2d-4d wall-crossing and hyperholomorphic bundles

Andrew Neitzke, UT Austin
(work in progress with Davide Gaiotto, Greg Moore)

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Preface

Wall-crossing is an annoying/beautiful phenomenon that occurs in many contexts where something that naively was supposed to be "invariant" instead suddenly jumps.

In the context of BPS degeneracies it's been explored at some length. Now well understood both for 2d $\mathcal{N} = (2, 2)$ theories and 4d $\mathcal{N} = 2$ theories. Turns out to have interesting geometric applications, e.g. new construction of hyperkähler metrics.

I'll review these two cases, and describe a sort of hybrid of the two: surface operators in 4d $\mathcal{N} = 2$ theories.

Wall-crossing

In supersymmetric field/string theories one is often interested in BPS states: 1-particle states whose energy is the minimum allowed by the supersymmetry algebra.

Such states are to some extent "protected" from quantum corrections, because of the rigidity of short SUSY representations.

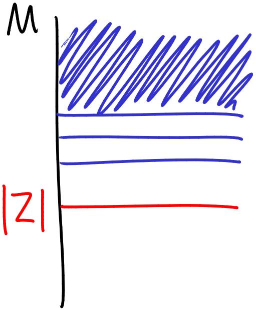
In particular, in many cases one can define an index which counts the number of such states (with some weights, like ± 1 for boson/fermion). This index is supposed to be a good invariant, and not to change when we vary parameters (e.g. vary from weak to strong coupling!)

It's exploited for e.g. the string theory approach to black hole entropy. [Strominger-Vafa]

Wall-crossing

There is an important loophole...

Invariance of the index only works as long as the 1-particle BPS Hilbert space doesn't mix with the multiparticle continuum.

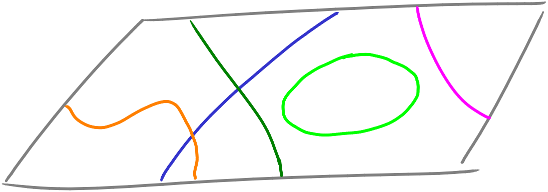


No mixing.

Mixing.

Wall-crossing

There are real-codimension-1 loci in parameter space where this mixing does occur. Call these loci walls.



At a wall, there are marginal bound states; as we cross the wall 1-particle states may decay, or conversely appear in the spectrum.

So index counting these states depends on parameters in a piecewise-constant fashion, jumps at the walls.

Question: How does it jump?

Wall-crossing

More precise question: the 1-particle Hilbert space is graded by conserved charges γ. So we can consider index

$$\Omega(\gamma, u) \in \mathbb{Z}$$

counting BPS states with charge γ in theory with parameters u.

Then ask: How does the collection $\{\Omega(\gamma, u)\}_{\gamma \in \Gamma}$ jump when u crosses a wall in parameter space?

Wall-crossing

This question has been answered in two important general settings:

- $\mathcal{N} = (2,2)$ theories in $d = 2$ [Cecotti-Vafa, Cecotti-Fendley-Intriligator-Vafa]
- $\mathcal{N} = 2$ theories in $d = 4$

  [Seiberg-Witten, Denef-Moore, Kontsevich-Soibelman, Gaiotto-Moore-Neitzke, Cecotti-Vafa, Dimofte-Gukov-Soibelma

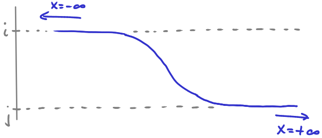
The two answers are remarkably parallel to one another, and both have some interesting associated geometry...

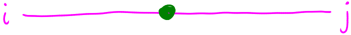
I'll review them in turn, and then discuss how they may be fused together.

Wall-crossing for $\mathcal{N} = (2,2)$ theories in $d = 2$

Suppose we have a massive $\mathcal{N} = (2,2)$ theory in $d = 2$, depending on parameters t.

Discrete vacua, labeled by $i = 1, \ldots, n$. Consider ij-solitons.





