1 Preface

I'll describe results from two papers: one with Gunaydin-Pioline from July 2006, one with Walcher from September 2007.

An idiosyncratic view of the geometry underlying the holomorphic anomaly of topological strings. (Nothing to do with moduli spaces of Riemann surfaces.) Explain how the anomaly equation of BCOV can be rewritten precisely in the form of a heat equation, and how Walcher's extension is a heat equation with convection.

At the end, some indications (from physics) that all this is connected to hyperkähler geometry and thereby to Kontsevich-Soibelman wall-crossing formulas.

2 Special geometry

Let $\mathcal{M}$ be moduli of complex structures on X: special Kähler manifold. $\mathcal{L}$ its Hodge line bundle, $\mathcal{L}_t = H^{3,0}(X_t)$ for each $t \in \mathcal{M}$. Then introduce $\widetilde{\mathcal{M}} = \mathcal{L} \to \mathcal{M}$. It's a submanifold of the complex symplectic space $V_{\mathbb{C}} = H^3(X, \mathbb{C})$. $T_\Omega \widetilde{\mathcal{M}} = (H^{3,0} \oplus H^{2,1})(X_\Omega, \mathbb{C})$. So $\widetilde{\mathcal{M}}$ is complex Lagrangian submanifold. Such a submanifold is determined by a holomorphic "generating" function $\mathcal{F}_0$ once we specify a symplectic frame for V (choose A and B cycles). Concretely, in this case we can write $X^I = \int_{A^I} \Omega$ and $F_I = \int_{B_I} \Omega$, and then we have

$$\mathcal{F}_0 = \frac{1}{2} X^I F_I, \quad F_I = \frac{\partial \mathcal{F}_0}{\partial X^I}.$$

This structure is equivalent to saying $\widetilde{\mathcal{M}}$ is rigid special pseudo-Kähler. (One negative direction.) It has a simple pseudo-Kähler potential,

$$\tilde{K} = i \int_X \Omega \wedge \bar{\Omega},$$

and Hermitian metric, given in the X^I coordinates as $\mathrm{Im}\,\tau$ where

$$\tau_{IJ} = \partial_I \partial_J \mathcal{F}_0.$$

Corresponding Kähler potential on $\mathcal{M}$ itself is

$$K = \log \tilde{K}.$$

3 Topological partition function

BCOV introduce a formal generating function:

$$\Psi(t, \bar{t}, x, \lambda) = \lambda^{\frac{\chi}{24}-1} \exp \left(\sum_{g=0}^{\infty} \sum_{k=0}^{\infty} \lambda^{2g-2+k} \langle O_{i_1} \cdots O_{i_k} \rangle_{g,(t,\bar{t})} x^{i_1} \cdots x^{i_k} \right)$$

The $\langle \cdots \rangle$ are supposed to be provided by the topological B model. (They are mirror to Gromov-Witten invariants.) This object is defined in a modular invariant way (no choice of A/B cycles breaking diff invariance). Low point functions ($2g - 2 + k \leq 0$) set to zero.

$\langle \cdots \rangle_g$ is a section of $\mathcal{L}^{2g-2}$. Convenient to make Ψ a pure number, so compensate by making λ^{-1} transform in $\mathcal{L}$.

O_i are (c,c) operators which correspond to infinitesimal deformations of the B model along $\mathcal{M}$, so x^i are coordinates of a point $x^i \frac{\partial}{\partial t^i} \in \mathcal{L} \otimes \mathcal{TM}$, and λx^i similarly a point of $\mathcal{TM}$.

Altogether then, Ψ is globally defined on the total space of $\mathcal{L} \oplus \mathcal{L} \otimes \mathcal{TM} \to \mathcal{M}$. (It's actually a section of a line bundle itself, but we'll suppress that in this talk.)

