

A wall-crossing formula for 2d-4d DT invariants

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String-Math, June 2011

Preface

In the last few years there has been a lot of progress in the theory of **generalized Donaldson-Thomas invariants**.

[Kontsevich-Soibelman, Joyce-Song, ...]

In many (all?) cases where they can be defined, these invariants have a physical meaning: **BPS state index in 4d $\mathcal{N} = 2$ SUSY quantum systems**.

[Ooguri-Strominger-Vafa, Denef, Denef-Moore, Gaiotto-Moore-N, Dimofte-Gukov-Soibelman, Cecotti-Vafa, ...]

Preface

This talk is motivated by some new progress in physics, involving new “2d-4d” BPS state indices, which can be defined when we add a 2d surface operator to our 4d quantum system.

We have learned a few basic facts about these 2d-4d indices — in particular we have learned what their wall-crossing formula is. The main aim of this talk is to explain these facts.

We conjecture that the 2d-4d indices should be part of a not-yet-formulated extension of Donaldson-Thomas theory.

As Davide explained in his talk, these 2d-4d indices seem to be useful tools even if your ultimate interest is only in the original generalized DT invariants.

Preface

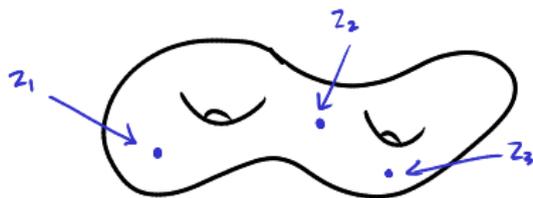
I'll focus on a specific class of situations where both the generalized DT invariants and the new 2d-4d invariants can be defined relatively easily, and the new 2d-4d wall-crossing formula can be directly checked.

In these examples, the generalized DT invariants (as well as their 2d-4d extensions) are counting special **trajectories** associated with a **quadratic differential** on a Riemann surface C .

First, I'll review the "old" story; then I'll give its 2d-4d extension.

Invariants of a quadratic differential

Fix **compact Riemann surface** C , with $n > 0$ marked points z_i , $i = 1, \dots, \ell$. Let C' be C with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ for each i .



Let \mathcal{B} be the space of **meromorphic quadratic differentials** φ_2 on C , with double pole at each z_i , residue m_i^2 :

$$\varphi_2 = \frac{m_i^2}{(z - z_i)^2} dz^2 + \dots$$

Full disclosure

In what follows, if you want to know what is really proven, stick to C of genus zero. All the phenomena I'll discuss occur already in that case.

Invariants of a quadratic differential

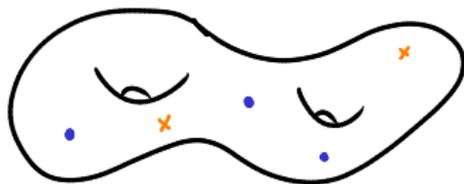
Fix a point of \mathcal{B} , i.e. fix a **meromorphic quadratic differential** φ_2 on C with double pole at each z_i , residue m_i .

This determines a **metric** h on C , in a simple way:

$$h = |\varphi_2|$$

(so if $\varphi_2 = P(z) dz^2$ then $h = |P(z)| dz d\bar{z}$.)

More precisely, h is a metric on only an open subset of C , where we delete both the **poles** of φ_2 (the z_i) and also the **zeroes** of φ_2 . h is **flat** on this open subset.

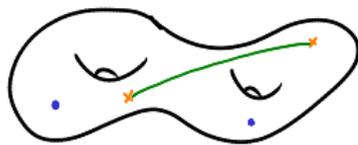


- poles of φ_2
- x zeroes of φ_2

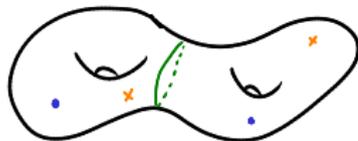
Invariants of a quadratic differential

Now we can consider **finite length inextendible geodesics** on C' in the metric h . These come in two types:

- ▶ **Saddle connections**: geodesics running between two **zeroes** of φ_2 . These are **rigid** (don't come in families).



- ▶ **Closed geodesics**. When they exist, these come in **1-parameter families**, sweeping out annuli on C' .