BPS bound is

$$M \geq |Z_{ij}|$$

where the "central charges" $Z_{ij}(t) \in \mathbb{C}$ obey

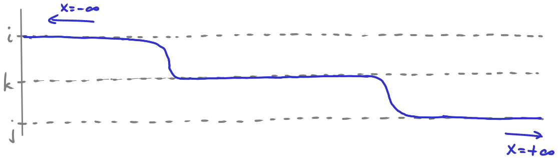
$$Z_{ij} + Z_{jk} = Z_{ik}$$

Index $\mu(ij, t) \in \mathbb{Z}$ counts BPS solitons.

Wall-crossing for $\mathcal{N} = (2,2)$ theories in $d = 2$

As we vary t, $\mu(ij, t)$ can jump when t crosses a wall.

Walls are loci where some $Z_{ik}/Z_{kj} \in \mathbb{R}_+$. Here a BPS ij-soliton can decay into ik-soliton plus kj-soliton.



The jump at the wall is

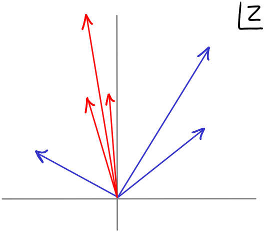
$$\mu(ij, t_+) - \mu(ij, t_-) = \pm\mu(ik)\mu(kj)$$

This is a wall-crossing formula (2d WCF). [Cecotti-Vafa]

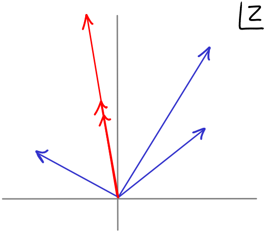
Wall-crossing for $\mathcal{N} = (2,2)$ theories in $d = 2$

Let's reformulate the 2d WCF.

At the wall, some collection of solitons become aligned, i.e. their central charges Z are all lying on the same ray in $\mathbb{C}$.



Near the wall.



On the wall.

Focus on these participating solitons only.

Wall-crossing for $\mathcal{N} = (2, 2)$ theories in $d = 2$

To each participating ij-soliton, assign an $n \times n$ matrix:

$$\mathcal{S}_{ij} = \mathbf{1} + e_{ij}$$

Now consider the object

$$: \prod_{ij} \mathcal{S}_{ij}^{\mu(ij)} :$$

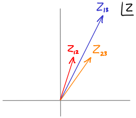
where :: means we multiply in order of the phase of Z_{ij}.

The WCF is the statement that this object is the same on both sides of the wall.

Wall-crossing for $\mathcal{N} = (2,2)$ theories in $d = 2$

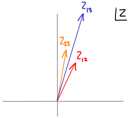
For example

$$\begin{pmatrix} 1 & \mu(12) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mu(13) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu(23) \\ 0 & 0 & 1 \end{pmatrix}$$



equals

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu'(23) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mu'(13) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu'(12) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



if and only if

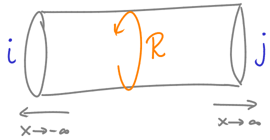
$$\begin{aligned} \mu'(12) &= \mu(12) \\ \mu'(23) &= \mu(23) \\ \mu'(13) &= \mu(13) + \mu(12)\mu(23) \end{aligned}$$

which is the WCF we wrote before.

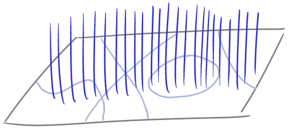
tt^* geometry

Original proof of the 2d WCF used tt^* geometry.

Complicated story, but simple basic idea: compactify the 2d theory on a circle and look at the metric $\langle i|j\rangle$ on the space of ground states, i.e. cylinder path integral.

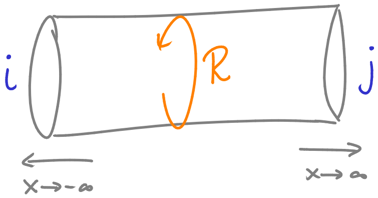


As we vary parameters t, the space of ground states forms a rank n complex vector bundle over parameter space.



