1 Preface

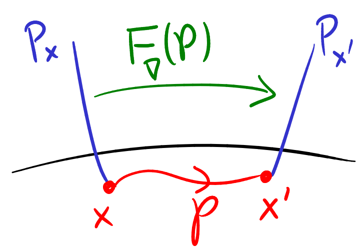
Two problems involving flat connections over manifolds:

- computing monodromy of Schrödinger operators on surfaces,
- computing Chern-Simons invariants of flat connections.

Unifying tool useful for both: *abelianization*. Arose in SUSY gauge theory (work with Gaiotto, Moore) but suppress that in this talk.

2 Setup

M a manifold: a G*-connection* ∇ over M is a principal G-bundle P plus a notion of parallel transport: $F_\nabla(\wp) \in Hom(P_x, P_{x'})$.



Call ∇ *flat* if $F_\nabla(\wp)$ depends only on homotopy class of $\wp$. Two connections *equivalent* if have map of principal bundles $P \to P'$ commuting with the F.

If we fix a section s of P ("fix a gauge") then $F_\nabla(\wp)$ takes $s(x)$ to $gs(x')$ for some $g \in G$; or in short, $F_\nabla(\wp) \in G$. In particular, fixing a basepoint x this gives a rep $\pi_1(M) \to G$. There is an identification of moduli spaces

$$\mathcal{M}(M,G) = \{\text{flat } G\text{-connections}\}/\sim \quad = \quad \{\text{reps } \pi_1(M) \to G\}\}/\sim$$

3 Abelianization on S^1

Take $G = GL_N\mathbb{C}$.

First $M = S^1$. Then a flat G-connection ∇ over M means a *matrix*,

$$A \in GL_N\mathbb{C}$$

If A is generic, it can be diagonalized, in finitely many ways. Eigenvalues give local coordinates on $\mathcal{M}(M, G) = GL_N\mathbb{C}/\sim$.

In terms of ∇ this means: we can trivialize P in such a way that all $F_\nabla(\wp)$ are diagonal.

4 Abelianization on punctured torus

How about two matrices?

$$A, B \in GL_N\mathbb{C}$$

modulo simultaneous conjugation. Let's think of A, B as giving a representation

$$\pi_1(M) \to GL_N\mathbb{C}$$

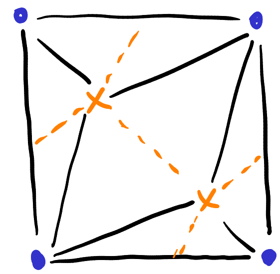
where M is the once-punctured torus. i.e. a flat connection ∇ over M.

If A, B don't commute, hopeless to simultaneously diagonalize them. So, we won't be able to find a global diagonal gauge. But there's something you can do.

Take $N = 2$ for a minute, so $G = GL_2\mathbb{C}$. Now draw a picture $\mathcal{W}$ on M:

We can find a gauge off $\mathcal{W}$ such that:

- parallel transports off $\mathcal{W}$ are diagonal;



- parallel transports across dashed lines are strictly off-diagonal;
- parallel transports across solid lines are upper-triangular unipotent.

Call such a gauge an *abelianization* (wrt $\mathcal{W}$). Two abelianizations are equivalent if they differ by a diagonal gauge transformation.

Facts:

- a generic A, B can be abelianized in 2 ways, up to equivalence.
- The diagonal/off-diagonal parallel transports. assemble into an almost-flat $GL_1(\mathbb{C})$-connection ∇^{ab} over a branched double cover $\Sigma \to M$.

  Thus we get an element $\mathcal{X}_\gamma \in \mathbb{C}^\times$ (up to sign) for each $\gamma \in H_1(\Sigma, \mathbb{Z})$. These are analogues of the *eigenvalues* for a single matrix. They give local coordinates on moduli space $\mathcal{M}(C, G)$ of pairs A, B (identify it with $(\mathbb{C}^\times)^5$).