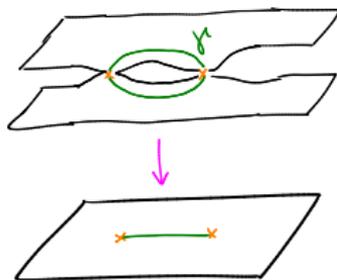


Invariants of a quadratic differential

To “classify” these finite length geodesics, introduce a little more technology: φ_2 determines a **branched double cover** $\Sigma \rightarrow C'$,

$$\Sigma = \{x : x^2 = \varphi_2\} \subset T^*C.$$

Each finite length geodesic can be **lifted** to a union of closed curves in Σ , representing some homology class $\gamma \in H_1(\Sigma, \mathbb{Z})^{\text{odd}}$.

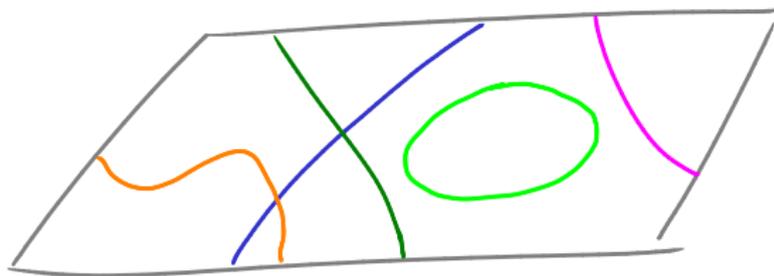


We define an invariant

$$\Omega(\gamma) = \begin{cases} 1 & \text{if there is a saddle connection w/ lift } \gamma \\ -2 & \text{if there is a closed loop w/ lift } \gamma \\ 0 & \text{if neither} \end{cases}$$

Walls

As we vary φ_2 , $\Omega(\gamma)$ can **jump**, when some finite-length geodesics appear or disappear. This occurs at some real-codimension-1 loci $W \subset \mathcal{B}$ (“walls”).



Walls

As we vary φ_2 , $\Omega(\gamma)$ can **jump**, when some finite-length geodesics appear or disappear.

Basic mechanism: decay/formation of **bound states**.



here, $\Omega(\gamma_1) = 1$ $\Omega(\gamma_2) = 1$ $\Omega(\gamma_1 + \gamma_2) = 1$	here, $\Omega(\gamma_1) = 1$ $\Omega(\gamma_2) = 1$ $\Omega(\gamma_1 + \gamma_2) = 0$
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\mathcal{B}

Where is the wall? Define a function Z_γ on \mathcal{B} (the parameter-space of quadratic differentials φ_2) by

$$Z_\gamma = \oint_\gamma \lambda$$

where λ is the canonical 1-form on T^*C . Then the wall is the locus in \mathcal{B} where

$$Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}.$$

Wall-crossing formula

The jump of the $\Omega(\gamma)$ at the wall is governed by the Kontsevich-Soibelman WCF.

To state that formula (which will take a few slides), we **axiomatize** our structure a bit: the data are

- ▶ Complex manifold \mathcal{B} (space of quadratic differentials on C)
- ▶ Lattice Γ w/ antisymmetric pairing \langle, \rangle ($H^1(\Sigma, \mathbb{Z})^{odd}$)
- ▶ Homomorphism $Z : \Gamma \rightarrow \mathbb{C}$ for each point of \mathcal{B} , varying holomorphically over \mathcal{B} ($Z_\gamma = \oint_\gamma \lambda$)
- ▶ “invariants” $\Omega : \Gamma \rightarrow \mathbb{Z}$ for each point of \mathcal{B} (counts of finite length geodesics)

Wall-crossing formula

Attach a “ \mathcal{K} -ray” in \mathbb{C} to each γ with $\Omega(\gamma) \neq 0$.

Slope of the \mathcal{K} -ray is given by the argument of Z_γ .

These rays move around as we vary the quadratic differential φ_2 ,
i.e. as we move in \mathcal{B} .

Walls in \mathcal{B} are loci where some set of rays become aligned:



Focus on these **participating** rays.