It carries a Hermitian metric obeying an interesting system of integrable PDEs — related e.g. to Hitchin equations.

tt^* geometry



Puzzle:

- This quantity receives quantum corrections from solitons going around the compactification circle.
- Solitons can appear and disappear as t varies.
- But the answer should be continuous as a function of t (since the theory has no phase transition).

What's going on?

tt^* geometry

Answer: multisoliton contributions become comparable to 1-soliton contributions at the wall, ensure smoothness.

But only if the WCF is satisfied!

This gives a (slightly indirect) proof of the WCF.

Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

Now take a 4d $\mathcal{N} = 2$ theory.

Such a theory (often) has a Coulomb branch (moduli space of vacua): complex manifold of dimension r.

IR physics on the Coulomb branch is pretty simple:
supersymmetric abelian gauge theory, gauge group $U(1)^r$, couplings determined by prepotential $\mathcal{F}$.

Particles carry electromagnetic charge γ.

(e.g. for rank $r = 1$, $\gamma = (p, q)$ for $p, q \in \mathbb{Z}$.)

DSZ pairing $\langle \gamma, \gamma' \rangle$: $\langle (p, q), (p', q') \rangle = pq' - qp'$.

Central charges $Z_\gamma \in \mathbb{C}$ obeying

$$Z_\gamma + Z_{\gamma'} = Z_{\gamma+\gamma'}$$

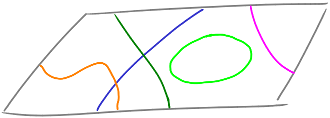
Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

BPS bound

$$M \geq |Z_\gamma|$$

Introduce index $\Omega(\gamma, u)$ counting BPS states of charge γ, in Coulomb branch vacuum u.

Walls occur at u for which $Z_\gamma / Z_{\gamma'} \in \mathbb{R}_+$ for some γ, γ' with $\langle \gamma, \gamma' \rangle \neq 0$.



Basically parallel to 2d case.

Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

Structure of WCF is also parallel to 2d case, but we need to replace the finite-dimensional matrices $\mathcal{S}_{ij}$ by something fancier:

Torus algebra with one generator X_γ for each γ,

$$X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}$$

Automorphism $\mathcal{K}_\gamma$ of this algebra:

$$\mathcal{K}_\gamma : X_{\gamma'} \mapsto (1 + X_\gamma)^{\langle \gamma, \gamma' \rangle} X_{\gamma'}$$

Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

At a wall, some group of BPS particles become aligned. (Maybe infinitely many.)

To each participating particle, assign the automorphism $\mathcal{K}_\gamma$.

Now consider the object

$$: \prod_\gamma \mathcal{K}_\gamma^{\Omega(\gamma)} :$$

where :: means we multiply in order of the phase of Z_γ.

The WCF is the statement that this object is the same on both sides of the wall.

[Kontsevich-Soibelman, ...]

Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

Simple example:

$$\mathcal{K}_{1,0}\mathcal{K}_{0,1} = \mathcal{K}_{0,1}\mathcal{K}_{1,1}\mathcal{K}_{1,0}$$

(electron and monopole form a single dyon bound state, which can appear/decay at the wall)

More interesting example (from $SU(2)$ Seiberg-Witten theory):

$$\mathcal{K}_{1,0}\mathcal{K}_{-1,2} = \left(\prod_{n=1}^{\infty} \mathcal{K}_{-1,2n}\right)\mathcal{K}_{0,2}^{-2}\left(\prod_{n=\infty}^{0} \mathcal{K}_{1,2n}\right)$$

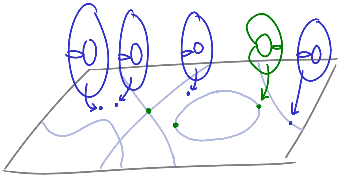
(monopole and dyon on one side, infinitely many dyons plus W boson on the other side)

Knowing one side determines the other by purely algebraic means!

Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

To prove this WCF: Compactify the theory on S^1 of radius R.

In the IR, the resulting theory looks 3-dimensional. Dualize all gauge fields into scalars to get a sigma model, whose target $\mathcal{M}$ is a fibration by compact $2r$-tori over the 4d Coulomb branch.



