1 Preface

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Recall the notion of *scattering diagram / wall-crossing structure / $\mathcal{K}$-wall network* which has appeared in the work e.g. of [Kontsevich-Soibelman, Auroux, Gross-Siebert, Gross-Hacking-Keel]. (Draw example.) It's a collection of codimension-1 subsets of some base manifold ("walls"). Walls labeled by automorphisms of a torus. The scattering diagram is a kind of "blueprint," which tells you how to glue together tori $(\mathbb{C}^\times)^{2n}$ to make a complex manifold $\mathcal{M}$.

In these talks I want to describe a certain extension of this notion, which arose naturally in physics. (Draw the extended diagram.) Edges will be labeled by automorphisms of a *vector bundle* over a torus. The extended scattering diagram is similarly a blueprint for gluing together a holomorphic vector bundle E over $\mathcal{M}$.

A few uses for the extended diagram:

1. By studying a (carefully designed) $\mathbb{C}^\times$-family of such gluings, we can get some extra structure. The complex manifold $\mathcal{M}$ gets upgraded to a hyperkähler manifold. The vector bundle E similarly gets upgraded, to a *hyperholomorphic* bundle over $\mathcal{M}$; i.e. a vector bundle E over $\mathcal{M}$, carrying a unitary connection, such that the curvature F is of type $(1,1)$ with respect to all of the $\mathbb{CP}^1$ worth of complex structures on $\mathcal{M}$. (If $\dim \mathcal{M} = 4$ this means $F = - \star F$, i.e. a Yang-Mills instanton.)

   So we get a new tool for studying these "hyper" things. The dream would be to use this tool to actually produce explicit formulas; at the moment it's at least useful e.g. for asymptotic estimates.

2. The hyperholomorphic bundles we obtain in this way often arise

in *families*, i.e. we get a bundle over some $\mathcal{M} \times C$ rather than just $\mathcal{M}$. In this case we get additional geometric structure "along C." For example, if C is complex 1-dimensional we will get a *solution to Hitchin equations*. The restriction of the extended scattering diagram to $\{u\} \times C \subset \mathcal{B} \times C$ contains a lot of information about the solutions. (Spectral network)

Talk plan:

1. Review of the scattering diagram and its use for building hyperkähler spaces;
2. The extended scattering diagram and its use for building hyperholomorphic vector bundles;
3. Application to Hitchin systems and spectral networks.

2 Data

Hyperkähler integrable system data:

- $\mathcal{B}$ a complex manifold, $\mathcal{B}'$ complement of a divisor in $\mathcal{B}$,
- Γ a local system of lattices over $\mathcal{B}'$, fiberwise given as an extension

$$0 \to \Gamma_u^f \to \Gamma_u \to \Gamma_u^g \to 0$$

  where Γ_u^g carries nondegenerate antisymmetric pairing $\langle , \rangle$, and Γ_u^f is a fixed lattice.
- $Z : \Gamma \to \mathbb{C}$ fiberwise homomorphism, varying holomorphically over $\mathcal{B}$, obeying $\langle \mathrm{d}Z \wedge \mathrm{d}Z \rangle = 0$ and $\langle \mathrm{d}Z \wedge \mathrm{d}\bar{Z} \rangle > 0$,
- *Scattering diagram*, a collection of codimension-1 "walls" in $\mathcal{B}$ (possibly coincident), with the following properties:

- Each wall carries a label $\gamma \in \Gamma$.
- A wall with label γ lies in $\{Z_\gamma \in e^{i\vartheta}\mathbb{R}_+\}$.
- Define complex torus, (shifted version of) $T = \mathrm{Hom}(\Gamma^g, \mathbb{C}^\times)$. A wall with label γ carries a Poisson automorphism of T, of the form $\mathcal{K}_\gamma^{\Omega(\gamma)}$, where

$$\mathcal{K}_\gamma : X_\mu \to X_\mu (1 - X_\gamma)^{\langle \gamma, \mu \rangle}$$

- The automorphisms we meet upon traveling around a loop in $\mathcal{B}$ multiply to the monodromy of Γ around that loop (e.g. to 1 if the loop is contractible).

3 Basic examples

From these data, one expects to be able to build a hyperkähler manifold. Let's describe a few examples.