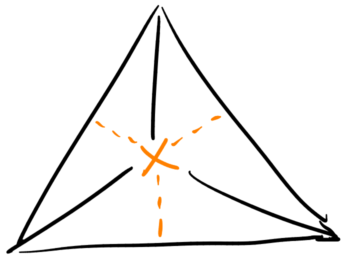
  The abelianization amounts to an isomorphism

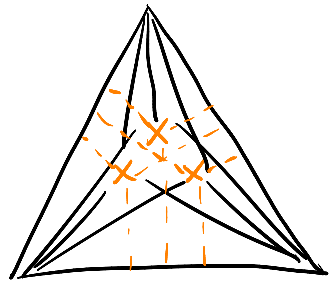
  $$\iota : \nabla \simeq \pi_* \nabla^{\mathrm{ab}}$$

  off the walls, which has unipotent jumps.

Similar statements for any punctured surface with an ideal triangulation, and for any N. eg for $N = 2, 3$:

Properties:





- the $\mathcal{X}_\gamma$ give *local Darboux coordinates* for Atiyah-Bott-Goldman Poisson structure on moduli spaces of representations: $\{\mathcal{X}_\gamma, \mathcal{X}_\mu\} = \langle\gamma, \mu\rangle \mathcal{X}_{\gamma+\mu}$. for $N = 2$, *complexified shear coordinates*. for higher N, related to *cluster coordinates* [Fock-Goncharov]
- changing triangulations gives coordinate systems related by *cluster transformation*, like $(x, y) \to (x(1 + y), y)$.
- invariants of the representation like $\mathrm{Tr}\, ABABA^{-1}$ are finite sums of $\mathcal{X}_\gamma$. (So for each loop on M there's corresponding finite bunch of loops on Σ.)

5 Opers

Now fix a *Riemann surface* $M = C$ and a meromorphic quadratic differential ϕ_2 on C. This determines *Schrödinger operators*: diff ops $D_{\hbar,\phi_2} : K_C^{-1/2} \to K_C^{3/2}$, locally like $\hbar^2\partial_z^2 + \phi_2(z)$; analytic continuation of solutions gives flat $SL_2\mathbb{C}$-connections

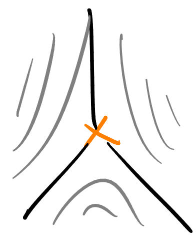
$$\nabla_{\hbar,\phi_2}$$

and corresponding monodromy representation

$$\rho_{\hbar,\phi_2} : \pi_1(C) \to SL_2\mathbb{C}$$

A basic question: *compute* $\rho_{\hbar,\phi_2}$. Concretely, e.g. this could mean computing traces $\mathrm{Tr}\, F_{\nabla_{\hbar,\phi_2}}(\wp)$.

For this: define *critical graph* $\mathcal{W}(\vartheta, \phi_2)$ as follows. C has foliation with the leaves defined by condition: $e^{-\mathrm{i}\vartheta}\sqrt{\phi_2}$ real. Singular at zeroes and poles of ϕ_2. Now, suppose ϕ_2 has only simple zeroes. Then 3-pronged singularity:



For ϑ generic, no critical leaf ends up on another zero. Then $\mathcal{W}(\vartheta, \phi_2)$ is union of critical leaves. For convenience, suppose ϕ_2 has at least one pole of order ≥ 2.

Then, facts (theorem in progress with Kohei Iwaki, related to work with Marco Gualtieri and Nikita Nikolaev):

- For $\hbar$ generic, $\nabla_{\hbar,\phi_2}$ admits a *canonical* abelianization, (with respect to the critical graph $\mathcal{W}(\vartheta = \arg\hbar, \phi_2)$), for which the double cover Σ is spectral curve

$$\Sigma = \{\lambda^2 + \phi_2 = 0\} \subset T^*C$$

- This abelianization has

$$\mathcal{X}_\gamma(\hbar) \sim \exp(Z_\gamma/\hbar + \cdots)$$

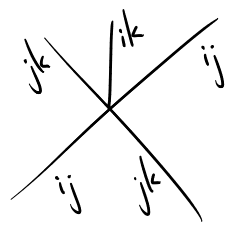
  as $\hbar \to 0$ (in $\frac{1}{2}$-plane centered around ϑ), where

  - $Z_\gamma = \oint_\gamma \lambda$, λ Liouville 1-form on T^*C,
  - $\cdots$ is a computable series in powers of $\hbar$.