Supersymmetry implies $\mathcal{M}$ is hyperkähler. [Seiberg-Witten]

(If our $d = 4$ theory was obtained from the $d = 6$ $(2, 0)$ SCFT, then $\mathcal{M}$ is the Hitchin moduli space which occurred in several earlier talks.)

Wall-crossing for $\mathcal{N} = 2$ theories in $d = 4$

How to calculate the metric on $\mathcal{M}$?

Naive dimensional reduction from $\mathbb{R}^4$ to $\mathbb{R}^3 \times S^1$ gives an approximation ("semiflat metric"), exact in the limit $R \to \infty$, away from singular fibers. (Like the SYZ picture of a Calabi-Yau 3-fold, exact in the large complex structure limit.) [Cecotti-Ferrara-Girardello]

Puzzle:

- The exact metric receives quantum corrections from BPS instantons: the 4d BPS particles going around S^1.
- These BPS particles appear and disappear as u varies.
- But the metric should be continuous (no phase transition).

Hyperkähler geometry

What's going on? To find out, develop technology for incorporating the instanton corrections.

Answer: multiparticle contributions become comparable to 1-particle contributions at the wall, ensure smoothness of the metric on $\mathcal{M}$...

But only if the WCF is satisfied!

Hyperkähler geometry

On the one hand this story gives an explanation of the 4d WCF, and somewhat demystifies the torus algebra: it's just the algebra of functions on a coordinate patch of $\mathcal{M}$.

On the other hand it provides a new way of describing hyperkähler metrics on total spaces of integrable systems. The input data is an $\mathcal{N} = 2$ theory, its Seiberg-Witten solution (i.e. 4d IR effective action), and its BPS spectrum.

Hyperkähler geometry and TBA

To write an explicit formula for the metric on $\mathcal{M}$, one has to solve some interesting integral equations: of the form

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf} \exp\left[\sum_{\gamma'} \Omega(\gamma')\langle\gamma,\gamma'\rangle \frac{1}{4\pi i}\int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'}\log(1-\mathcal{X}_\gamma(\zeta'))\right]$$

Here $\mathcal{X}_\gamma$ are "holomorphic Darboux coordinates" on $\mathcal{M}$, also functions of twistor parameter $\zeta \in \mathbb{C}^\times$ which keeps track of the complex structures on $\mathcal{M}$.

These equations have exactly the form of the thermodynamic Bethe ansatz for a 2d theory w/ factorized scattering. [Zamolodchikov]

Open questions:

- Where did this 2d theory come from? (Why on earth would the rapidity be related to the twistor parameter?)
- What does wall-crossing mean in the 2d theory?

A connection to $\mathcal{N} = 4$

Suppose we take our $\mathcal{N} = 2$ theory to be the "n-th Argyres-Douglas-type SCFT," characterized by Seiberg-Witten curve [Argyres-Douglas, Argyres-Plesser-Seiberg-Witten]

$$y^2 = x^{n+2} + (\text{lower order})$$

This is one of the examples where $\mathcal{M}$ is a Hitchin system (with irregular singularity).

This Hitchin system is the same one that governs strings in AdS_3 with the "polygon boundary conditions" that appeared in the strong-coupling $\mathcal{N} = 4$ SYM computations from several previous talks. $(2n + 8 = \text{number of gluons})$ [subsets of Alday-Gaiotto-Maldacena-Sever-Vieira]

Our integral equation in that case is the one which appeared in those talks.

Open question: Does this connection mean anything?

Wall-crossing for combined 2d-4d theories

2d and 4d stories were very parallel. Now let's combine them.

Consider a 4d $\mathcal{N} = 2$ theory with a surface defect preserving $d = 2$, $\mathcal{N} = (2, 2)$ supersymmetry. [...]

Example:

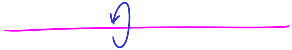
- 4d $\mathcal{N} = 2$ $SU(2)$ gauge theory in $\mathbb{R}^4$
- 2d supersymmetric sigma model into $\mathbb{CP}^1$, supported on $\mathbb{R}^2 \subset \mathbb{R}^4$

Couple the two by using 4d gauge fields to gauge the global $SU(2)$ isometry group of $\mathbb{CP}^1$.