1. - $\mathcal{B} = \mathbb{C}$, $\mathcal{B}' = \mathcal{B}$
   - Γ a rank 2 lattice, with generators γ_e, γ_m, $\langle \gamma_e, \gamma_m \rangle = 1$
   - Fix τ in upper half-plane, $Z_{\gamma_e}(a) = a$, $Z_{\gamma_m}(a) = a\tau$
   - Scattering diagram empty

   (Could also replace $\mathcal{B}$ by any open subset of $\mathbb{C}$, take any holomorphic Z_{γ_e}, Z_{γ_m} as long as $dZ_{\gamma_m}/dZ_{\gamma_e}$ stays in upper half-plane.)

2. - $\mathcal{B} = \{|u| < |\Lambda|\} \subset \mathbb{C}$, $\mathcal{B}' = \mathcal{B} \setminus \{u = 0\}$
   - $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$ a rank 2 lattice, with generators γ_e, γ_m, $\langle \gamma_e, \gamma_m \rangle = 1$, monodromy $\gamma_m \mapsto \gamma_m + \gamma_e$ around $u = 0$
   - $Z_{\gamma_e}(u) = u$, $Z_{\gamma_m}(u) = \frac{1}{2\pi i}(u \log \frac{u}{\Lambda} - u)$
   - Scattering diagram has two rays, at $u/\zeta > 0$ and $u/\zeta < 0$, carrying automorphisms $X \to X(1 - Y)$, $X \to X(1 - Y^{-1})^{-1}$ (composition is $X \to XY$ as needed)

(Could also shift Z_{γ_m} by any holomorphic function.)

3. • $\mathcal{B} = \mathbb{C}$, $\mathcal{B}' = \mathcal{B} \setminus \{u = \pm 2\}$
   • Introduce complex curves
   $$\Sigma_u = \{y^2 = z^3 - 3z + u\} \subset \mathbb{C}^2$$
   For $u \in \mathcal{B}'$, Σ_u is a noncompact smooth genus 1 curve. $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$.
   • Introduce the 1-form $\lambda = y\,\mathrm{d}z$. For $\gamma \in \Gamma_u$,
   $$Z(\gamma) = \frac{1}{\pi} \oint_\gamma \lambda$$
   • Scattering diagram constructed by shooting out rays from each of two singular points.

The scattering diagram is constructed similarly in "generic" cases. In our examples:

1. The hyperkähler space $\mathcal{M}$ is simply $\mathbb{R}^2 \times T^2$. For $\zeta \in \mathbb{C}^\times$, $\mathcal{M}_\zeta$ is just a torus $\mathbb{C}^\times \times \mathbb{C}^\times$. For $\zeta = 0$, $\mathcal{M}_\zeta$ is $\mathbb{C} \times \Sigma_\tau$ with Σ_τ a compact complex torus; similarly $\zeta = \infty$.
2. $\mathcal{M}$ is the "Ooguri-Vafa space", an explicitly known hyperkähler integrable system over the disc, used by [Ooguri-Vafa, Seiberg-Shenker, Gross-Wilson] For $\zeta = 0$, $\mathcal{M}_\zeta$ is the "Tate curve."
3. $\mathcal{M}$ is a complete hyperkähler metric fibered over the plane, (also known as a moduli space of Higgs bundles; see next lecture)

4 Mirror perspective

Our construction builds a family of complex structures J_ζ on $\mathcal{M}$. For $\zeta = 0$ we get a complex integrable system, for $\zeta \in \mathbb{C}^\times$ something which is locally like $(\mathbb{C}^\times)^n$.

How to think about the construction, at least for $\zeta = e^{\mathrm{i}\vartheta}$: begin with a *complex integrable system*, i.e. a holomorphic fibration $X \to \mathcal{B}$, where X has holomorphic symplectic form ϖ and fibers are Lagrangian. Then get hyperkähler integrable system data:

- $\mathcal{B}$ is the base of X, $\mathcal{B}'$ is the complement of the discriminant locus.
- For $u \in \mathcal{B}'$,

  $$\Gamma_u = H_2(X; X_u, \mathbb{Z})$$

  It is an extension

  $$0 \to \Gamma_u^f = H_2(X, \mathbb{Z})/[X_u] \to \Gamma_u \to \Gamma_u^g = H_1(X_u, \mathbb{Z}) \to 0$$

  The Γ_u fit together into a local system $\Gamma \to \mathcal{B}'$. Γ_u has pairing $\langle , \rangle$ pulled back from $H_1(X_u, \mathbb{Z})$.
- $Z : \Gamma \to \mathbb{C}$,

  $$Z_\gamma = \int_\gamma \varpi$$

  Well defined because $\partial\gamma$ is constrained to lie on a ϖ-Lagrangian fiber; holomorphic on total space of Γ.