Proof: a rigorous version of "exact WKB" method [Voros, Ecalle, ...] using recent results on Borel summability [Koike, Schafke]. Walls are "Stokes curves."

This result determines the asymptotic expansion of the traces. (They have Stokes phenomena, i.e. it depends on *how* $\hbar \to 0$: dominated by the term with largest $\mathrm{Re}(Z_\gamma/\hbar)$.) With more work, can try to bootstrap into an actual computation of the traces themselves.

Conjecture: similar picture for higher-degree equations. In this case each wall is carrying a label ij which tells what kind of unipotent matrix goes there. New phenomenon: scattering of the walls.



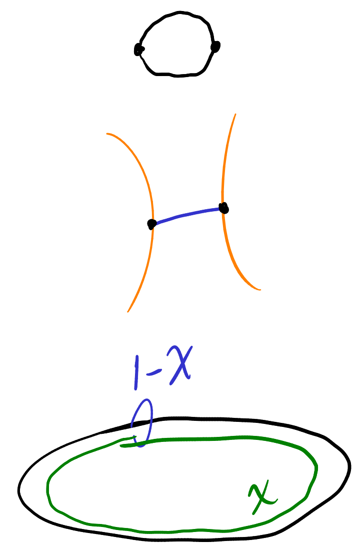
In general the networks which appear here are *not* related to triangulations. Some interesting new combinatorics.

6 3-manifolds

Abelianization seems to be useful for 3-manifolds too (work in progress with Dan Freed).

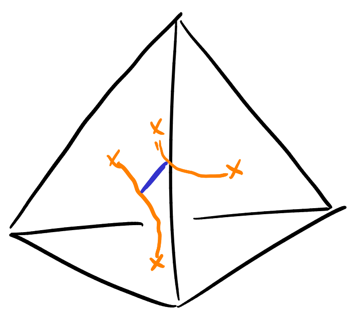
The picture is similar to before: 3-manifold M, branched double covering $\Sigma \to M$, network $\mathcal{W} \subset M$ of codimension-1 walls, relation $\iota : \nabla \simeq \pi_* \nabla^{\mathrm{ab}}$ off $\mathcal{W}$. But, a new phenomenon: the connections ∇^{ab} which we consider have to be *singular* at some circles $S_i \subset \Sigma$. (Double-covering pieces of wall connecting branch loci.)

Holonomy obeys relation:



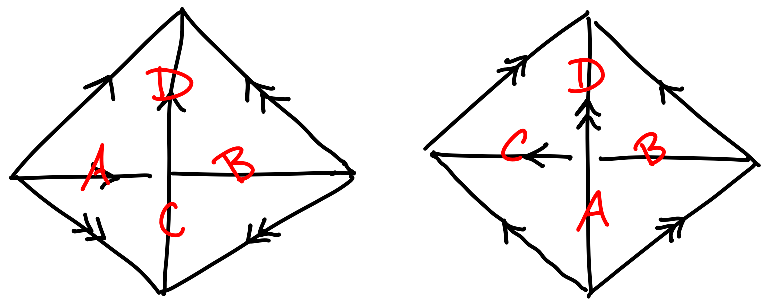
7 Triangulated 3-manifolds

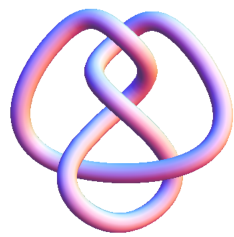
Basic example: an ideal triangulation of a 3-manifold (glue together tetrahedra along their faces, with vertices deleted.) For every such manifold there is an associated $\mathcal{W}$. Double cover $X \to M$ now branched over a 1-manifold which threads through the tetrahedron. In the interior of the tetrahedron, two walls *meet head-on*, in a line segment, double-covered by a circle S_i.



e.g. M = figure-eight knot complement $S^3 \setminus K$, glued together from 2 tetrahedra.

