1 Preface

Work partly finished (1110.1619), partly ***in progress*** with [Freed]. Builds on joint work with [Gaiotto-Moore].

Closely related to physics work of [Alexandrov-Persson-Pioline] and mathematical work of [Goncharov, Fock-Goncharov, Kontsevich-So Haydys, Hitchin].

Another application of $\mathcal{N} = 2$ supersymmetric QFT in four dimensions. If you study such a QFT on $\mathbb{R}^3 \times S^1$, at low energies, you see a sigma model whose target is a hyperkähler integrable system $\mathcal{M}$. Also see a recipe for constructing $\mathcal{M}$.

I'll describe an extension of the story, related to the physics of the theory on a circle *fibration* over $\mathbb{R}^3$. In this setting we meet $\mathcal{M}$ as before, but also a construction of a canonically defined line bundle V over $\mathcal{M}$, with a hyperholomorphic connection.

Then ask: why does this construction exist? Say, in "theories of class S." Not a complete answer yet, but partial answer: abelianization of Chern-Simons theory.

To ease the notation, I'll specialize to class S from the beginning.

2 Hitchin moduli space

Fix a compact complex curve C, and group G, say $G = U(K)$. The *Hitchin equations* for a tuple (E, h, φ, D) — E a C^∞ bundle, h a Hermitian metric in E, φ a $(1,0)$-form with values in $End(E)$, D a h-unitary connection in E — are

$$F + [\varphi, \varphi^\dagger] = 0, \qquad \bar\partial_D \varphi = 0.$$

The moduli space

$$\mathcal{M} = \mathcal{M}(C, G)$$

parameterizing solutions modulo equivalence is a hyperkähler space.

2.1 Twistor family

Like any hyperkähler space, $\mathcal{M}$ carries a family of complex structures J_ζ and holomorphic symplectic forms ϖ_ζ, for $\zeta \in \mathbb{CP}^1$. Organize them into *twistor family*

$$\mathcal{Z} \to \mathbb{CP}^1.$$

- Fiber over $\zeta = 0$: $\mathcal{M}^{Higgs}(G, C)$.
- Fiber over $\zeta \in \mathbb{C}^\times$: $\mathcal{M}^\flat(G, C)$.
- Fiber over $\zeta = \infty$: $\overline{\mathcal{M}}^{Higgs}(G, C)$.

Points of $\mathcal{M}$ give distinguished holomorphic sections of $\mathcal{Z}$.

Having $\mathcal{Z}$ as a complex manifold, with fiberwise symplectic structure and real structure, is equivalent to having $\mathcal{M}$ as a hyperkähler space.

2.2 The line bundle

Fact [Haydys, Hitchin]: $\mathcal{Z}$ is carrying an interesting holomorphic line bundle $\mathcal{V}$.

- Over the special fiber $\mathcal{M}^{Higgs}$, $\mathcal{V}$ is the *determinant line bundle*, i.e. fiber over (E, φ) is $\det H^*(E)$.
- Over the generic fibers $\mathcal{M}^\flat$, $\mathcal{V}$ is the *Chern-Simons line bundle*: fiber over ∇ is $CS(\nabla)$.

Recall that for any flat $GL(K)$-connection ∇ over C there is a corresponding line $CS(\nabla)$, "Chern-Simons line." If ∇^3 is a $GL(K)$-connection over a compact 3-manifold X then we can define a corresponding "classical Chern-Simons invariant" $CS(\nabla^3) \in \mathbb{C}^\times$; if it's on a trivial bundle, $\nabla^3 = \mathrm{d} + A$, then

$$CS(\nabla^3) = \exp\left[\frac{1}{4\pi\mathrm{i}}\int_X \mathrm{Tr}(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A)\right]$$

independent of trivialization. If X has boundary C, and ∇^3 restricts to a flat connection ∇ over C, there is a similar story except that $CS(\nabla^3)$ depends on the trivialization at the boundary, in a fixed way; this defines a line, which we call $CS(\nabla)$.