Wall-crossing formula

Introduce **torus** $T \simeq (\mathbb{C}^\times)^{\text{rank } \Gamma}$ with coordinate functions $X_\gamma : T \rightarrow \mathbb{C}^\times$ for each $\gamma \in \Gamma$, obeying

$$X_\gamma X_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}.$$

To each γ , assign a (birational) **automorphism** \mathcal{K}_γ of T :

$$\mathcal{K}_\gamma : X_{\gamma'} \mapsto (1 - X_\gamma)^{\langle \gamma, \gamma' \rangle} X_{\gamma'}$$

Now consider a product over all participating γ ,

$$: \prod_{\gamma} \mathcal{K}_\gamma^{\Omega(\gamma)} :$$

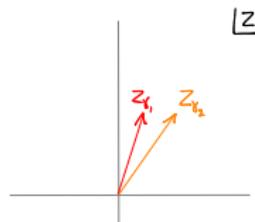
where $::$ means we multiply in **order** of the phase of Z_γ .

The Kontsevich-Soibelman WCF is the statement that this automorphism is **the same** on both sides of the wall.

Wall-crossing formula

For example: if $\langle \gamma_1, \gamma_2 \rangle = 1$,

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2}$$



equals

$$\mathcal{K}_{\gamma_2}^{\Omega'(\gamma_2)} \mathcal{K}_{\gamma_1 + \gamma_2}^{\Omega'(\gamma_1 + \gamma_2)} \mathcal{K}_{\gamma_1}^{\Omega'(\gamma_1)}$$



if and only if

$$\Omega'(\gamma_1) = 1$$

$$\Omega'(\gamma_2) = 1$$

$$\Omega'(\gamma_1 + \gamma_2) = 1$$

Wall-crossing formula

More interesting example: if $\langle \gamma_1, \gamma_2 \rangle = 2$,

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2} = \left(\prod_{n=0}^{\infty} \mathcal{K}_{n\gamma_1 + (n+1)\gamma_2} \right) \mathcal{K}_{\gamma_1 + \gamma_2}^{-2} \left(\prod_{n=\infty}^0 \mathcal{K}_{(n+1)\gamma_1 + n\gamma_2} \right)$$

So,

- ▶ on one side of the wall we have only $\Omega(\gamma_1) = 1$ and $\Omega(\gamma_2) = 1$, all others zero;
- ▶ on the other side we have **infinitely** many $\Omega(\gamma) = 1$, and also $\Omega(\gamma_1 + \gamma_2) = -2$.

Wall-crossing formula

Key fact: KS WCF holds for our integer invariants $\Omega(\gamma)$!

So e.g.

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_2} \mathcal{K}_{\gamma_1 + \gamma_2} \mathcal{K}_{\gamma_1}$$

for $\langle \gamma_1, \gamma_2 \rangle = 1$ says that if we have two saddle connections that intersect at 1 point, then after wall-crossing a **third** saddle connection will appear.

Similarly in the formula

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2} = \left(\prod_{n=0}^{\infty} \mathcal{K}_{n\gamma_1 + (n+1)\gamma_2} \right) \mathcal{K}_{\gamma_1 + \gamma_2}^{-2} \left(\prod_{n=0}^0 \mathcal{K}_{(n+1)\gamma_1 + n\gamma_2} \right)$$

for $\langle \gamma_1, \gamma_2 \rangle = 2$, on one side we have two saddle connections intersecting at two points; on the other side we have infinitely many **saddle connections** plus a single **closed geodesic**.

Segue

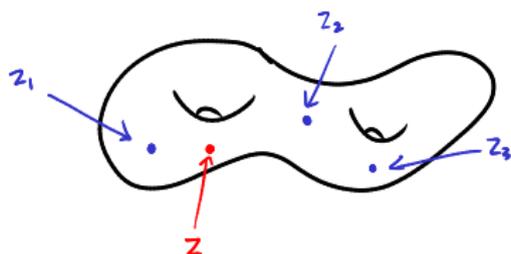
So far, so good: we described a simple class of enumerative invariants $\Omega(\gamma) \in \mathbb{Z}$ attached to a punctured curve C , and explained that they give examples of the general wall-crossing formula of Kontsevich-Soibelman.