It is neither holomorphic nor antiholomorphic; obeys the anomaly equations of BCOV,

$$\left[\partial_{\bar{i}} - \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j \partial x^k} - G_{j\bar{i}}x^j\frac{\partial}{\partial \lambda^{-1}}\right]\Psi = 0\,,$$

$$\left[\partial_i - \Gamma^k_{ij}x^j\frac{\partial}{\partial x^k} + \partial_i K\left(\lambda^{-1}\frac{\partial}{\partial\lambda^{-1}} + x^k\frac{\partial}{\partial x^k} + \frac{\chi}{24} - 1\right)\right.$$
$$\left. - \lambda^{-1}\frac{\partial}{\partial x^i} + \partial_i F_1 + \frac{1}{2}C_{ijk}x^j x^k\right]\Psi = 0\,.$$

Here G is the metric on $\mathcal{M}$ (with Kähler potential $\log \int \Omega\wedge\bar{\Omega}$), Γ its Christoffel symbols, C the Yukawa coupling $C_{ijk} = \int_X \Omega\wedge\delta_i\delta_j\delta_k\Omega$, F_1 is the 1-loop partition function. These equations are the key tool in one popular approach to computation of the partition function.

Want to rewrite/reinterpret these equations in a more familiar way.

Pass to "big phase space": First pull everything back through $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ (just ask Ψ to be invariant under rescaling.) Then for the fiber directions, note that $\pi^*(\mathcal{L}\oplus\mathcal{L}\otimes\mathcal{T}\mathcal{M}) \simeq H^{3,0}\oplus H^{2,1} \simeq \mathcal{T}\widetilde{\mathcal{M}}$ as a complex vector bundle (but not holomorphically!) So can view Ψ as a function on $\mathcal{T}\widetilde{\mathcal{M}}$.

(In coordinates: introduce periods of $\omega = \lambda^{-1}\Omega + z^i\delta_i\Omega$:

$$z^I = \int_{A^I}\omega = \lambda^{-1}X^I + z^i(\partial_i X^I + \partial_i K X^I).$$

These are complex coordinates on $\mathcal{T}\widetilde{\mathcal{M}}$. So we're writing Ψ as a function of $(X^I, \bar{X}^I, z^I)$. Still neither holomorphic nor antiholomorphic.)

Now pass from $\mathcal{T}\widetilde{\mathcal{M}}$ to $\mathcal{T}^*\widetilde{\mathcal{M}}$. Coordinates on the latter space are

$$y_I = (\tau - \bar{\tau})_{IJ}\bar{z}^J.$$

Also restore 0, 1 point functions in Ψ (breaking modular invariance):

$$\Psi \to \frac{1}{\sqrt{\det \operatorname{Im} \tau_{IJ}}} \exp\Big[-\frac{i}{4} z^J (\operatorname{Im} \tau_{JK}) z^K + f_1\Big] \Psi$$

where f_1 is the holomorphic part of F_1, defined by

$$F_1 = -\frac{1}{2} \log \det \operatorname{Im} \tau_{IJ} - \Big(\frac{\chi}{24} - 1\Big) K + f_1 + \bar{f}_1 \, .$$

Then $\bar{\Psi}$ is purely holomorphic, defined on $\mathcal{T}^* \widetilde{\mathcal{M}}$, obeying

$$\left(\frac{\partial}{\partial X^I} - \frac{\mathrm{i}}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K} \right) \bar{\Psi}_{closed} = 0.$$

Here $C_{IJK} = \partial_I \partial_J \partial_K \mathcal{F}_0$. It's exactly the equation of a theta function (formally).

4 Open topological string data: normal functions

The B model is supposed to have an open version which integrates both over open and closed Riemann surfaces, with "boundary conditions" specified by objects in the bounded derived category of coherent sheaves on X (plus orientifold data!) This should be mirror to the open A model which includes strings ending on Lagrangian submanifolds.

Suppose we have a pair of cohomologous 2-cycles $C_\pm$ (brane and orientifold in the B model). Choose an interpolating 3-cycle E. Then we can integrate 3-forms from $H^{3,0} \oplus H^{2,1}$ over E: define

$$\nu_I = \int_E \delta_I \Omega \tag{4.1}$$

(This is well defined, because elements in $H^{2,1}$ can be represented in $\bar{\partial}$ cohomology, so that the ambiguity is a $(2,0)$ form.) Invariantly, ν

gives a section of $\mathcal{T}^*\mathcal{M}$. In fact, it's a holomorphic and Lagrangian section (wrt $dX \wedge dy$) like $\mathcal{M}$ itself was.

