplus \mathbb{H}$, $\mathrm{Cliff}^0(3) \simeq \mathbb{H}$, $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$.

Definition A.5 (Spin structures). Fix an oriented Riemannian manifold X, and let P be its principal $\mathrm{SO}(n)$-bundle of orthonormal frames. A *spin structure on* X is a lift of P to a $\mathrm{Spin}(n)$-bundle. A *spinnable manifold* is a Riemannian oriented manifold X which admits a spin structure. A *spin manifold* is a Riemannian oriented manifold X equipped with a choice of spin structure.

Example A.6 (Spin structures on the circle). When $X = S^1$ with a fixed metric and orientation, the bundle P of oriented frames is just X itself, so a spin structure is just a double cover of S^1. Up to equivalence there are two double covers of S^1 (one connected and one disconnected), thus two spin structures.

Exercise A.6. For any Riemannian oriented X which admits a spin structure, show that the spin structures on X up to equivalence form a *torsor* for the group $H^1(X, \mathbb{Z}/2\mathbb{Z})$. (Hint: $H^1(X, \mathbb{Z}/2\mathbb{Z})$ classifies double covers of X up to equivalence, and given a spin structure on X and a double cover of X, one can *twist* the spin structure by the double cover, to get another spin structure.)

Proposition A.7. Every manifold of dimension 1, 2 or 3 is spinnable.

Proposition A.8. $\mathbb{CP}^2$ is not spinnable.

Proof. Let H denote a hyperplane in $\mathbb{CP}^2$, i.e. an embedded $\mathbb{CP}^1$. Then, $T\mathbb{CP}^2$, restricted to H, is $\mathcal{O}(1) \oplus \mathcal{O}(2)$: $\mathcal{O}(2)$ is $T\mathbb{CP}^1$ and $\mathcal{O}(1)$ is the normal bundle.[24]

$\mathbb{CP}^1$ has two charts, so we can explicitly write down transition functions for $\mathcal{O}(1) \oplus \mathcal{O}(2)$, which are maps $S^1 \to \mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{SO}(4)$. Explicitly, let $R_\theta \in \mathrm{SO}(2)$ denote the matrix which acts through rotation by θ; then one of the transition functions is

$$\theta \mapsto \begin{pmatrix} R_\theta & \\ & R_{2\theta} \end{pmatrix}. \tag{A.8}$$

A spin structure is a lift of this map to $\mathrm{Spin}(4)$, the universal cover of $\mathrm{SO}(4)$. But the loop defined by (A.8) is the nontrivial element of $\pi_1(\mathrm{SO}(4))$, and therefore this map cannot lift to $\mathrm{Spin}(4)$. □

From now on we specialize to even dimensions, since this is what we will use in the main text.

Proposition A.9 (Spin representations). $\mathrm{Cliff}(2n)$ admits a $\mathbb{Z}/2\mathbb{Z}$-graded irreducible complex representation, $S = S^0 \oplus S^1$. Each of S^0 and S^1 has complex dimension 2^{n-1}. S carries a natural Hermitian metric, with respect to which S^0 and S^1 are orthogonal, V acts by skew-adjoint endomorphisms, and $\mathrm{Spin}(V)$ acts unitarily. Up to equivalence and reversal of grading, S is the unique graded irreducible representation of $\mathrm{Cliff}(2n)$.

[24] A generic section of the normal bundle intersects itself at one point, which is the reason why the normal bundle is $\mathcal{O}(1)$; a similar argument gets you $\mathcal{O}(2)$ for the tangent bundle.

Example A.10 (Spin representation in dimension 2). The spin representation S of $\mathrm{Cliff}(2)$ is 2-dimensional. The generators e_1 and e_2 act by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{A.9}$$

Thus $e_1 e_2$ acts by $\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$.