Now, we consider our “2d-4d” extension. $\Omega(\gamma) \in \mathbb{Z}$ will be slightly refined to some new objects $\omega(\gamma, \gamma_{ij}) \in \mathbb{Z}$, and we will also introduce new $\mu(\gamma_{ij}) \in \mathbb{Z}$.

Invariants of a quadratic differential plus a point

As before: Fix compact Riemann surface C , with $n > 0$ marked points z_i , $i = 1, \dots, n$. Let C' be C with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ for each i .

New 2d-4d datum: Fix a **point** $z \in C$.



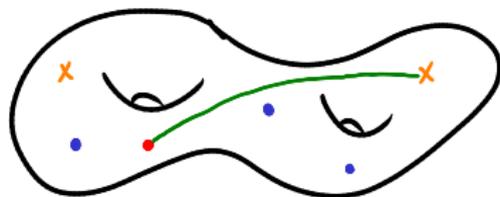
As before: let \mathcal{B} be the space of meromorphic quadratic differentials φ_2 on C , with double pole at each z_i , residue m_i^2 :

$$\varphi_2 = \frac{m_i^2}{(z - z_i)^2} dz^2 + \dots$$

Invariants of a quadratic differential plus a point

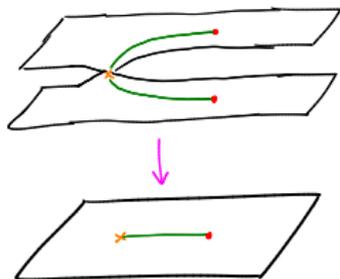
As before: we are interested in **counting finite-length geodesics** on C' , in the flat metric h determined by φ_2 .

However, now we allow them to be **open**, i.e. to have one **end** on the point z . (And the other end on a zero of φ_2 as before.)



Invariants of a quadratic differential plus a point

To categorize these **open** geodesics, we again consider their **lifts** to the double cover Σ :



These give 1-chains γ_{ij} with $\partial\gamma_{ij} = z_i - z_j$; let Γ_{ij} be set of such 1-chains modulo boundaries.

Γ_{ij} is a **torsor** over the homology Γ . For any $\gamma_{ij} \in \Gamma_{ij}$, we define new invariant $\mu(\gamma_{ij})$ by

$$\mu(\gamma_{ij}) = \begin{cases} 1 & \text{if there is an open geodesic w/ lift } \gamma_{ij} \\ 0 & \text{if not} \end{cases}$$

Invariants of a quadratic differential plus a point

In the presence of the extra point z , we can also keep track of slightly **more information** about the ordinary finite geodesics: measure their homology classes on Σ **punctured** at the preimages of z .

To encode that information: for any $\gamma \in \Gamma$ and $\gamma_{ij} \in \Gamma_{ij}$, we define new invariant $\omega(\gamma, \gamma_{ij})$, by

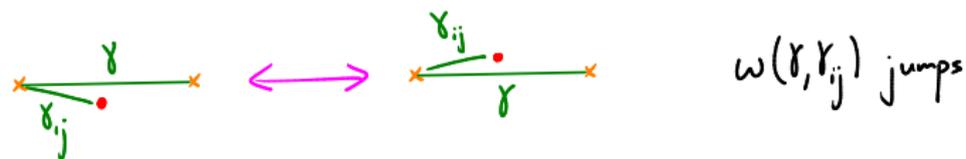
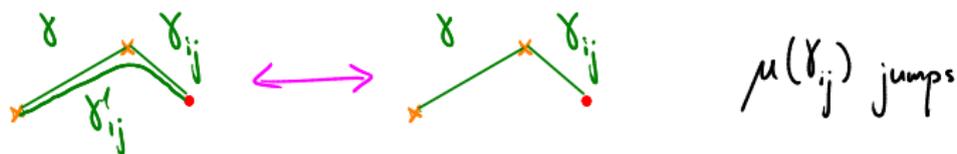
$$\omega(\gamma, \gamma_{ij}) = \Omega(\gamma) \langle \gamma, \gamma_{ij} \rangle$$

(To define the intersection number with the open path γ_{ij} , if γ is an isolated geodesic, use the actual geodesic representative for γ . If γ not isolated, use the two ends of the family, take the average.)