Fixing any $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, X becomes a real integrable system, with symplectic form

$$\omega = \mathrm{Re}(e^{\mathrm{i}\vartheta}\varpi)$$

and the desired object is the complex *mirror* $\mathcal{M}_\vartheta$ of this integrable system.

Then the scattering diagram reflects the places where quantum corrections appear. Given a point u on a wall with label γ, $\Omega(\gamma)$ is a count of *holomorphic discs* ending on X_u, in the homology class γ.

5 Mirror perspective, redux

To motivate the extension, think again of the mirror symmetry setup. [Fukaya] Recall that points of the fiber X_u are supposed to correspond to line bundles on the dual fiber $\mathcal{M}_u$. If we view X_u as a symplectic space with respect to $\omega(\vartheta)$ then Lagrangian sections of X_u with respect to ω are supposed to correspond to holomorphic line bundles on the mirror. Thus, if we have a *complex* Lagrangian section ν, i.e. one which is Lagrangian with respect to ϖ, then it should give a holomorphic line bundle on each mirror $\mathcal{M}_\vartheta$.

The interesting examples will not involve line bundles but rather vector bundles. So, suppose we are given a complex Lagrangian *multi*-section of X_u. Locally, represent the multi-section as k sections, $(\nu_1, \ldots, \nu_k)$. Mirror should be a holomorphic vector bundle $E_{\nu,\vartheta}$ of rank k on each $\mathcal{M}_\vartheta$.

Now what are the quantum corrections here? Given a point u we will have a new kind of object: *holomorphic triangle* with boundary on X_u and ν.

To measure their "homology classes": For each i, j let $\Gamma_{ij,u}$ consist of 2-chains in X whose boundary restricted to F_u is a path from $\nu_i(u)$ to $\nu_j(u)$, and whose boundary away from F_u lies completely on ν; these chains are considered modulo boundaries of 3-chains. $\Gamma_{ij,u}$ is a Γ_u-torsor (at least if ν simply connected; even if not, let's extend Γ^f as necessary to make it true). Not canonically isomorphic to Γ_u, e.g. it can have additive monodromies when $\nu_i(u)$ winds around some cycle of the torus. Local system of torsors Γ_{ij}. Now our new data:

- $\mathcal{B}''$ is the locus where the multi-section ν is regular.
- Γ_{ij}.

- $Z : \Gamma_{ij} \to \mathbb{C}$ given by
$$Z_a = \int_a \varpi$$

6 Data, redux

So, in addition to hyperkähler integrable system data we had before, now add:

- $\mathcal{B}'' \subset \mathcal{B}'$ complement of divisor,
- Finite (K-fold) cover $N \to \mathcal{B}''$,
- Γ_{ij} a local system of Γ-torsors over $N \times_{\mathcal{B}''} N$, with compatible additions,
- $Z : \Gamma_{ij} \to \mathbb{C}$ affine-linear, compatible with additions, varying holomorphically.
- *Extended scattering diagram.* As before, $T = \mathrm{Hom}(\Gamma^g, \mathbb{C}^\times)$ (more precisely a fixed lift of this to $\mathrm{Hom}(\Gamma, \mathbb{C}^\times)$). Now *split* Γ_{ij} (locally!) as
$$\Gamma_{ij} = \Gamma_i - \Gamma_j$$
with $Z : \Gamma_i \to \mathbb{C}$.

