1 Preface

Maxwell theory: electromagnetism. Realized in nature at least approximately. (Have to look at very low energies, $E < m_e = 2 \times 10^{-27}$ g.)

$U(1)$ gauge theory — analogous to Yang-Mills theory which will be discussed later. But simpler, because $U(1)$ is abelian; for general G, the structure constants of $\mathfrak{g}$ give interactions, absent here. On the other hand, more complicated because not supersymmetric.

References: Witten lecture notes (Sec 8-10), Witten paper hep-th/9505186.

2 Classical theory

M Riemannian or Lorentzian manifold (spacetime). Electromagnetic field strength: a two-form F over M. Maxwell's equations in vacuum say F is harmonic,

$$dF = 0, \quad d \star F = 0.$$

Some vacuum solutions in $M = \mathbb{E}^{3,1}$:

$$F(x) = (\epsilon \wedge k)\, e^{ik(x)}, \quad \epsilon, k \in (\mathbb{R}^{3,1})^*, \quad \|k\|^2 = 0,\ \langle \epsilon, k \rangle = 0.$$

So electromagnetic waves have 2 physical polarizations ($\epsilon \sim \epsilon + \lambda k$), propagate at the speed of light.

More generally could include electric and magnetic current densities: $j_e, j_m \in \Omega^3_{closed}(M, \mathbb{R})$. Then Maxwell's equations become

$$dF = j_m, \quad d \star F = j_e.$$

Typical example: electrically charged particle, $j_e = 2\pi Q_e \delta(\gamma)$ for γ some path in M. Suppose the particle sits at rest, i.e. $\gamma(t) = (t, x_0) \in$

$\mathbb{E}^{3,1}$ for some fixed $x_0 \in \mathbb{E}^3$. Then the solution looks like

$$F = Q_e \frac{dt \wedge dr}{2r^2}$$

where $r = |x - x_0|$. Similarly the field of a magnetic monopole is

$$F = Q_m 2\pi dvol_{S^2}$$

The operation $F \leftrightarrow \star F$ exchanges $Q_e \leftrightarrow Q_m$.

If we have a 2-cycle with $\int \star F \neq 0$ then we say there is "electric flux" through that cycle, similarly "magnetic flux". If $H_2(M, \mathbb{R}) \neq 0$, then can have fluxes without sources.

3 Action

Now pass to the quantum theory for the electromagnetic field: keep the sources classical.

Naively might try to write the theory with F as the "fundamental" field. Turns out this is wrong: instead the data are a principal $U(1)$ bundle P over M, and a connection ∇ on P. F is now the curvature of ∇. This description breaks the symmetry between electric and magnetic: $dF = 0$ is automatic, while $d \star F = 0$ will come from extremization.

Action

$$S = \frac{1}{2g^2} \int_M F \wedge \star F$$

Equations of motion are linear: $d \star F = 0$.

Gauge symmetry: two equivalent (P, ∇) give the same S. Imagine P fixed for a moment, then $\{\nabla\}$ is affine modeled on $\Omega^1(M, \mathbb{R})$, the symmetry ("gauge group") is $\nabla \to \nabla + d\chi$, for any $\chi : M \to \mathbb{R}/\mathbb{Z}$. Regard equivalent configurations as physically the same.

Action is purely quadratic, so the theory is *free*: no interactions. Many subtleties of QFT absent.

4 Path integral

From now on, take M compact Euclidean, and oriented.

Provisional definition of the action, including theta angle:

$$S = \frac{1}{2g^2}\int_M F \wedge \star F + \frac{i\theta}{4\pi^2}\int_M F \wedge F$$

θ doesn't enter the equations of motion, since $\int_M F \wedge F$ is locally constant. Note orientation reversal takes $\theta \to -\theta$ (CP violation). Introduce

$$\tau = \frac{\theta}{\pi} + i\frac{2\pi}{g^2}$$

and then

$$\begin{aligned} S &= \frac{i\bar{\tau}}{4\pi}\int_M F_+ \wedge F_+ + \frac{i\tau}{4\pi}\int_M F_- \wedge F_- \\ &= \frac{i\bar{\tau}}{4\pi}\int_M \|F_+\|^2 - \frac{i\tau}{4\pi}\int_M \|F_-\|^2 \end{aligned}$$

where

$$F_\pm = \frac{1}{2}(F \pm \star F)$$

To compute partition function (or correlation functions) want to "integrate" S over the space $\mathcal{C}/\sim$ of all (P, ∇) up to equivalence,

$$Z = \int_{\mathcal{C}/\sim} \mu\, e^{-S}$$

(Z would have been infinite if we didn't divide by $\sim$.)

