1 Aim

I want to describe some recent progress in 4-dimensional quantum field theory.

Our universe seems to be well described by 4-dimensional quantum field theory. By now (thanks to enormous effort in the 20th century) we think we know the rules of the QFT game pretty well, and we know a lot about *which* QFT describes our particular universe. But even easy-sounding questions are hard to answer — e.g. to go from the fundamental equations of the theory to "observables" like the spectrum of particles one would actually detect.

Theorists often study "toy models" — examples of QFT's which are *not* directly realized in our universe — as a way to get general lessons about what can happen in QFT.

Recently our class of toy models has been dramatically expanded, thanks to a surprising trick. The trick, in short, is that in order to study *four*-dimensional physics, it is convenient to start from a *six*-dimensional perspective, then take two of the dimensions to be small. [Witten, Gaiotto, Gaiotto-Moore-Neitzke] This is the origin of the "$4 + 2 = 6$" in my title. From this perspective, many "mysterious" phenomena in 4d physics become easier to understand, or at least easier to predict.

2 Maxwell theory

Let's start with *vacuum Maxwell's equations* (in units where $c = 1$):

$$\operatorname{div} \vec{E} = 0, \tag{2.1}$$

$$\operatorname{curl} \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0, \tag{2.2}$$

$$\operatorname{div} \vec{B} = 0, \tag{2.3}$$

$$\operatorname{curl} \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{2.4}$$

These equations are *linear* in the fields. So, we can actually *solve* them exactly: basis for solutions given by propagating electromagnetic waves (light). Linearity means that we can *superpose* these waves: no interactions. Linearity also means that the *quantum* version of this theory is understandable. For example, the "Hilbert space of 1-particle states" in this theory just consists of states $|\vec{k}, \vec{\epsilon}\rangle$ representing quantized electromagnetic waves (photons) with momentum $\vec{k}$ and polarization $\vec{\epsilon}$.

These equations also have a remarkable *symmetry* to them: they would be the same if we exchange

$$\vec{E} \to \vec{B}, \vec{B} \to -\vec{E} \tag{2.5}$$

This is the fact of *electric-magnetic duality.*

What is the *use* of this duality? Suppose we couple the theory to external charges. The duality says for example that if we have two *monopoles* separated by some distance, with charge p_1, p_2, we can compute the potential energy between them: by duality, it must be the same as the potential energy between two *electric charges*,

$V = -p_1 p_2/r$! And similarly for any other question we ask about complicated combinations of electric and magnetic charges.

This duality also survives into the *quantum* version of Maxwell theory – not too surprising. But, there's a little twist. Recall Dirac's quantization law: fix an electrically charged particle of charge q, and a magnetic monopole of charge p; then pq must be an integer (in units where $\hbar = 1$.) This means that if q_0 and p_0 are the minimum quanta of electric and magnetic charge then $p_0 = 1/q_0$. Thus, if the force between elementary electric charges is small (weak coupling), the force between elementary magnetic charges will be large (strong coupling). So, electric-magnetic duality is in some sense a strong/weak duality.

3 Gauge theory

The Standard Model of particle physics is founded on equations which are in some ways analogous to Maxwell's equations. The electromagnetic field-strengths $\vec{E}$, $\vec{B}$ are replaced by vectors of *matrices*. For the weak interactions we should take 2×2 skew-Hermitian matrices, for strong interactions 3×3 ones. Then we can write the equations of *classical Yang-Mills theory*:

$$\operatorname{div} \vec{E} = g[\vec{A} \cdot \vec{E}], \tag{3.1}$$

$$\operatorname{curl} \vec{B} - \frac{\partial \vec{E}}{\partial t} = g[V\vec{E}] - g[\vec{A} \times \vec{B}], \tag{3.2}$$

$$\operatorname{div} \vec{B} = -\frac{1}{2} g \operatorname{div}[\vec{A} \times \vec{A}], \tag{3.3}$$

$$\operatorname{curl} \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\frac{1}{2} g \partial_t [\vec{A} \times \vec{A}] + g \operatorname{curl}[V\vec{A}]. \tag{3.4}$$

These equations involve both the fields $\vec{E}$, $\vec{B}$ and their potentials $(V, \vec{A})$. $\vec{E}$ and $\vec{B}$ are determined by $(V, \vec{A})$ by equations similar to the usual ones, with corrections. But, no way to write the equations just using $\vec{E}$, $\vec{B}$. The notation $[\vec{A} \cdot \vec{E}]$ means $\sum_{i=1}^{3} A_i E_i - E_i A_i$, and similarly for the other [] terms.

The equations are *nonlinear*: much more difficult to solve! Reflects the fact that these "gauge fields" actually *interact* with one another. The parameter g controls the strength of this interaction. One could hope to solve the equations by some kind of perturbation expansion in g.

Similarly for the quantum theory: perturbation in g is possible in principle (not easy), but as we increase g things will get harder and harder.