  Then have a principal $\mathbb{C}^\times$-bundle L_i which is the set of affine-linear maps
$$\mathrm{Hom}(\Gamma_i, \mathbb{C}^\times) \to T$$
  Each $a \in \Gamma_{ij} = \Gamma_i - \Gamma_j$ gives a homomorphism
$$e_a : L_i \to L_j$$
  Let
$$V = \bigoplus_{i=1}^{k} L_i \to T$$

Have all the $\mathcal{K}$-walls as before, but enhanced: the Poisson automorphisms $\mathcal{K}_\gamma^{\Omega(\gamma)}$ get lifted to act on the whole $V \to T$. Also add new *$\mathcal{S}$-walls*, such that

– Each $\mathcal{S}$-wall carries a label $a \in \Gamma_{ij}$.

– A wall with label a lies in $\{Z_a/\zeta \in \mathbb{R}_+\}$.

– A wall with label $a \in \Gamma_{ij}$ carries an automorphism of V (leaving T fixed), of the form $\mathcal{S}_a^{\mu(a)}$, where $\mathcal{S}_a = 1 + e_a$.

– The automorphisms we meet upon traveling around a contractible loop in $\mathcal{B}$ multiply to the identity.

– The automorphisms we meet upon traveling around a component of D or D' multiply to the monodromy of (Γ, Γ_i) around that component.

7 Basic examples

1.
   - Hyperkähler integrable system data as in #1 above
   - $K = 1$, N trivial cover
   - $\Gamma_i = \Gamma$,
   - $Z : \Gamma_i \to \mathbb{C}$ determined by $Z(0) = W$, a holomorphic function $\mathbb{C} \to \mathbb{C}$ (generating function of $\mathbb{C}$ Lagrangian section)
   - Scattering diagram trivial,
   - Hyperholomorphic line bundle E over $\mathbb{R}^2 \times T^2$ is trivial bundle, with curvature given by "semiflat" formula: $F = \mathrm{d}\nu \cdot \mathrm{d}\theta$ (ν a section of $\Gamma \otimes \mathbb{R}$, defined by $\nu \cdot \mathrm{d}Z = \mathrm{d}W$)
2.
   - Hyperkähler integrable system data as in #1 above
   - $K = 2$, N branched over $a = 0$
   - $\Gamma_i = \Gamma$ $(i = 1, 2)$

- $Z : \Gamma_i \to \mathbb{C}$ determined by $Z(0) = W = a^{3/2}$ (could also shift by a holomorphic function vanishing at $a = 0$)
- Scattering diagram given by 3 rays $a^{3/2}/\zeta \in \mathbb{R}_+$. (Corresponding to 3 holomorphic discs, cf. the 2 emerging from a bad fiber in the "usual" scattering picture). Has correct monodromy thanks to the identity

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Hyperholomorphic rank-2 bundle E over $\mathbb{R}^2 \times T^2$ is a certain $SU(2)$ instanton, radially symmetric on $\mathbb{R}^2$, invariant under shifts along T^2. Involves Painleve III transcendents. Differs from semiflat F by corrections of order $e^{-R|Z|}$.

Remark: this case (only $\mathcal{S}$-walls) was essentially studied by [Cecotti-Vafa, Dubrovin]; goes under the name of tt^* geometry.

Remark: this picture was also discussed by [Fukaya] in mirror symmetric context (one complex structure at a time).

In more general examples, walls will interact. New kinds of interaction which can occur now:

- Two $\mathcal{S}$-walls meeting, e.g. one carrying $a \in \Gamma_{ij}$ and one carrying $b \in \Gamma_{jk}$ scatter to produce a wall carrying $a + b \in \Gamma_{ik}$. In local coordinates, to understand this:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A $\mathcal{K}$-wall and an $\mathcal{S}$-wall meeting, e.g. $\mathcal{S}_a \mathcal{K}_\gamma = \mathcal{K}_\gamma \mathcal{S}_a \mathcal{S}_{a+\gamma}$ (with lifts of $\mathcal{K}_\gamma$ to V on each side.) In local coordinates, to understand this: represent $\mathcal{S}_a$ as a matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and $\mathcal{K}_\gamma : x \mapsto x(1+y)$, acting trivially on the fibers.