In Walcher's framework, such a ν specifies the low energy open string data. Walcher also proposed holomorphic anomaly equations to be obeyed by the corresponding open topological partition function. I don't write these equations here. But even without knowing the equations, geometrically, there is an obvious guess for how to solve them: just shift along the fibers by ν, i.e.

$$\Psi_{open}(X^I, y_I) = \Psi_{closed}(X^I, y_I + i\nu_I)?$$

It turns out that this shift indeed transforms the *equations* into one another — after properly defining Ψ_{open}, very similar to what we did above,

$$\Psi \to \frac{1}{\sqrt{\det \operatorname{Im} \tau_{IJ}}} \exp\Big[-\frac{i}{4}(z^J - i\delta^J) \operatorname{Im} \tau_{JK}(z^K - i\delta^K) + f_{1,0} + f_{0,2}\Big]\Psi$$

where

$$\delta^J = \operatorname{Im} \tau^{JK}(\bar{\nu}_K - \nu_K)$$

But the shift does not directly relate *solutions*: can check this just by looking at the $\lambda \to 0$ limit of the shifted open string partition function.

So the open and closed string partition functions both fit into the same structure but they don't seem to be exactly the same object.

5 Wave functions

First review Witten's wave function interpretation of the anomaly equations: the wave function is formally a state obtained from quantization of the symplectic vector space $V = H^3(X, \mathbb{R})$. So both the closed topological string and the open-closed topological string (in each D-brane sector) provide states in this (formal!) Hilbert space.

6 Dimensional reduction

Next note, the geometry of $\mathcal{T}^*\widetilde{\mathcal{M}}$ was important in the above. This space also arises physically in the dimensional reduction from 4 to 3 dimensions. In particular, it is a hyperkähler manifold, because of constraints from supersymmetry in $d = 3$.

Complex structure I is that of $\mathcal{T}^*\widetilde{\mathcal{M}}$. But also interesting is complex structure J: this one has complex coordinates

$$\xi^I = \operatorname{Re} X^I + i\zeta^I, \qquad \tilde{\xi}_I = \operatorname{Re} F_I + i\tilde{\zeta}_I$$

These peculiar complex coordinates are exactly what enters naturally in OSV!

7 Kontsevich-Soibelman formula

There is another interesting story about the reduction to $d = 3$. Kontsevich-Soibelman concerns degeneracies of BPS states in $d = 4$. Very naively these would be integers attached to classes $C \in H_3(X, \mathbb{Z})$. But even defining these degeneracies is subtle since they're known to have *wall-crossing*: the index can jump as the moduli are varied. So a full understanding of these invariants has to take this into account. Turns out that one can sometimes understand what will happen at the wall just from the low energy supergravity.

Kontsevich-Soibelman have proposed a very powerful and mysterious wall-crossing formula. They say: consider some abstract algebraic torus, $(\mathbb{C}^\times)^{2n}$, as a holomorphic symplectic manifold with $\Omega = \frac{dx^I}{x^I} \wedge \frac{dy_I}{y_I}$. Then for every BPS hypermultiplet in $d = 4$ of charge (a, b) write a symplectomorphism of this torus,

$$T_{a,b} : (x, y) \to (x(1 - (-1)^{ab} x^a y^b)^b, y(1 - (-1)^{ab} x^a y^b)^{-a}) \quad (7.1)$$

Then consider formally the product T of all these symplectomorphisms (squared). Typically an infinite product, but convergent in some sense. They don't commute, so order them with respect to the phase of the central charge; and make some choice of particle-antiparticle split, e.g. only take Z in the upper half-plane.

Claim: T is invariant under wall-crossing!

This bizarre formula deserves some physical explanation. We propose that it has to do with the construction of the quantum-corrected hyperkähler moduli space of the exact $d = 3$ theory: naive semi-flat metric corrected by instantons coming from BPS particles in $d = 4$. But this then suggests a connection with the topological string, because the semi-flat metric itself is just the one on (covered by) $\mathcal{T}^*\mathcal{M}$.