Enumerative invariants and Hitchin systems

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(work with Davide Gaiotto, Greg Moore)

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Preface

In recent joint work with Davide Gaiotto and Greg Moore, we discovered an unexpected connection between hyperkähler geometry and the theory of (generalized) Donaldson-Thomas invariants.

Roughly: Donaldson-Thomas invariants are the key ingredient in a new construction of hyperkähler metrics.

In this talk I describe this connection, focusing on a special case in which the whole story is especially concrete. This special case is related to the geometry of Hitchin's integrable system (recently of interest for Geometric Langlands).

The work was originally motivated by the physics of $\mathcal{N} = 2$ supersymmetric gauge theories, but I will mostly suppress that in the talk.

Outline

Calabi-Yau manifolds

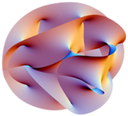
According to Yau's proof of Calabi's conjecture, any Kähler manifold $\mathcal{M}$ with $c_1(\mathcal{M}) = 0$ (Calabi-Yau manifold) admits a Ricci-flat Kähler metric.

The theorem is a triumph of hard analysis. It implies that lots of Ricci-flat Kähler metrics exist. Famous example: quintic threefold in $\mathbb{CP}^4$

$$\{x_1, \ldots, x_5 \in \mathbb{C} : P_5(x_1, x_2, x_3, x_4, x_5) = 0\} \,/\, \mathbb{C}^\times$$

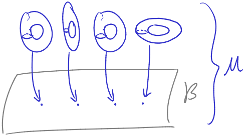
with P_5 homogeneous polynomial of degree 5.

However, it gives very little guidance about what these metrics actually look like.

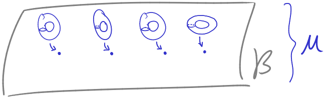


SYZ picture of Calabi-Yau manifolds

Motivated by mirror symmetry, Strominger-Yau-Zaslow proposed a simple picture: a Calabi-Yau manifold $\mathcal{M}$ of complex dimension n is fibered by special Lagrangian tori of real dimension n.



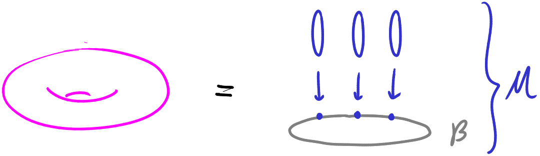
Such $\mathcal{M}$ typically come in families, i.e. depend on parameters. Gross and Wilson proposed that in a certain limit of these parameters ("large complex structure"), the torus fibers shrink and $\mathcal{M}$ collapses to the base $\mathcal{B}$ of this fibration.



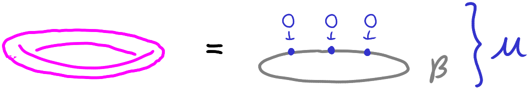
SYZ picture of Calabi-Yau manifolds

The only Calabi-Yau of complex dimension 1 is a 2-torus.

SYZ picture here is trivially correct: a 2-torus is a (trivial) circle fibration over a circle.



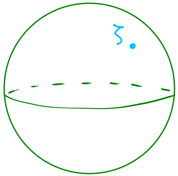
Gross and Wilson's degeneration picture is also trivially correct: the relative size of the two circles is a parameter of the flat metric; there is a limit of this parameter in which $\mathcal{M}$ collapses to a single circle.



SYZ picture of K3

A 2-dimensional Calabi-Yau is either a 4-torus or a K3 surface. Take $\mathcal{M}$ to be K3.

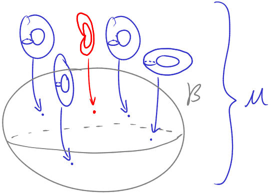
$\mathcal{M}$ is not only Kähler but hyperkähler. This means it is Kähler with respect to a whole $\mathbb{CP}^1$ worth of complex structures. Call them $J^{(\zeta)}$.



Moreover, in each of these complex structures $\mathcal{M}$ has a holomorphic symplectic form, $\varpi^{(\zeta)}$.

SYZ picture of K3

In one of its complex structures (say $J^{(\zeta=0)}$), $\mathcal{M}$ is elliptically fibered. The base of the fibration is $\mathcal{B} = \mathbb{CP}^1$.



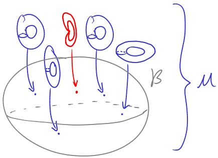
Generic fiber is a compact complex torus.

Unlike the 1-dimensional case, here we have to allow the fibration to have singular fibers (although the total space is smooth.) Generically, 24 of them.