Walls

As before: as we vary φ_2 and z , $\mu(\gamma)$ and $\omega(\gamma, \gamma_{ij})$ can **jump**.

Two sample pictures:



2d-4d wall-crossing formula

To state our extended “2d-4d” WCF, **axiomatize** our new structure: to our old list

- ▶ Complex manifold \mathcal{B} (space of quadratic differentials on C)
- ▶ Lattice Γ w/ antisymmetric pairing \langle, \rangle ($H^1(\Sigma, \mathbb{Z})^{odd}$)
- ▶ Homomorphism $Z : \Gamma \rightarrow \mathbb{C}$ for each point of \mathcal{B} , varying holomorphically over \mathcal{B} ($Z_\gamma = \oint_\gamma \lambda$)
- ▶ “invariants” $\Omega : \Gamma \rightarrow \mathbb{Z}$ for each point of \mathcal{B} (counts of finite geodesics)

we now add

- ▶ Γ -torsors Γ_{ij} for $i, j = 1, \dots, n$, with addition operations $\Gamma_{ij} \times \Gamma_{jk} \rightarrow \Gamma_{ik}$, satisfying associativity ($n = 2$, spaces of 1-chains with boundary $z_i - z_j$)
- ▶ Maps $Z : \Gamma_{ij} \rightarrow \mathbb{C}$ obeying additivity ($Z_{\gamma_{ij}} = \int_{\gamma_{ij}} \lambda$)
- ▶ “invariants” $\omega : \Gamma \times \Gamma_{ij} \rightarrow \mathbb{Z}$, satisfying $\omega(\gamma, \gamma' + \gamma_{ij}) = \omega(\gamma, \gamma_{ij}) + \Omega(\gamma)\langle \gamma, \gamma' \rangle$ (refined counts of finite geodesics)
- ▶ “invariants” $\mu : \Gamma_{ij} \rightarrow \mathbb{Z}$ for each i, j with $i \neq j$, and each point of \mathcal{B} (counts of open geodesics)

2d-4d wall-crossing formula

As before, attach a ray in \mathbb{C} to each nonzero invariant: “ \mathcal{K} -ray” for each γ with $\omega(\gamma, \cdot) \neq 0$, “ \mathcal{S} -ray” for each γ_{ij} with $\mu(\gamma_{ij}) \neq 0$.

Slope of the rays given by the argument of Z_γ or $Z_{\gamma_{ij}}$.

The rays move around as we vary the quadratic differential φ_2 and the point z , i.e. as we move in $\mathcal{B} \times \mathcal{C}$.

As before, **walls** in $\mathcal{B} \times \mathcal{C}$ are loci where some set of rays become aligned:



Focus on these **participating** rays.

2d-4d wall-crossing formula

To formulate the KS WCF, we used an auxiliary gadget, the **torus** $T \simeq (\mathbb{C}^\times)^{\text{rank } \Gamma}$. The $\Omega(\gamma)$ got encoded into automorphisms $\mathcal{K}_\gamma^{\Omega(\gamma)}$ of T .

For the 2d-4d WCF, we **decorate** that story a bit: add a trivializable holomorphic rank- n **vector bundle** V over T . The 2d-4d invariants get encoded into automorphisms of that object:

- ▶ The $\omega(\gamma, \cdot)$ contain slightly more information than $\Omega(\gamma)$; correspondingly, they determine an object $\mathcal{K}_\gamma^\omega$, which lifts $\mathcal{K}_\gamma^{\Omega(\gamma)}$ to act on V .
- ▶ The new invariants $\mu(\gamma_{ij})$ give new automorphisms $\mathcal{S}_{\gamma_{ij}}^{\mu(\gamma_{ij})}$ which leave points of T fixed, act only on the fiber of V . (Unipotent matrices with one off-diagonal entry, in the ij place.)

2d-4d wall-crossing formula

Just to show you that the formulas are concrete:

$\mathcal{K}_\gamma^\omega$ and $\mathcal{S}_{\gamma ij}$ are automorphisms of a vector bundle V over T . I'll give their action on a basis of sections of $End(V)$: so along with the X_γ we had before (functions on T), now also have sections $X_{\gamma ij}$ ("elementary matrix" sections of $End(V)$), and the automorphisms act by

$$\begin{aligned}\mathcal{K}_\gamma^\omega : X_{\gamma ij} &\mapsto (1 - X_\gamma)^{\omega(\gamma, \gamma ij)} X_{\gamma ij} \\ \mathcal{S}_{\gamma kl}^\mu : X_{\gamma ij} &\mapsto (1 + \mu X_{\gamma kl}) X_{\gamma ij} (1 - \mu X_{\gamma kl})\end{aligned}$$

This is enough for our purposes.