Definition A.11 (Complex spinor bundles). Fix a spin manifold X. The spin structure is a principal $\mathrm{Spin}(n)$-bundle Q over X. The associated bundle $SX = Q \times_{\mathrm{Spin}(n)} S$ is the *complex spinor bundle* over X.

SX carries several nice structures:

- Since Q has a canonical connection induced by the Levi-Civita connection, SX also has a canonical connection.
- Because the action of $\mathbb{R}^n \subset \mathrm{Cliff}(n)$ on S, $\rho : \mathbb{R}^n \to \mathrm{End}(S)$, is equivariant for the action of $\mathrm{Spin}(n)$ on both sides, it transfers to an action $\rho : TX \to \mathrm{End}(SX)$.

A.2 Dirac operator

Definition A.12 (Dirac operator). Fix a spin manifold X. The *Dirac operator* is the operator $\partial\!\!\!/ : C^\infty(SX) \to C^\infty(SX)$ given by

$$\partial\!\!\!/ = \sum_{i=1}^{n} \rho(e_i) \circ \nabla_{e_i} \tag{A.10}$$

where $\{e_i\}$ form an orthonormal basis for TX.

Example A.13 (Dirac operator on $\mathbb{R}^2$). If $X = \mathbb{R}^2$, then SX is the 2-dimensional complex trivial bundle, and

$$\partial\!\!\!/ = e_1\partial_1 + e_2\partial_2 = \begin{pmatrix} 0 & \partial_1 + \mathrm{i}\partial_2 \\ -\partial_1 + \mathrm{i}\partial_2 & 0 \end{pmatrix}. \tag{A.11}$$

Proposition A.14. $\partial\!\!\!/$ is a symmetric operator with respect to the Hermitian L^2 pairing on SX, i.e.

$$\langle \psi, \partial\!\!\!/ \psi' \rangle = \langle \partial\!\!\!/ \psi, \psi' \rangle \tag{A.12}$$

Definition A.15 (Spinor Laplacian). The *spinor Laplacian* is the operator $\Delta : C^\infty(SX) \to C^\infty(SX)$ given by

$$\Delta = -\partial\!\!\!/^2. \tag{A.13}$$

Note that on $X = \mathbb{R}^2$, relative to the trivialization of SX we gave above, Δ is just the identity matrix times the usual Laplacian on $C^\infty(\mathbb{R}^2)$. What is interesting is that $\partial\!\!\!/$ provides a *square root* of $-\Delta$. This is one of the nice features of spinors: in contrast, for ordinary functions, the square root of $-\Delta$ cannot be realized as a local differential operator.

A.3 Index of the Dirac operator

Proposition A.16. If X is a compact spin manifold, then $\not\partial : C^\infty(SX) \to C^\infty(SX)$ is an elliptic operator.

In particular, if we decompose $\not\partial$ with respect to the grading of S,

$$\not\partial = \begin{pmatrix} 0 & \not\partial^1 \\ \not\partial^0 & 0 \end{pmatrix}, \tag{A.14}$$

then each of $\not\partial^0$, $\not\partial^1$ is also elliptic. Since $\not\partial$ is symmetric, these two elliptic operators are formally adjoint to one another. Now we consider the *index*

$$\operatorname{ind} \not\partial^0 = \dim\ker \not\partial^0 - \dim\operatorname{coker} \not\partial^0 = \dim\ker \not\partial^0 - \dim\ker \not\partial^1. \tag{A.15}$$

We are going to give a formula for $\operatorname{ind} \not\partial^0$ in terms of characteristic classes. Recall that given a symmetric function $C(\{y_i\})$ where the y_i are formal variables, we can define a corresponding characteristic class of $\mathrm{SO}(n)$-bundles over X: to construct it, we locally block-diagonalize the curvature $F = \bigoplus_i \begin{pmatrix} 0 & F_i \\ -F_i & 0 \end{pmatrix} \in \mathfrak{so}(n)$ and then take the form $C(\{\frac{F_i}{2\pi \mathrm{i}}\}) \in \Omega^*(X)$.[25]