Note, Z has a trivial symmetry $\tau \to \tau + 2$, since $\int_M F \wedge F \in 4\pi^2\mathbb{Z}$ so that e^{-S} is unchanged by $\theta \to \theta + 2\pi$.

To integrate over the quotient, first fix one P in each equivalence class: labeled by $x = c_1(P)$. Also fix a harmonic $\nabla_h^{(x)}$. Now $\nabla = \nabla_h^{(x)} + A$. Action splits,

$$S(\nabla) = S(\nabla_h^{(x)}) + S(A)$$

(view A as connection on trivial bundle). So Z involves a sum over x, multiplied by a functional determinant.

Ignore torsion in $H^2(X, \mathbb{Z})$. Then lattice sum over x gives a Narain-Siegel theta function depending on $(\tau, \bar{\tau})$:

$$\sum_{x \in H^2(X,\mathbb{Z})} q^{\frac{1}{4}(-(x,x)+(x,\star x))} \bar{q}^{\frac{1}{4}((x,x)+(x,\star x))}, \quad q = e^{2\pi i \tau}$$

modular of weight $\frac{1}{2}(b_+, b_-)$.

Functional determinant is tricky. Roughly it should be $1/\sqrt{\det A}$ with A the quadratic form appearing in the action. First get rid of the modes which are pure gauge, and the honest zero modes. There is still a problem, $\det A$ is infinite — too many short-distance modes. It reflects the infinite vacuum energy due to the quantum fluctuations. This can be regulated e.g. by putting the theory on a lattice or Pauli-Villars, then adding a purely local counterterm to eliminate the cutoff dependence. If you put the theory on the lattice, get a total of $B_1 - B_0 + 1 - b_1$ nonzero modes, so the τ dependence of the determinants is

$$(\operatorname{Im} \tau)^{\frac{1}{2}(B_0 - B_1 + b_1 - 1)}$$

After adding a local counterterm, the τ dependence then comes out to $(\operatorname{Im} \tau)^{\frac{1}{2}(b_1 - 1)}$.

So including the functional determinant the weight of the full partition function becomes $\frac{1}{4}(\chi - \sigma, \chi + \sigma)$.

5 Duality

Theta function has modular property. Where does it come from? *Electric-magnetic duality*: theory originally formulated in terms of A can also be formulated in terms of A_D, related roughly by $dA = F$,

$dA_D = \star F$. The dual description has $\tau_D = -1/\tau$. Exchanges the strong coupling regime $\tau \to 0$ with weak coupling $\tau \to \infty$.

To prove it: introduce new theory with more fields, then reduce path integral to the original one and also a "dual" one. New theory has (P, ∇) as before, but also a new (P_D, ∇_D) and a 2-form G. Design it so that it has a symmetry under

$$(P, \nabla) \to (P, \nabla) \otimes (P', \nabla'), \quad G \to G + F'$$

(some kind of 2-form analogue of gauge symmetry). Then $\mathcal{F} = F - G$ is invariant. Invariant action

$$S = \frac{i\bar{\tau}}{4\pi} \int_M \|\mathcal{F}_+\|^2 - \frac{i\tau}{4\pi} \int_M \|\mathcal{F}_-\|^2 - \frac{i}{2\pi} \int_M F_D \wedge G$$

(the last term is invariant because of quantization of F_D, G). Treat this action in two ways:

- Integrate over (P_D, ∇_D). In each topological sector, integrate over A_D: $\int dx e^{ixy} = \delta(y)$ gives the condition $dG = 0$. Then summing over sectors, $\sum_n e^{iny} = \sum_m \delta(y - 2\pi m)$ gives $[G/2\pi]$ integral. So G is the curvature of some connection (P', ∇'). Use the extra symmetry to set $G = 0$, reducing to the original theory.