Wall-crossing for combined 2d-4d theories

In the IR: 4d abelian gauge theory, as before. Assume surface defect is massive in the IR.

Factor out the time direction. Surface defect looks like a string in space.

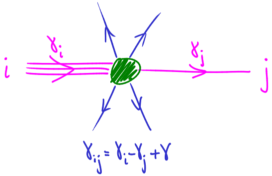


It creates a boundary condition for the gauge fields: fixed holonomy around the string. So particles transported around the string pick up a phase; like the Aharonov-Bohm effect created by a solenoid.

Wall-crossing for combined 2d-4d theories

The flux through the solenoid depends on the IR data: both the Coulomb branch modulus u and the discrete choice of vacuum i on surface defect.

In particular, if we have a soliton on the surface defect, the flux changes across the soliton, in a non-quantized way: some of it must have escaped into the 4d bulk! In other words, 2d solitons carry (fractional) 4d gauge charge.



So they can form bound states with / decay into 4d particles as well as other 2d particles: 2d-4d wall-crossing.

Wall-crossing for combined 2d-4d theories

2d-4d wall-crossing phenomena are governed by a kind of hybrid of the two WCF we had before. Each BPS state corresponds to a certain automorphism:

Automorphism of what?

- In 2d case, a vector space.
- In 4d case, a complex torus.
- In 2d-4d case, a vector bundle over a complex torus.

A 2d BPS state of charge γ_{ij} gives an endomorphism $\mathcal{S}_{\gamma_{ij}}$ of the bundle. A 4d BPS state of charge γ gives (roughly!) an automorphism $\mathcal{K}_\gamma$ of the torus, lifted to act on the bundle.

Wall-crossing for combined 2d-4d theories

2d-4d WCF says that

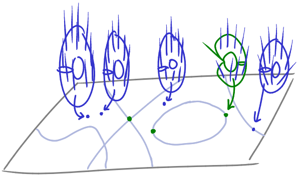
$$: \prod_{i,j,\gamma_{ij}} \mathcal{S}_{ij}^{\mu(\gamma_{ij})} \prod_{\gamma} \mathcal{K}_{\gamma}^{\Omega(\gamma)} :$$

remains constant as we cross wall.

Hyperholomorphic geometry

Upon compactification to 3d, we get a sigma model into hyperkähler manifold $\mathcal{M}$ as before, now with an extra line operator inserted.

The line operator couples to a connection A in a vector bundle V over $\mathcal{M}$. V is just the bundle of vacua of the surface defect on S^1.



Supersymmetry requires that A is a hyperholomorphic connection. (Curvature of type $(1,1)$ in all complex structures.)

Hyperholomorphic geometry

Hyperholomorphic connections are the same kind of objects that appeared in Gukov's talk. There they appeared as D-branes in a 2-dimensional sigma model into $\mathcal{M}$.

To recover that picture here, compactify from 3d to 2d on a circle surrounding the line operator: get 2d sigma model on a half-space, with boundary condition coming from the line operator.

(In fact, can also understand our construction of the brane as mirror symmetry: the IR Lagrangian fixes a certain BAA brane which is mirror to the hyperholomorphic bundle, i.e. BBB brane, which we are constructing.)

Hyperholomorphic geometry

On the one hand this story explains why the 2d-4d WCF is true.

On the other hand it provides a new way of describing hyperkähler spaces with hyperholomorphic vector bundles. The input data is an $\mathcal{N} = 2$ theory with surface defect, its IR effective action (Seiberg-Witten solution in 4d plus effective superpotential in 2d), and its 2d-4d BPS spectrum.

A special case of this is constructing solutions to Hitchin equations. This relates to some classical geometric questions! For example, revisiting the application to strong-coupling $\mathcal{N} = 4$ computations, it would give not just the minimal area of the string worldsheet but also the actual minimizing configuration. It is also related to the classical problem of uniformization.

To write an explicit formula for the hyperholomorphic connection, one again has to solve some interesting integral equations: generalization of the TBA we had before.