In complex structure $J^{(\zeta=1)}$, the torus fibers are special Lagrangian. This realizes the SYZ picture.

SYZ picture of K3

How about the metric?



Locally, identify the torus fiber with $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, where τ varies holomorphically over $\mathcal{B}$. Also construct local coordinate a on $\mathcal{B}$ using $\varpi^{(\zeta=0)}$.

There's a simple explicit metric g^{sf}, which locally looks like:

$$g^{sf} = (\operatorname{Im}\tau(a))|da|^2 + \frac{1}{R^2(\operatorname{Im}\tau(a))}|dz|^2$$

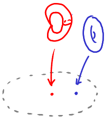
Depends on a real parameter R; as $R \to \infty$, the fibers shrink to zero size. g^{sf} is Ricci-flat and hyperkähler, but horribly singular at the 24 degenerate fibers.

[Greene-Shapere-Vafa-Yau]

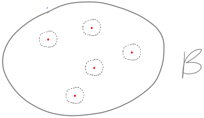
SYZ picture of K3

How to correct g^{sf} to the desired g?

There is a nice “model” for the behavior near each bad fiber: Ooguri-Vafa metric on a torus fibration over the disc with a single degenerate fiber. It’s hyperkähler and smooth.



So, simplest idea: start with g^{sf}, cut out a neighborhood of each bad fiber and glue in the Ooguri-Vafa metric.



SYZ picture of K3

Gross-Wilson show the resulting metric $g^{GW}(R)$ is smooth, not exactly Ricci-flat, but "extremely close": there is a Ricci-flat metric $g(R)$ such that

$$g^{GW}(R) - g(R) \to 0 \text{ exponentially as } R \to \infty.$$

This is enough to prove their conjecture about collapsing as $R \to \infty$.

SYZ picture of K3

What if we want to do better: get an asymptotic series for the exact $g(R)$ around $R \to \infty$?

This is the problem we address — not for K3 but for some simpler noncompact (but complete) hyperkähler spaces $\mathcal{M}$.

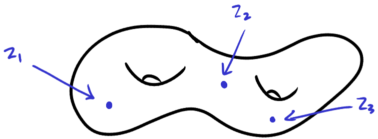
(We hope that the difficulties in extending to K3 are "only" technical...)

Next, let's describe the $\mathcal{M}$ we study.

Outline

Flat connections

Fix compact Riemann surface C, with $n > 0$ marked points z_i, $i = 1, \dots, n$. Let C' be C with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ and $m_i^{(3)} \in \mathbb{R}$ for each i.



Let $\mathcal{M}$ be the moduli space of flat $SL(2,\mathbb{C})$-connections over C', such that the holonomy around z_i is conjugate to $\begin{pmatrix} \mu_i & 0 \\ 0 & \mu_i^{-1} \end{pmatrix}$ where

$$\mu_i = \exp(Rm_i + im_i^{(3)} + R\bar{m}_i)$$

(Such a connection is determined by its monodromy representation, i.e. a homomorphism $\pi_1(C') \to SL(2,\mathbb{C})$, and determines that representation up to equivalence.)

Flat connections

By construction, $\mathcal{M}$ is a complex manifold, of dimension $6g - 6 + 2n$.

In fact $\mathcal{M}$ has an additional, rather unexpected structure (due to Hitchin): a hyperkähler metric g! So in particular $\mathcal{M}$ has a canonical $\mathbb{CP}^1$ worth of complex structures. Call them $J^{(\zeta)}$; the original one is $J^{(\zeta=1)}$.

In complex structure $J^{(\zeta=0)}$, $\mathcal{M}$ is a fibration over a complex base $\mathcal{B}$. The generic fiber is a compact complex torus.

This is just like the picture of K3 which Gross-Wilson exploited, except $\mathcal{B}$ is an affine space instead of S^2. Namely $\mathcal{B}$ is the space of meromorphic quadratic differentials φ_2 on C with double pole at each z_i, residue m_i.

hyperkähler metric on space of flat connections

As with K3, we can "easily" write down a hyperkähler metric g^{sf} on $\mathcal{M}$, which is smooth in most places but singular at the bad fibers.

The interesting part of the story is the corrections that modify g^{sf} to g. Want to describe these corrections exactly.

Where do they come from? Turns out they can be explicitly described in terms of certain integer invariants...

Outline

- Calabi-Yau manifolds and SYZ
- Flat connections
- Invariants of a quadratic differential
- Constructing the hyperkähler metric
- Wall-crossing
- A little about the proof

Invariants of a quadratic differential

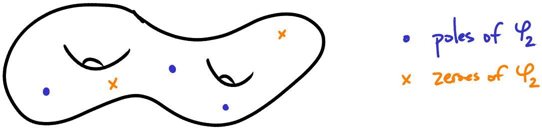
Fix a point of $\mathcal{B}$, i.e. fix a meromorphic quadratic differential φ_2 on C with double pole at each z_i, residue m_i.