Concrete consequence: having $\mathcal{V}$ over $\mathcal{Z}$, trivial over sections, means that over $\mathcal{M}$ we get a *hyperholomorphic* line bundle V [Ward, Hitchin, Feix, Goncharov], i.e. bundle with a Hermitian metric and a single unitary connection, compatible with *all* the holomorphic structures. In particular, F is of type $(1,1)_\zeta$ for all ζ, i.e. a $U(1)$ instanton over $\mathcal{M}$.

3 Direct construction

The $\mathcal{N} = 2$ SUSY QFT leads to a picture of $\mathcal{Z}$ and $\mathcal{V}$ as "classical plus quantum corrections": build a simple explicit hyperkähler space $\mathcal{M}^{\mathrm{sf}}$ with explicit hyperholomorphic line bundle V^{sf}, then construct $\mathcal{Z}$ and $\mathcal{V}$ as C^∞ objects, plus a map

$$\Psi : \mathcal{Z} \to \mathcal{Z}^{\mathrm{sf}}$$

and lift it to a map of line bundles. All structure pulled back. [Gaiotto-Moore-Neitzke]

The map Ψ turns out to be *discontinuous* — these discontinuities give the "quantum corrections" to the metric in $\mathcal{M}$ and the hyperholomorphic structure in V. The precise construction of Ψ involves solving a Riemann-Hilbert problem.

The space $\mathcal{Z}^{\mathrm{sf}}$, fiber by fiber, locally looks like a *torus* $(\mathbb{C}^\times)^n$. Gluing maps like $(x, y) \to (x, y(1 + x))$. So Ψ is like cluster coordinates [Fock-Goncharov].

The line bundle $\mathcal{V}^{\mathrm{sf}}$ has explicit connection form

$$\frac{\mathrm{i}}{4\pi}(\log x \, d\log y - \log y \, d\log x) + d\log\psi$$

Gluing then involves the dilogarithm:

$$\psi \to \psi e^{\frac{1}{2\pi\mathrm{i}} Li_2(x)}$$

[Fock-Goncharov]

4 Abelianization

Now, how should we understand this construction? "Abelianization."

- For $\zeta = 0$: for each $(E, \varphi, \bar\partial)$ in $\mathcal{M}^{Higgs}$ there is a branched K-fold cover $\Sigma \subset T^*C$, holomorphic line bundle $(\mathcal{L}, \bar\partial)$ over Σ, and an isomorphism $\iota : \pi_*\mathcal{L} \simeq E$ of holomorphic line bundles.
- For $\zeta \in \mathbb{C}^\times$: for each $(E, \nabla) \in \mathcal{M}^\flat$ there is a branched K-fold cover $\Sigma \subset T^*C$, line bundle $\mathcal{L}$ with flat $GL(1)$-connection ∇^{ab} over Σ, and an isomorphism $\iota : \pi_*\mathcal{L} \simeq E$ — *but*, only on the complement of some "walls" $\mathcal{W}$, with unipotent jumps. So, concretely speaking, ∇ can be put in a local diagonal "gauge" everywhere away from the walls. Draw a "typical" example.

- For $\zeta = \infty$: complex conjugate of $\zeta = 0$.

Now what we claim for the lines:

- For $\zeta = 0$, have

$$\det H^*(\mathcal{L}, \bar{\partial}) \simeq \det H^*(E, \bar{\partial})$$

- For $\zeta \in \mathbb{C}^\times$, also have (in progress!)

$$CS(\mathcal{L}, \nabla^{\mathrm{ab}}) \simeq CS(E, \nabla)$$

  Natural context for this: a broader story involving also 3-manifolds. So let's suppose we have a 3-manifold, and the same structure: then

  - If you have a *diagonal* connection A then

$$\mathrm{Tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A) = \sum_{i=1}^{K} (A_i \wedge \mathrm{d}A_i)$$

    which is what you would get from integrating over the fibers.
  - The contribution from "gluing in" the unipotent matrices vanishes.

- For $\zeta = \infty$: complex conjugate of $\zeta = 0$.

5 Gluing

This picture now "explains" the gluing construction from before: the tori that we considered are the tori of *abelian* connections over the spectral curves, and the explicit line bundle with connection that we used is the line bundle associated to $GL(1)$ Chern-Simons.

To understand the dilogarithms we have to *compare* the abelianizations coming from two different spectral networks. This comparison gets most naturally interpreted in 3d, but this is a subject for another talk.