3. • Hyperkähler integrable system data as in #2 above, but think of Γ as $H_1(\Sigma_u, \mathbb{Z})$ where $\Sigma_u = \{y^2 = u - z^2\} \subset \mathbb{C}^2$; γ_e is the basic cycle winding around both points, while γ_m is virtual cycle running to "large distance", say from $(y, z) \approx (\Lambda, \Lambda)$ to $(y, z) \approx (-\Lambda, \Lambda)$.
   - $K = 2$, $N = \{y^2 = u - z_0^2\}$ for some fixed z_0
   - Γ_i is the set of homology classes of paths on Σ_u running from some (y, z_0) to $\approx (\Lambda, \Lambda)$.
   - $Z : \Gamma_i \to \mathbb{C}$ is $Z_a = \int_{(y,z_0)}^{(\Lambda,\Lambda)} y \, \mathrm{d}z$.
   - Scattering diagram given by (draw picture here, for $\zeta \in \mathbb{R}$ and $z_0 = 1 + 2\mathrm{i}$).
   - Hyperholomorphic rank-2 bundle E over $\mathbb{R}^2 \times T^2$ is a self-dual $SU(2)$ connection over the Ooguri-Vafa space.

4. Fix a compact complex curve C, with punctures z_ℓ; a group $G = U(K)$; and a semisimple $m_\ell \in \mathfrak{g}_\mathbb{C}$ for each puncture.

   - $\mathcal{B}$ is the set of tuples $u = (\varphi_1, \cdots, \varphi_K)$ such that each φ_r is a meromorphic section of $K_C^{\otimes r}$, holomorphic away from the z_ℓ, $Res_{z_\ell}\varphi_r$ is the r-th Casimir of m_ℓ. Define

$$\Sigma_u = \{\lambda^K + \sum \varphi_r \lambda^{K-r} = 0\} \subset T^*C$$

   Then

$$\mathcal{B}' = \{u \in \mathcal{B} : \Sigma_u \text{ is smooth}\}$$

   - $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$, with intersection pairing $\langle , \rangle$. Γ^f is the kernel of this pairing.

- $Z_\gamma = \oint_\gamma \lambda$ (λ the tautological 1-form).
- Fix a point $z \in C$. $\mathcal{B}'' = \{u : z$ is not ramification point of $\Sigma_u\}$
- $\Gamma_{ij,u}$ is the set of paths from $z^{(i)}$ to $z^{(j)}$ on C, up to homotopy.
- $Z_a = \int_a \lambda$.
- Scattering diagram constructed by the general "shooting out lines" procedure.
- $\mathcal{M}$ is a moduli space of solutions to Hitchin's equations over C (with singularities at the poles), and E is the universal bundle over $\mathcal{M} \times C$, restricted to $\mathcal{M} \times \{z\}$. We'll explain why momentarily.

8 Gluing

The expected picture of the hyperkähler space $\mathcal{M}$:

Underlying manifold $\mathcal{M}$ is (a twisted version of) $\mathrm{Hom}(\Gamma_g, U(1))$.

For each $\zeta \in \mathbb{C}^\times$, have a section $\mathcal{X}(\zeta) : \mathcal{M} \to T$, which locally identifies $\mathcal{M}$ with T. Use this map to pull back holomorphic and symplectic structures. Thus we get a $\mathbb{C}^\times$-family of holomorphic symplectic structures on $\mathcal{M}$, but also we get *distinguished local Darboux coordinates* $\mathcal{X}_\gamma$, the components of the map $\mathcal{X}$.

$\mathcal{X}(\zeta)$ is characterized as solution of a *Riemann-Hilbert problem*:

- $\mathcal{X}(\zeta)$ depends piecewise holomorphically on ζ,
- $\mathcal{X}(\zeta)$ *jumps* at the walls of the scattering diagram, according to the given automorphisms,
- As $\zeta \to 0$ or ∞, it has asymptotic behavior

$$\mathcal{X}_\gamma(\zeta) \sim c_\gamma \exp\left(RZ_\gamma/\zeta + \mathrm{i}\theta_\gamma + R\bar{Z}_\gamma\zeta\right),$$

  where $c : \Gamma \to \mathbb{R}$, and $R > 0$ is a fixed constant.