This determines a metric h on C, in a simple way:

$$h = |\varphi_2|$$

(so if $\varphi_2 = P(z)\,dz^2$ then $h = |P(z)|\,dz\,d\bar{z}$.)

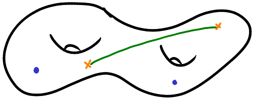
More precisely, h is a metric on only an open subset of C, where we delete both the poles of φ_2 (the z_i) and also the zeroes of φ_2. h is flat on this open subset.



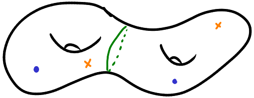
Invariants of a quadratic differential

Now we can consider finite length inextendible geodesics on C' in the metric h. These come in two types:

- Saddle connections: geodesics running between two zeroes of φ_2. These are rigid (don't come in families).



- Closed geodesics. When they exist, these come in 1-parameter families, sweeping out annuli on C'.

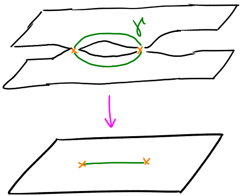


Invariants of a quadratic differential

To "classify" these finite length geodesics, introduce a little more technology: φ_2 determines a branched double cover $\Sigma \to C$,

$$\Sigma = \{\lambda : \lambda^2 = \varphi_2\} \subset T^*C.$$

Each finite length geodesic can be lifted to a union of closed curves in Σ, representing some homology class $\gamma \in H_1(\Sigma, \mathbb{Z})$.



We define an invariant $\Omega(\gamma)$ which counts these finite length geodesics: every saddle connection with lift γ contributes $+1$, every closed loop with lift γ contributes -2.

$\Omega(\gamma)$ are the key ingredients in our construction of the metric g.

Outline

Twistor description of hyperkähler metrics

How to describe the hyperkähler metric we're going to construct on $\mathcal{M}$? Main technical tool: twistor picture of hyperkähler geometry.

If $\mathcal{M}$ is any hyperkähler manifold, we can reconstruct the hyperkähler metric g if we know all the holomorphic symplectic structures $(J^{(\zeta)}, \varpi^{(\zeta)})$ on $\mathcal{M}$.

More precisely: the holomorphic symplectic form $\varpi^{(\zeta)}$ has an expansion

$$\varpi^{(\zeta)} = \zeta^{-1}(\omega_1 + i\omega_2) + \omega_3 + \zeta(\omega_1 - i\omega_2)$$

and the metric is just

$$g = \omega_1(\omega_2)^{-1}\omega_3$$

The corrected hyperkähler metric

We construct $\varpi^{(\zeta)}$ by producing "holomorphic Darboux coordinates" $\mathcal{X}_\gamma(\zeta)$.

Obtained as solutions of an integral equation

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf}(\zeta) \exp\left[\sum_{\gamma'} \frac{\Omega(\gamma')\langle\gamma,\gamma'\rangle}{4\pi i} \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta+\zeta'}{\zeta-\zeta'} \log(1-\mathcal{X}_{\gamma'}(\zeta'))\right]$$

where $\mathcal{X}_\gamma^{sf}$ is a simple explicit function of the form

$$\mathcal{X}_\gamma^{sf} = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]$$

Here θ_γ are angular coordinates on the torus fibers of $\mathcal{M}$, $Z_\gamma = \frac{1}{\pi}\oint_\gamma \lambda$, and $\ell_\gamma = Z_\gamma \mathbb{R}_- \subset \mathbb{C}$.

The corrected hyperkähler metric

Theorem: [Gaiotto-Moore-Neitzke]

Let $\mathcal{M}$ be a moduli space of flat $SL(2, \mathbb{C})$ connections as above, and define $\Omega(\gamma)$ as above. For R large enough, the above construction yields $\varpi^{(\zeta)}$ corresponding to an hyperkähler metric g on $\mathcal{M}$. g coincides with the hyperkähler metric defined by Hitchin.

(In particular, writing an iterative solution to the integral equation should lead to (at least) an asymptotic series representation for the metric.)

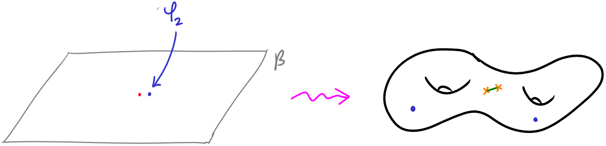
The corrected hyperkähler metric

In the large R limit,

$$g = g^{sf} + O(e^{-RL})$$

where L is the length of the shortest finite geodesic.