Wall-crossing formula

Now consider a product over all participating rays

$$: \prod_{\gamma, \gamma_{ij}} \kappa_{\gamma}^{\omega} S_{\gamma_{ij}}^{\mu} :$$

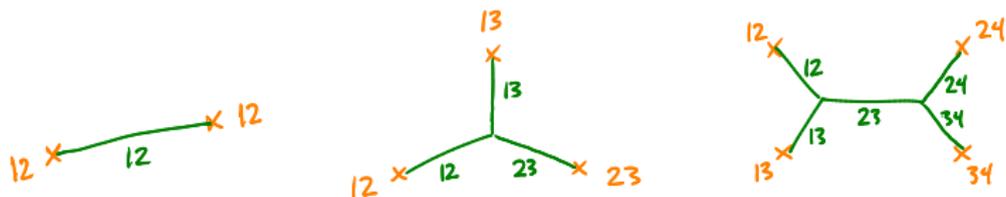
where $::$ means we multiply in **order** of the phase of Z_{γ} , $Z_{\gamma_{ij}}$.

This object is an automorphism of the torus T , lifted to act on the vector bundle V .

The 2d-4d WCF is the statement that this automorphism is **the same** on both sides of the wall.

Networks

Story so far might seem a little lame: 2d-4d WCF involves an integer n , but my only example of 2d-4d invariants had $n = 2$. There is a more general version: the quadratic differential φ_2 is replaced by a tuple of k -differentials $(\varphi_k)_{2 \leq k \leq n}$, and instead of counting geodesics on C , we count certain **networks**:

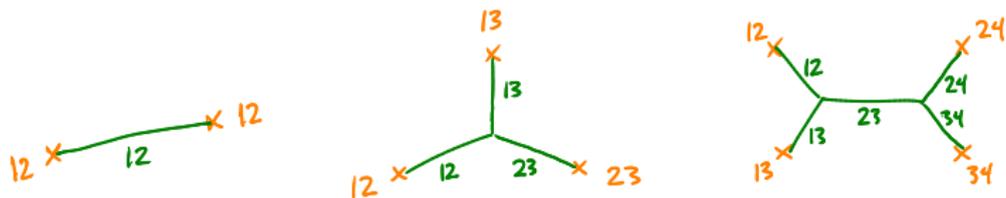


Each leg is labeled by a pair of sheets (x_i, x_j) of the n -fold covering

$$\Sigma = \left\{ x^n + \sum_{k=0}^{n-2} x^k \varphi_{n-k} = 0 \right\} \subset T^*C$$

and is a straight line in the coordinate $\int x_i - x_j$, with inclination ϑ , the same for all legs.

Networks



We don't know anywhere that these **networks** have been studied before (would be very curious to learn a reference!)

They are a natural generalization of the trajectories of quadratic differentials, which are somewhat better-studied. [Strebel, Masur, ...]

We expect that the counting of these objects — and their open analogues, with one leg ending on a marked point z — is governed by our 2d-4d WCF (details still being checked).

2d-4d wall-crossing formula

The 2d-4d WCF does not come out of nowhere.

Cecotti-Vafa previously studied purely 2d field theories and gave a WCF with structure similar to the KS WCF and the 2d-4d WCF.

In that WCF, the relevant group of “automorphisms” which appears is just $GL(n)$.

The 2d-4d WCF is a kind of **hybrid** between that WCF and the KS WCF.

(It fits into a general formalism where the “invariants” are allowed to belong to a general graded Lie algebra.)