Definition A.17. If X is a compact Riemannian manifold, $p(X) \in \Omega^*(X)$ is the characteristic class of TX associated to the symmetric function

$$\prod_i (1 + y_i^2). \tag{A.16}$$

It is a sum

$$p(X) = 1 + p_1(X) + p_2(X) + \cdots, \qquad p_k(X) \in \Omega^{4k}(X). \tag{A.17}$$

Example A.18. We have $p_1(X) = -\frac{1}{4\pi^2} \operatorname{Tr} R \wedge R$, where $R \in \Omega^2(\mathfrak{so}(TX))$ is the Riemann curvature, and Tr denotes the usual trace, thinking of elements of $\mathfrak{so}(n)$ as $n \times n$ matrices.

The forms $p_k(X)$ are actually de Rham representatives of classes in $H^{4k}(X, \mathbb{Z})$; thus integrating them, or their products, gives integers. But as we will see momentarily, they have considerably stronger divisibility properties than mere integrality!

Example A.19. The first nontrivial examples of Pontryagin classes arise in dimension 4, where we have:

$$\int_{S^4} p_1(S^4) = 0, \qquad \int_{\mathbb{CP}^2} p_1(\mathbb{CP}^2) = 3, \qquad \int_{K3} p_1(K3) = -48. \tag{A.18}$$

These integers have a simple interpretation:

Theorem A.20. If X is a 4-manifold then $\int_X p_1(X) = 3 \operatorname{sign}(X)$.

[25]We will only use functions invariant under $y_i \to -y_i$, so we actually get characteristic classes of $\mathrm{O}(n)$-bundles.

Exercise A.7. Use (A.18) to show that there is no orientation-reversing diffeomorphism of $\mathbb{CP}^2$ or $K3$: these manifolds are, in some essential way, *chiral* — in contrast to S^4 for which the antipodal map is orientation-reversing.

Definition A.21. If X is a compact Riemannian manifold, $\hat{A}(X) \in \Omega^*(X)$ is the characteristic class of TX associated to the symmetric function

$$\prod_i \frac{y_i/2}{\sinh(y_i/2)} = \prod_i \left(1 - \frac{y_i^2}{24} + \frac{7y_i^4}{5760} + \cdots\right) \tag{A.19}$$

So $\hat{A}(X)$ is a sum of forms in degrees $4k$. We can expand $\hat{A}(X)$ in terms of Pontryagin classes:

$$\hat{A}(X) = 1 - \frac{1}{24}p_1(X) + \frac{7p_1(X)^2 - 4p_2(X)}{5760} + \cdots \tag{A.20}$$

Theorem A.22 (Atiyah-Singer index theorem for the Dirac operator on complex spinors). We have

$$\operatorname{ind} \not{\partial}^0 = \int_X \hat{A}(X). \tag{A.21}$$

This implies in particular that $\int_X \hat{A}(X)$ is actually an integer, so e.g. for a spinnable 4-manifold, $\int_X p_1(X)$ is a multiple of 24.[26]

A.4 Dirac operator coupled to a vector bundle

[characteristic classes for $U(n)$-bundles]

Definition A.23 (Chern character). $\mathrm{ch}(E)$ is the characteristic class of a $U(n)$-bundle E associated to the function [check factors]

$$\sum_i e^{z_i}. \tag{A.22}$$

It is concentrated in even degrees, expanding as

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots \tag{A.23}$$

Definition A.24 (Twisted Dirac operator). The *Dirac operator twisted by* E is the operator $\not{\partial}_E : C^\infty(SX \otimes E) \to C^\infty(SX \otimes E)$ given by

$$\not{\partial}_E = \sum_{i=1}^n \rho(e_i) \circ \nabla_{e_i} \tag{A.24}$$

where $\{e_i\}$ form an orthonormal basis for TX, and ∇ denotes the induced connection on $SX \otimes E$.