In particular, the biggest corrections arise in regions of $\mathcal{M}$ corresponding to φ_2 that yield a very short geodesic. These are the regions near the bad fibers of $\mathcal{M}$.



Including only the correction coming from this short geodesic and ignoring all others would give an analogue of Gross-Wilson's approximate metric on K3.

Outline

Calabi-Yau manifolds and SYZ

Flat connections

Invariants of a quadratic differential

Constructing the hyperkähler metric

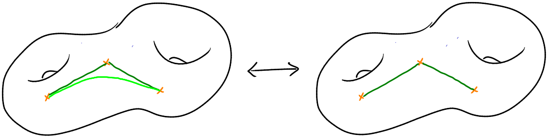
Wall-crossing

A little about the proof

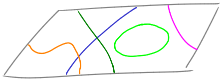
Wall-crossing

One of the key ingredients of the proof is a careful understanding of how the integers $\Omega(\gamma)$ vary as we move around in $\mathcal{B}$.

Indeed, as we vary the quadratic differential φ_2 and hence our metric on C, the finite geodesics on C counted by $\Omega(\gamma)$ can appear or disappear. The mechanism is "formation of bound states" or "decay into constituents".



This phenomenon occurs at codimension-1 loci in $\mathcal{B}$ (walls).



So $\Omega(\gamma)$ is only piecewise constant on $\mathcal{B}$.

Wall-crossing

$\Omega(\gamma)$ is only piecewise constant on $\mathcal{B}$.

Since $\Omega(\gamma)$ entered into our construction, this looks dangerous: will it make g discontinuous?

It turns out that the jumping of $\Omega(\gamma)$ is completely determined by a wall-crossing formula (WCF); and this jumping behavior is exactly what's needed to ensure that g is continuous.

Moreover this WCF is actually identical to one written down by Kontsevich-Soibelman in a very different context: the theory of Donaldson-Thomas invariants. [Bridgeland, Kontsevich-Soibelman, Joyce-Song, ...]

A surprising connection!

Wall-crossing formula

Above we considered classes $\gamma \in H_1(\Sigma, \mathbb{Z})$ and we defined $Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda$. The Z_γ vary as we move in $\mathcal{B}$.

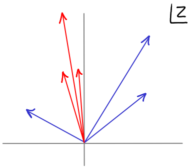
To state Kontsevich-Soibelman WCF, axiomatize that structure a bit:

- Complex manifold $\mathcal{B}$
- Lattice Γ w/ antisymmetric pairing $\langle , \rangle$
- Homomorphism $Z : \Gamma \to \mathbb{C}$ for each point of $\mathcal{B}$, varying holomorphically over $\mathcal{B}$
- "invariants" $\Omega : \Gamma \to \mathbb{Z}$ for each point of $\mathcal{B}$

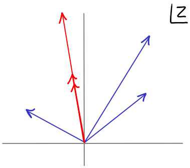
WCF tells how $\Omega(\gamma)$ vary as we move around on $\mathcal{B}$.

Wall-crossing formula

Walls in $\mathcal{B}$ are loci where some set of Z_γ (for lin. indep. γ with $\Omega(\gamma) \neq 0$) become aligned:



Near the wall.



On the wall.

Focus on these participating γ.

Wall-crossing formula

Introduce torus algebra with one generator X_γ for each γ,

$$X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}$$

To each participating γ, assign an automorphism $\mathcal{K}_\gamma$ of torus algebra:

$$\mathcal{K}_\gamma : X_{\gamma'} \mapsto (1 + X_\gamma)^{\langle\gamma,\gamma'\rangle} X_{\gamma'}$$

Now consider a product over all participating γ,

$$: \prod_\gamma \mathcal{K}_\gamma^{\Omega(\gamma)} :$$

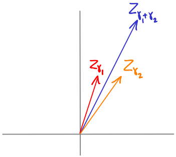
where :: means we multiply in order of the phase of Z_γ.

The Kontsevich-Soibelman WCF is the statement that this automorphism is the same on both sides of the wall.