[Kontsevich-Soibelman]

Applications

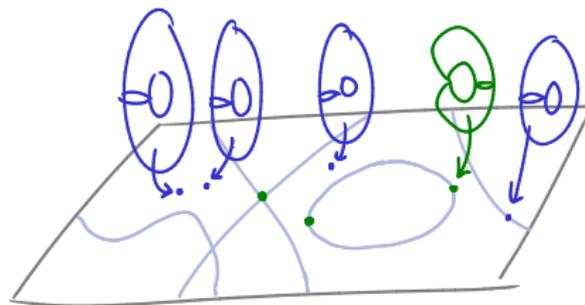
What are the 2d-4d invariants **good for**?

As mentioned in Davide's talk, if you are in a situation where the 2d-4d invariants can be defined, knowing μ is enough to give information about the original generalized DT invariants Ω .

Indeed in many cases μ actually determines Ω ! And μ seem to be somewhat more easily computable...

A geometric application

One application of the KS WCF: a construction of a family of **hyperkähler metrics** on the total space \mathcal{M} of a **torus bundle** over \mathcal{B} .

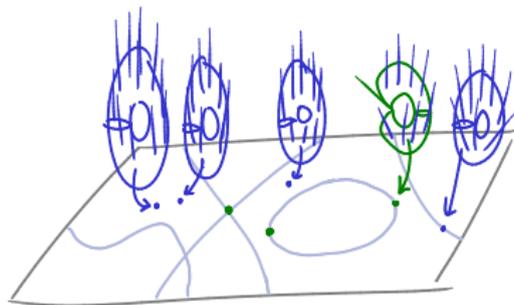


Very roughly, construction proceeds by **gluing together** patches which look like the torus T , with the gluing maps given by the automorphisms $\mathcal{K}_\gamma^{\Omega(\gamma)}$. [Strominger-Yau-Zaslow, Kontsevich-Soibelman, Auroux, Gross-Siebert, ...]

In the examples I have been discussing here, \mathcal{M} is a **moduli space** of solutions of **Hitchin equations** over C with gauge group $SU(n)$.

A geometric application

The 2d-4d WCF gives a new construction of a **hyperholomorphic bundle** \mathcal{V} over this hyperkähler \mathcal{M} (think of this as a generalization of a **Yang-Mills instanton**.) Very roughly, construction proceeds by gluing together patches which look like the vector bundle V over T , with gluing maps given by $\mathcal{K}_\gamma^\omega$ and $\mathcal{S}_{\gamma ij}^{\mu ij}$.



In the examples I have been discussing here, \mathcal{M} is a moduli space of solutions of Hitchin equations over C with gauge group $SU(k)$, \mathcal{V} is the **universal bundle** restricted to $z \in C$.

2d-4d invariants in general

We conjecture that 2d-4d invariants obeying the 2d-4d WCF exist in other “Donaldson-Thomas situations” as well.

Roughly speaking, whenever we have a (A or B) brane whose moduli space is 0-dimensional, we should be able to use it to define a 2d-4d version of the counting of (B or A) branes.

(A funny-looking mixing of the two mirror-dual categories!)

2d-4d invariants in general

For example, suppose we have a Calabi-Yau threefold X . The standard dogma is that there should be generalized Donaldson-Thomas invariants $\Omega(\gamma)$ which “count” **special Lagrangian cycles (A branes)** on X , for $\gamma \in H_3(X, \mathbb{Z})$.

Now fix a class in $H_2(X, \mathbb{Z})$ supporting finitely many **holomorphic curves** Y_1, \dots, Y_n (**B branes**).

In this situation, with luck, there should be **2d-4d invariants** $\omega(\gamma, \gamma_{ij})$ and $\mu(\gamma_{ij})$, with $i, j = 1, \dots, n$, and γ_{ij} lying in torsors Γ_{ij} for $H_3(X, \mathbb{Z})$ (roughly **3-chains with boundary** $Y_i - Y_j$).

I am not bold enough to try to say today how they should be defined.

Conclusions

Motivated by physics, we propose that there should exist a new “2d-4d” extension of the usual theory of generalized Donaldson-Thomas invariants. $\Omega(\gamma)$ replaced by $\mu(\gamma_{ij})$ and $\omega(\gamma, \gamma_{ij})$.

In particular, we claim this new theory governs the counting of certain open and closed **networks** of trajectories on Riemann surfaces.

While we don't know how to define this theory in general, we do know what its **wall-crossing** behavior should be.