[26] Even more is true: if X is a spinnable manifold of dimension $8n+4$, $\int_X \hat{A}(X)$ is an *even* integer, because in these dimensions $\not{\partial}$ commutes with an additional conjugate-linear symmetry $j : SX \to SX$ with $j^2 = -1$, i.e. $\not{\partial}$ is $\mathbb{H}$-linear rather than merely $\mathbb{C}$-linear.

Theorem A.25 (Atiyah-Singer index theorem for twisted Dirac operator on complex spinors). We have

$$\operatorname{ind} \partial\!\!\!/_E^0 = \int_X \hat{A}(X)\,\mathrm{ch}(E). \tag{A.25}$$

The simplest example of Theorem A.25 arises when $\dim X = 2$. In this case the theorem becomes

$$\operatorname{ind} \partial\!\!\!/_E^0 = \int_X c_1(E) = \deg E. \tag{A.26}$$

If we choose E to be $V \otimes (\mathbb{C} \oplus T^*X)$ [... explain relation to Riemann-Roch]

A.5 Hodge theory

Definition A.26 (Formal adjoint of d). If X is a Riemannian manifold of dimension n, the formal adjoint of d is the operator

$$\mathrm{d}^* : \Omega^k(X) \to \Omega^{k-1}(X) \tag{A.27}$$

given by

$$\mathrm{d}^* = (-1)^{n(k+1)+1} \star \mathrm{d} \star. \tag{A.28}$$

If X is a compact Riemannian manifold, we have the L^2 pairing on $\Omega^*(X)$ given by

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha(x), \beta(x) \rangle \, \mathrm{d}vol_X = \int_X \alpha \wedge \star\beta. \tag{A.29}$$

Lemma A.27 (Formal adjoint is actual adjoint on compact manifold). If X is a compact Riemannian manifold, d^* is the actual adjoint with respect to the L^2 pairing, i.e.

$$\langle \mathrm{d}^*\alpha, \beta \rangle = \langle \alpha, \mathrm{d}\beta \rangle. \tag{A.30}$$

Definition A.28 (Laplace operator on Riemannian manifold). If X is a Riemannian manifold, we define the form Laplacian

$$\Delta : \Omega^k(X) \to \Omega^k(X) \tag{A.31}$$

by

$$\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}. \tag{A.32}$$

[...]

A.6 Symplectic manifolds

Definition A.29 (Nondegenerate skew pairing). Suppose V is a vector space over $\mathbb{R}$ or $\mathbb{C}$. We say $\omega \in \wedge^2(V)$ is *nondegenerate* if the map

$$V \to V^* \tag{A.33}$$
$$v \mapsto \iota_v \omega = \omega(v, \cdot) \tag{A.34}$$

is an isomorphism.

Proposition A.30 (Standard basis for a nondegenerate skew pairing). If V is a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and $\omega \in \wedge^2(V)$ is nondegenerate, then V has dimension $2n$ for some n, and there exists a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ for V such that

$$\omega(e_i, f_j) = \delta_{ij}, \tag{A.35}$$
$$\omega(e_i, e_j) = 0, \tag{A.36}$$
$$\omega(f_i, f_j) = 0. \tag{A.37}$$

Definition A.31 (Symplectic manifold). A *symplectic manifold* is a pair (X, ω) where X is a manifold and $\omega \in \Omega^2(X)$, such that

$$\mathrm{d}\omega = 0 \tag{A.38}$$

and $\omega(x)$ is nondegenerate for every $x \in X$.

Definition A.32 (Exact symplectic manifold). An *exact symplectic manifold* is a tuple (X, ω, λ) where (X, ω) is a symplectic manifold and $\lambda \in \Omega^1(X)$ has $\mathrm{d}\lambda = \omega$.