Wall-crossing formula

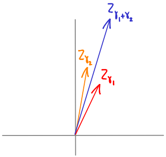
For example: if $\langle \gamma_1, \gamma_2 \rangle = 1$,

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2}$$



equals

$$\mathcal{K}_{\gamma_2}^{\Omega'(\gamma_2)} \mathcal{K}_{\gamma_1+\gamma_2}^{\Omega'(\gamma_1+\gamma_2)} \mathcal{K}_{\gamma_1}^{\Omega'(\gamma_1)}$$



if and only if

$$\Omega'(\gamma_1) = 1$$
$$\Omega'(\gamma_2) = 1$$
$$\Omega'(\gamma_1 + \gamma_2) = 1$$

Wall-crossing formula

More interesting example: if $\langle \gamma_1, \gamma_2 \rangle = 2$,

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \left(\prod_{n=0}^{\infty} \mathcal{K}_{n\gamma_1+(n+1)\gamma_2}\right)\mathcal{K}^{-2}_{\gamma_1+\gamma_2}\left(\prod_{n=\infty}^{0} \mathcal{K}_{(n+1)\gamma_1+n\gamma_2}\right)$$

So,

- on one side of the wall we have only $\Omega(\gamma_1) = 1$ and $\Omega(\gamma_2) = 1$, all others zero;
- on the other side we have infinitely many $\Omega(\gamma) = 1$, and also $\Omega(\gamma_1 + \gamma_2) = -2$.

Wall-crossing formula

Key fact: the WCF holds for our $\Omega(\gamma)$!

So e.g.

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1}$$

for $\langle\gamma_1,\gamma_2\rangle = 1$ says that if we have two saddle connections that intersect at 1 point, then after wall-crossing a third saddle connection will appear.

Similarly in the formula

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \Big(\prod_{n=0}^{\infty}\mathcal{K}_{n\gamma_1+(n+1)\gamma_2}\Big)\mathcal{K}_{\gamma_1+\gamma_2}^{-2}\Big(\prod_{n=\infty}^{0}\mathcal{K}_{(n+1)\gamma_1+n\gamma_2}\Big)$$

for $\langle\gamma_1,\gamma_2\rangle = 2$, on one side we have two saddle connections intersecting at two points; on the other side we have infinitely many saddle connections plus a single closed geodesic.

Outline

A little about the proof

How do we prove our theorem? (Maybe better to say we sketch a proof; papers are part of the physics literature so far.)

The main task is to get some geometric understanding of the functions $\mathcal{X}_\gamma$ which obey our integral equation, and see explicitly that they are indeed Darboux coordinates for the ϖ coming from Hitchin's hyperkähler metric.

Fock-Goncharov defined a Darboux coordinate system $\mathcal{X}^T$ on (open patch of) $\mathcal{M}$ for any ideal triangulation of C.

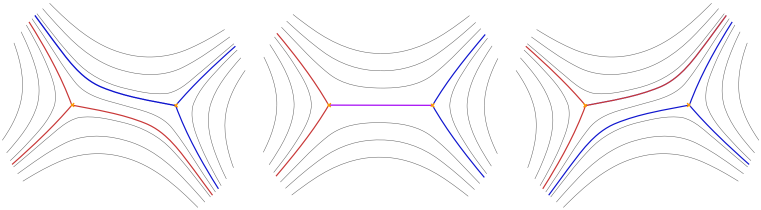
We construct a canonical triangulation T_{WKB} depending only on $\vartheta = \arg \zeta$ and φ_2. The edges of T_{WKB} are geodesics on C in the flat metric $|\varphi_2|$, along which $e^{-2i\vartheta}\varphi_2$ is real.

Then identify $\mathcal{X}_\gamma$ with Fock-Goncharov's functions $\mathcal{X}^{T_{WKB}}$.

A little about the proof

The desired integral equation is equivalent to two properties of $\mathcal{X}_\gamma$:

- They jump by the automorphism $\mathcal{K}_\gamma^{\Omega(\gamma)}$ when ζ crosses one of the rays ℓ_γ. This follows directly from corresponding jump of T_{WKB}.



$\vartheta < \vartheta_c$ $\vartheta = \vartheta_c$ $\vartheta > \vartheta_c$

- They have asymptotics $\sim e^{\pi R Z_\gamma / \zeta}$ as $\zeta \to 0$; these are obtained by careful application of WKB approximation to a connection of the form

$$\nabla^{(\zeta)} = \zeta^{-1}\varphi + D + \zeta\bar{\varphi}$$

Summing up

We have a new scheme for constructing hyperkähler metrics, giving more explicit information than has been previously available.

A crucial ingredient in this scheme is a set of integer "invariants" obeying the Kontsevich-Soibelman wall-crossing formula.

A concrete example of the story constructs the hyperkähler metric on Hitchin's integrable system with punctures. (In this talk I described the rank 2 case; there is a natural extension to higher rank, but not yet proven.)