Knowing the whole family of holomorphic symplectic structures on $\mathcal{M}$ is equivalent to knowing the hyperkähler metric on $\mathcal{M}$ [Hitchin], via the expansion

$$\varpi(\zeta) = \frac{\omega_+}{\zeta} + \omega_3 + \zeta\omega_-$$

with $\omega_\pm = \omega_1 \pm \mathrm{i}\omega_2$. Note, the asymptotics we fixed are what guarantees ϖ has such an expansion, with only 3 terms.

The expected picture of the vector bundle E is very similar: we first fix a C^∞ bundle $E = \oplus_i \operatorname{Hom}(\Gamma_i, U(1))$ over $\mathcal{M}$ and then give an identification between it and $\mathcal{X}^*V$, jumping at the walls by the appropriate automorphisms. This amounts to giving a section of E for each $a \in \Gamma_i$, with asymptotics as $\zeta \to 0, \infty$

$$\mathcal{X}_a(\zeta) \sim c_a \exp(RZ_a/\zeta + \mathrm{i}\theta_a + R\bar{Z}_a\zeta)$$

(where we use $e^{\mathrm{i}\theta_a}$ to denote a tautological section of E).

These *distinguished local sections* give in particular a holomorphic structure $\bar{\partial}_\zeta$ on E for each $\zeta \in \mathbb{CP}^1$, with control over asymptotics of $\bar{\partial}_\zeta$ as $\zeta \to 0, \infty$, which guarantees that these $\bar{\partial}_\zeta$ indeed come from a single unitary connection on E.

Now consider the Hitchin example; then we have a family of bundles E_z, depending on the parameter $z \in C$, or better, a single bundle E on $\mathcal{M} \times C$. The distinguished local sections induce a connection $\nabla(\zeta)$ in E, of the form

$$\nabla(\zeta) = R\varphi/\zeta + D + R\varphi^\dagger\zeta$$

This means a solution of Hitchin equations!

9 Geometric SYZ picture

Practical consequence: the $\mathcal{X}$ are given by solution of Riemann-Hilbert problem, recode this in terms of integral equation, then can actually find $\mathcal{X}$ on a computer; and can write a formal series representation for a solution. In principle this means you can get all the things which are supposed to be derived from $\mathcal{X}$: hyperkähler metrics, hyperholomorphic connections, solutions of Hitchin equations.

A consequence of that: an extension of *geometric SYZ* picture of hyperkähler integrable systems. The idea: both the hyperkähler metrics and hyperholomorphic bundles come in 1-parameter families, labeled by R. In the $R \to \infty$ limit, it's expected that the hyperkähler metric becomes very "simple" away from the singularities, while near the singularities one glues in a specific "fiducial" hyperkähler metric, generalization of Ooguri-Vafa [Gross-Wilson, Todorov, Kontsevich-Soibelman]; this is indeed what you get from the scattering diagram approach. Similarly, in $R \to \infty$ limit, the hyperholomorphic bundle becomes very "simple" away from the singularities: like the semiflat example we had above. Finally, get a similar picture for the solutions of Hitchin equations.

10 Extended DT invariants

The $\Omega(\gamma)$ which appear in the construction of hyperkähler metrics have an interpretation as DT-type invariants. Namely, fix some point $u \in \mathcal{B}$, want to get a collection of DT-type invariants $\Omega(u, \gamma)$; to do so, for each γ, draw the diagram at $Z_\gamma(u)/\zeta \in \mathbb{R}_+$. Then physically we expect these numbers to be dimensions of spaces of BPS states; moreover "by construction" they will obey WCF for DT invariants; and in many cases one knows the appropriate category.

The $\mu(a)$ which appear in our extended construction have a similar *physical* meaning: they are dimensions of spaces of BPS states. But these are BPS states which involve both an "A" and a "B" object in a sense. Something *like* a special Lagrangian with boundary on a complex surface, or a gradient flow of holomorphic Chern-Simons. From the point of view of physics, they do not represent simply a particle, rather a particle which is stuck on a string.

The $\Omega(\gamma)$ also get "improved" to $\omega(\gamma, a)$ for $a \in \Gamma_{ij}$, with

$$\omega(\gamma, a + \mu) - \omega(\gamma, a) = \Omega(\gamma)\langle\gamma, \mu\rangle$$

We don't know the categorical meaning of these enhanced invariants, however we do know what their WCF should be, and we know how to compute some examples, namely those attached to Hitchin systems.