Example A.33 (Cotangent bundle is an exact symplectic manifold). If X is any manifold and $Y = T^*X$, then Y carries a canonical 1-form ("Liouville form"), $\lambda \in \Omega^1(Y)$, defined as follows:

$$\lambda(x, p) \cdot v = p \cdot \pi_* v \qquad x \in X, p \in T_x^* X, v \in TY. \tag{A.39}$$

Then there is a canonical symplectic form on Y given by

$$\omega = \mathrm{d}\lambda. \tag{A.40}$$

Exercise A.8. Show that, in the canonical coordinate system (p_i, q_i) on T^*X induced by a coordinate system (q_i) on X, we have $\lambda = \sum_{i=1}^n p_i \mathrm{d}q_i$, and $\omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$.

Definition A.34 (Moment map). Suppose X is a symplectic manifold, with symplectic form ω, acted on by a real Lie group G. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and let

$$\rho : \mathfrak{g} \to \mathrm{Vect}(X) \tag{A.41}$$

be the infinitesimal action. Suppose given a function

$$\mu : X \to \mathfrak{g}^* \tag{A.42}$$

and for $Z \in \mathfrak{g}$ write $\mu_Z = \mu \cdot Z$. We say μ is a *moment map* for the G-action if for all $Z \in \mathfrak{g}$ we have

$$\iota_{\rho(Z)}\omega = \mathrm{d}\mu_Z, \tag{A.43}$$

and in addition the map (A.42) is G-equivariant (for the G-action on X and the coadjoint G-action on $\mathfrak{g}^*$).

In particular, the moment map μ *determines* the G-action.

Note that moment maps do not always exist. At the very least, the existence of a moment map requires that $\iota_{\rho(Z)}\omega$ is closed for all $Z \in \mathfrak{g}$, by (A.43). Using Cartan's "magic formula"

$$\mathcal{L}_v\omega = \mathrm{d}(\iota_v\omega) + \iota_v(\mathrm{d}\omega) \tag{A.44}$$

and the fact that $\mathrm{d}\omega = 0$, this is equivalent to requiring $\mathcal{L}_{\rho(Z)}\omega = 0$, i.e. the G-action preserves ω infinitesimally. But even if the G-action preserves ω, a moment map still may not exist.[27]

Conversely, if a moment map μ does exist and $\mathfrak{g}$ has nontrivial center, we can get another moment map by taking $\mu' = \mu + c$, where c is fixed by the *coadjoint* action of G, i.e. $c \in [\mathfrak{g}, \mathfrak{g}]^\perp \subset \mathfrak{g}^*$.

Exercise A.9. Suppose $X = \mathbb{R}^2$ with $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2$, and $G = SO(2) = U(1) = \{e^{\mathrm{i}\alpha} : \alpha \in \mathbb{R}\}$. Then $\mathfrak{u}(1)$ is 1-dimensional, spanned by ∂_α. Show that the counterclockwise rotation action of $U(1)$ on X, given by the matrices

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{A.45}$$

has a moment map $\mu : \mathbb{R}^2 \to \mathfrak{u}(1)^*$, given by

$$\mu(x_1, x_2) \cdot \partial_\alpha = -\frac{1}{2}(x_1^2 + x_2^2). \tag{A.46}$$

Thus if we identify $\mathfrak{u}(1) \simeq \mathbb{R}$ using the generator ∂_α, we can think of μ just as an $\mathbb{R}$-valued function on X,

$$\mu(x_1, x_2) = -\frac{1}{2}(x_1^2 + x_2^2). \tag{A.47}$$

Exercise A.10. Suppose X is any manifold, with a compact group G acting. Then T^*X is a symplectic manifold which also has a canonical action of G. Verify that

$$\mu_Z(x, p) = -p \cdot (\rho(Z)(x)) \qquad x \in X, p \in T_x^*X \tag{A.48}$$

gives a moment map for this action.

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[27] When G is compact, moment maps exist at least *locally* on X, though maybe not globally. When G is not compact there can even be a local obstruction.

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