- Use the symmetry to set (P, ∇) to be trivial, then completing the square $G' = G - \frac{1}{\bar{\tau}}(F_D)_+ + \frac{1}{\tau}(F_D)_-$, the action becomes

$$-\frac{i\bar{\tau}}{4\pi} \int_M \|G'_+\|^2 + \frac{i\tau}{4\pi} \int_M \|G'_-\|^2 + \frac{i}{4\pi\bar{\tau}} \int_M \|(F_D)_+\|^2 - \frac{i}{4\pi\tau} \int_M \|(F_D)_-\|^2$$

  Then integrating over G' just gives normalization factors. The last two terms are a theory for (P_D, ∇_D), like the original theory but with $\tau \to -1/\tau$.

So we have a subgroup of $SL(2, \mathbb{Z})$ acting on τ, generated by $\tau \to \tau + 2$ and $\tau \to -1/\tau$. Naively our analysis says Z is invariant under this; but we didn't keep track of determinants coming from Gaussian integrals. Regulating and studying the determinants carefully one finds Z is modular of weights $\frac{1}{4}(\chi - \sigma, \chi + \sigma)$ as before.

Also transforms the point operators $F(x)$, $\star F(x)$ (valued in $\wedge^2(T_x M)$): inserting $\mathcal{F}_\pm = F_\pm - G_\pm$ in the big theory, show that duality acts by $F_+ \to (-1/\bar{\tau})(F_D)_+$, $F_- \to (1/\tau)(F_D)_-$. At $\theta = 0$ this means $F \to \frac{g^2}{2\pi} \star F_D$.

6 Operators

Most interesting gauge invariant operators are not localized at a point.

For example consider a closed curve $\gamma \subset M$, R_n the n-th representation of $U(1)$, and $W_n(\gamma) = \mathrm{Tr}_{R_n} Hol(\gamma)$ ("Wilson loop"). In the Lorentzian theory, this can be thought of as the track of an electrically charged particle with charge $Q_e = \frac{g^2 n}{2\pi}$: inserting $W_n(\gamma) = e^{in \oint_\gamma A}$ gives $d \star F = g^2 n \delta(\gamma)$. These operators are the analogue of the order operators from Sergei's second talk.

Trying to expand around a saddle point with $W_n(\gamma)$ included (or to compute perturbation theory around the loop), discover the action is infinite: e.g. in $\mathbb{E}^{3,1}$,

$$\int_M F \wedge \star F \sim \int dt \int \frac{dr}{r^2}$$

Regulate this by blowing up the curve to a tubular neighborhood of radius ϵ. Then the integral is cut off, $\sim \frac{1}{\epsilon}$. "Renormalize" the operator by including an explicit factor $e^{-\frac{L}{\epsilon}}$, then take the limit $\epsilon \to 0$. Could also try lattice regularization or perturbative cutoff regularization.

Also have 't Hooft operators: these change the rules of path integration. Choose $n \in \mathbb{Z}$, then $T_n(\gamma)$ means (P, ∇) have a singularity along γ with $\oint_{S^2} F = 2\pi n$. So this S^2 supports some magnetic flux, corresponds to a particle with $Q_m = n$. Again here have to regulate by cutting off spacetime near the loop, appropriately renormalize, then take $\epsilon \to 0$. These operators are the analogue of the disorder operators from Sergei's second talk.

For T_n to be a local operator, boundary term in the variation of the action should vanish. Trouble comes from $\frac{\theta}{4\pi^2}\int_M F \wedge F$, which varies to

$$\frac{\theta}{2\pi^2}\int_M d(\delta A \wedge F) = \frac{\theta}{2\pi^2}\int_{\partial M} \delta A \wedge F = \frac{n\theta}{\pi}\int \delta A\, dt$$

This boundary term is cancelled by the one coming from $\frac{1}{2g^2}\int F \wedge \star F$ if we require $\oint_{S^2} \star F = \frac{g^2 n\theta}{\pi}$, so this particle also carries $\frac{n\theta}{\pi}$ times the quantum of electric charge. In particular, under $\tau \to \tau + 2$, the electric charge of a particle with magnetic charge n is shifted by $2n$ units. This is the simple part of the $SL(2,\mathbb{Z})$ action on charges.