Now want to study flat $SL_2\mathbb{C}$-connections over M, by abelianization. For example, M is hyperbolic [Riley-Thurston], so it has a





particular flat $PSL_2\mathbb{C}$-connection ∇. Can lift it to $SL_2\mathbb{C}$. Can we construct it? Not so trivial, since

$$\pi_1(M) = \langle A, B | A^{-1}BAB^{-1}AB = BA^{-1}BA \rangle$$

How to make matrices obeying these conditions? Idea: build it by starting with ∇^{ab}. This means giving a class $\in H^1(\Sigma \setminus S, \mathbb{C})$ obeying the constraints at each S_i. Determined by its values $\mathcal{X}_i$ on loops around S_i.

But the $\mathcal{X}_i$ have to obey our constraints. This leads to a system of algebraic equations (Thurston gluing equations). e.g. for figure-eight knot complement, if we also ask for $PSL_2\mathbb{C}$-unipotent holonomy around the boundary torus:

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}, \qquad \mathcal{X}(1 - \mathcal{X}) = 1$$

thus

$$\mathcal{X} = e^{\pi \mathrm{i}/3}$$

(NB, this is different from the surface case where we got a moduli space: here expected dimension is zero.)

8 Chern-Simons

Let $G = SL_N\mathbb{C}$. Suppose given a closed 3-manifold M carrying a G-connection ∇.

Then we can consider the (level 1) *Chern-Simons invariant*. Since G has $\pi_0 = \pi_1 = \pi_2 = 0$ every G-bundle over M is trivial, thus we can represent ∇ as $d + A$ for $A \in \Omega^1(\mathfrak{g})$, and then the Chern-Simons invariant is

$$CS(\nabla) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A)$$

The choice of trivialization makes a difference, but changes $CS(\nabla)$ by something in $\mathbb{Z}$, so what's well defined is

$$CS(\nabla) \in \mathbb{C}/\mathbb{Z}$$

The figure-eight knot just described was slightly different: M has boundary, $PSL_2\mathbb{C}$-unipotent holonomy on boundary. Here, can similarly define $CS(\nabla) \in \mathbb{C}/\frac{1}{4}\mathbb{Z}$.

For the ∇ coming from a hyperbolic structure,

$$CS(\nabla) = (real) + \mathrm{i}\,\mathrm{vol}(M)/4\pi^2$$

i.e. this is a "complexified volume."

Question: how to actually *compute* $CS(\nabla)$?

9 Chern-Simons by abelianization

Theorem (in progress with Dan Freed): if we have $\mathcal{W}$ on M and a flat ∇ over M, abelianized by some ∇^{ab} over Σ, can compute $CS(\nabla)$. This computation is "easy" and gives $CS(\nabla) = CS(\nabla^{\mathrm{ab}})$, *except* for additional contributions from the S_i. The additional contribution is, *loosely speaking* $\frac{1}{4\pi^2} Li_2(\mathcal{X}_i)$.

Upon choosing a trivialization of the line bundle L over X this recovers dilogarithmic formulas for hyperbolic volumes. Choices of branch of Li_2 get dictated by trivialization of L.

For example, figure-eight knot complement: get

$$CS(\nabla) = \frac{1}{4\pi^2}(Li_2(e^{\pi i/3}) + Li_2(e^{\pi i/3})) \approx 2.02988\text{i}$$

recovering the hyperbolic volume (NB we take different branches for the two dilogs!)

Related to [Dupont, Neumann, Zickert]; higher rank version related to [Garoufalidis-Thurston-Zickert] in case of triangulations.

10 Chern-Simons and topological strings

What we said: $SL(N,\mathbb{C})$ Chern-Simons theory on M related to $GL(1,\mathbb{C})$ Chern-Simons on $\Sigma \to M$, *deformed* by some funny singular behavior.

This deformation would arise naturally in A model topological string: if $\Sigma \subset T^*M$, and equip T^*M with almost complex structure, get a deformation of Chern-Simons, where the S_i are the boundaries of holomorphic discs in T^*M [Witten, Ooguri-Vafa].

So hopefully all the story of abelianization has a natural meaning in topological string (Floer theory).

Key open question here: can we say something about *quantum* complex Chern-Simons using this idea?