In particular, questions like "what are the 1-particle states in the quantum version of this theory?" are very difficult indeed! It's generally *believed* that there are *no* massless particles in this theory: this is the phenomenon of *confinement*. But a really satisfactory theoretical understanding of confinement is lacking (1 million dollars for mathematical proof). In some sense the trouble is that one doesn't adequately understand the physics at large g.

4 Electric-magnetic duality, redux

OK, so how can we understand the physics at large g?

Electric-magnetic duality symmetry is far from obvious in the Yang-Mills equations. It was proposed [Montonen-Olive] that there is actually a hidden symmetry in the *quantum* theory, which again exchanges $\vec{E} \leftrightarrow \vec{B}$, and *simultaneously* exchanges $g \leftrightarrow 1/g$ (strong/weak duality again!)

A bizarre statement: says that the theory “becomes weakly coupled again,” when we go to very large g.

It’s so bizarre that nobody believed it. And for good reason: it is not true.

But, the idea was too good to be completely wrong. It turns out that it is true in a *supersymmetric extension* of Yang-Mills theory. This is a theory in which we have not only the gauge fields but also some additional matter built in, in a very specific way. The version we want is $N = 4$ *supersymmetric Yang-Mills theory*, where the N is measuring how much extra symmetry the theory has, and also roughly measuring how much extra matter we had to add — multipled the physical polarizations by 8. Think of this as a kind of "toy model" for the non-supersymmetric Yang-Mills theory we really want to undersand.

This "non-abelian" version of electric-magnetic duality is called S-duality. Lots of evidence by now that S-duality is actually true [Sen, Vafa-Witten]; but still somewhat mysterious.

5 A new picture

Now, a new way of making the electric-magnetic duality look "easy".

It requires us to swallow one claim which comes from string theory (but much less than the full string machine), as follows.

- There exists an interacting, scale invariant *six*-dimensional supersymmetric quantum field theory. ("Theory X").

- Formulate Theory X on a space-time which is $S^1 \times \mathbb{R}^5$, where S^1 has radius R, and then look at the theory at low energies ($E \ll 1/R$). We get a five-dimensional version of supersymmetric Yang-Mills theory, with coupling strength $g_5 = \sqrt{R}$.

Then, standard QFT method: further compactification on another S^1 of radius R' gives $N = 4$ supersymmetric Yang-Mills theory with coupling strength $g = g_5/\sqrt{R'} = \sqrt{R/R'}$.

But this is strange — the same theory can also be viewed as $N = 4$ super Yang-Mills with coupling $g = \sqrt{R'/R}$! (Draw the diagram.)

Conclusion: the "easy" geometrical symmetry *exchanging the two circles in* T^2 corresponds to the "deep" S-duality. [Vafa]

6 Closer to reality

How about *less supersymmetric* theories?

In the last few years it's been understood that a very similar picture can be applied to theories with "$N = 2$ supersymmetry," too. (Still too much for the real world, but getting closer. Only multiplied the number of physical polarizations by 4.) [Gaiotto]

Just replace the 2-torus by other surfaces C! i.e. take Theory X in a spacetime of the form $C \times \mathbb{R}^4$. In this way we can realize all kinds of different 4-dimensional quantum field theories just by varying our choice of C. For example: suppose we take C to be a sphere with 4 punctures, (and let our matrices be 2×2). Then we obtain $N = 2$ supersymmetric Yang-Mills theory (coupled to some matter). The coupling of the theory is determined by the position of the 4 punctures.

In fact, just as for the torus, we can obtain $N = 2$ super Yang-Mills in many different ways. Each choice of a curve which separates the 4 punctures into 2 pairs gives an example, with a *different* value of the coupling. Thus we get a large number of "S-dualities" relating different versions of the theory, with different couplings.

More generally, get lots of QFT's which *hadn't* been known before: stuff much wilder than Yang-Mills, only beginning to be explored. And a huge number of S-dualities between them. [Gaiotto, Distler-Chacaltana] And at the same time we get a lot of new tools for exploring them...

7 The spectrum of particles

This stuff can be used to address some down-to-earth questions. Given any complex physical system, one of the basic questions you always want to ask is: *what does it do*? e.g. *what is the ground state* and *what are the excitations around the ground state*?

In the supersymmetric theories we have been discussing, the question *what are the ground states* was basically answered in 1995 [Seiberg-Witten]. The question *what are the excitations* is harder.

The geometric picture I have been describing leads to a sharp answer to this question, at least for the *stable* excitations, i.e. particles with the smallest possible mass consistent with their charge. Namely: such particles correspond to certain *networks of strings* on the surface C [Klemm-Lerche-Mayr-Vafa-Warner, Gaiotto-Moore-Neitzke]. This is something you can really study on the computer. (Show movies?)

This already leads to some surprises. For example: in joint work with Tom Mainiero (student in Austin physics dept) and several collaborators at Rutgers [Galakhov-Longhi-Moore], we found that in the $N = 2$ supersymmetric version of the Yang-Mills theory with 3×3 matrices, there are vacua where the number of particles with mass $\leq M$ grows *exponentially* with M.

It's surprising to us that this can happen at all in field theory! Thermodynamic consequences not clear at the moment...