Natural guess: electric-magnetic duality will relate W_n to T_n.

Rough argument: say $\gamma = \partial E$ (otherwise the correlator is zero anyway, using invariance of S under tensoring by any flat bundle), then define the big theory including extra factor $in \int_E \mathcal{F}$:

- Integrating out (P_D, ∇_D) and gauging $G = 0$ recovers the Wilson loop amplitude.
- Integrating out (P, ∇) and then integrating over G recovers the dual theory with action

  $$\frac{i}{4\pi\bar{\tau}}\int \|(F_D - 2\pi n\delta(E))_+\|^2 - \frac{i}{4\pi\tau}\int \|(F_D - 2\pi n\delta(E))_-\|^2$$

  This expression $F_D - 2\pi n\delta(E)$ can be thought of as the curvature of a singular connection on a twist P'_D of P_D. Then letting ∇_D run over all connections on P_D, recover the dual theory at coupling $-1/\tau$, with the 't Hooft operator $T_n(\gamma)$ inserted.

Altogether, $\Gamma \subset SL(2,\mathbb{Z})$ transforms the electric and magnetic charges in the doublet representation.

7 Hamiltonian quantization

Formulate the theory on $M = \mathbb{R} \times X$. Let $C(X)$ denote the space of connections on principal $U(1)$ bundles over X up to equivalence. Choosing axial gauge and looking at Cauchy data: phase space of classical solutions is $TC(X)$, equipped with a symplectic structure (use metric on X to identify $TC(X) \simeq T^*C(X)$). So the connection on Σ is the canonical coordinate, while its time derivative is the canonical momentum.

The electric and magnetic fluxes are some classical functions $E(\Sigma)$, $B(\Sigma)$ on $TC(X)$,

$$E(\Sigma) = \int_\Sigma \star F, \quad B(\Sigma) = \int_\Sigma F.$$

When we quantize the theory, $B(\Sigma)$ will remain the same (it is just a function of the connection on Σ) while $E(\Sigma)$ will become an operator acting by infinitesimal diffeomorphisms.

Hilbert space $\mathcal{H}(X)$ obtained by quantizing $TC(X)$. $TC(X)$ is disjoint union of pieces $TC_x(X)$ labeled by conserved magnetic fluxes $B(\Sigma)$, so $\mathcal{H}(X)$ decomposes:

$$TC(X) = \bigcup_{x \in H^2(X,\mathbb{Z})} TC_x(X), \quad \mathcal{H}(X) = \bigoplus_{x \in H^2(X,\mathbb{Z})} \mathcal{H}_x(X)$$

Each $\mathcal{H}_x(X)$ is roughly $L^2(C_x(X))$. x is the collective eigenvalue of (the set of) $B(\Sigma)$.

Ignore torsion in $H^2(X,\mathbb{Z})$. Then $C_x(X)$ has an action of the space of flat connections on the trivial bundle over X, i.e. $H^1(X, U(1))$.

Decomposing in representations of $H^1(X, U(1))$ gives

$$\mathcal{H}(X) = \bigoplus_{x,y \in H^2(X,\mathbb{Z})} \mathcal{H}_{x,y}(X)$$

Here y is the collective eigenvalue of (the set of) $E(\Sigma)$.

Duality exchanges $(x, y) \to (-y, x)$.

8 Coupling to matter

Can also consider coupling the theory to dynamical matter. This just means sections of some bundles associated to P — for example, add a field $\phi \in \Gamma(P \otimes_G E_n)$ (complex line bundle) and terms

$$\frac{1}{2} \int_M \|\nabla \phi\|^2 - m^2 |\phi|^2$$

in the action. Then writing $\nabla = \partial + inA$ we find the equations of motion

$$d \star F = j_\phi, \quad (\Delta - m^2)\phi = 0,$$

where the current is

$$j_\phi = n \star \mathrm{Im}(\bar{\phi} d_\nabla \phi).$$

In this context one can talk about the Noether charge carried by ϕ particles, associated to global G transformations. (Photons have no charge, so didn't make sense to talk about this before.) Also get less pleasant things like vertex renormalization, need for regularization, running coupling.