Data

Fix the group $G = U(K)$, $G_{\mathbb{C}} = GL(K, \mathbb{C})$. Fix a real surface C with $n \geq 0$ punctures, and two semisimple orbits $m_i \subset \mathfrak{g}_{\mathbb{C}}$, $m_i^{\mathbb{R}} \subset \mathfrak{g}$ at each puncture z_i. Fix a rank K Hermitian bundle E over C, of degree 0.

The Hitchin system

Associated to these data there is a hyperkähler space: *Hitchin integrable system*, [Hitchin, Simpson]

$$\mathcal{M} = \mathcal{M}_d(G, C)$$

Defined as space of solutions of a hard PDE. Look at unitary connections D in E and $\varphi \in \Omega^{1,0}(\operatorname{End} E)$, with simple poles at the z_i (residues $m_i^{\mathbb{R}}$, m_i resp), obeying:

$$\begin{aligned} F_D + [\varphi, \varphi^\dagger] &= 0, \\ \bar{\partial}_D \varphi &= 0, \end{aligned}$$

modulo G-gauge.

$\mathcal{M}$ is hyperkähler, so it has a family of Kähler structures

$$(\mathcal{M}, I_\zeta, \omega_\zeta)$$

parameterized by $\zeta \in \mathbb{CP}^1$. For all $\zeta \in \mathbb{CP}^1$, $(\mathcal{M}, I_\zeta, \omega_\zeta)$ is a Calabi-Yau manifold (even holomorphic symplectic). For $|\zeta| = 1$,

- $(\mathcal{M}, \omega_\zeta)$ is a paradigm example of SYZ philosophy: it's a special Lagrangian fibration

$$\mathcal{M} \to \mathcal{B}$$

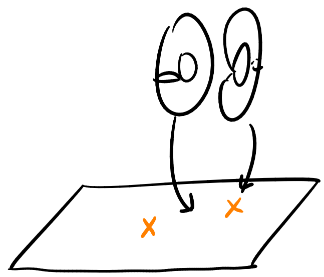
where $\mathcal{B}$ is the space of *spectral curves*

$$\Sigma = \{\det(\varphi - \lambda) = 0\} \subset T^*C$$

So a generic $u \in \mathcal{B}$ means a K-fold cover $\pi : \Sigma_u \to C$ with $\Sigma_u \subset T^*C$. Fiber over $u \in \mathcal{B}$ is

$$\mathcal{M}_u = Pic^d \, \Sigma_u$$

Singular fibers when Σ_u is singular.



- $(\mathcal{M}, I_\zeta)$ is concretely accessible space: for $\zeta \in \mathbb{C}^\times$ have

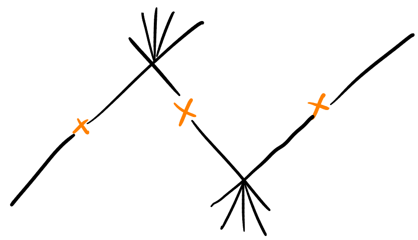
$$(\mathcal{M}, I_\zeta) \simeq \mathcal{M}^\flat(GL(K, \mathbb{C}), C)$$

moduli space of flat $G_\mathbb{C}$-connections, with monodromy around z_i given by

$$\mu_i = \exp(m_i/\zeta + \mathrm{i} m_i^\mathbb{R} + \bar{m}_i \zeta)$$

Scattering diagram

In joint work with [Gaiotto-Moore] we investigated $\mathcal{M}(G, C)$ from the point of view of supersymmetric quantum field theory. We draw a "scattering diagram" $\mathcal{D}$ on $\mathcal{B}$. Countably many walls of real codimension 1, maybe dense. Walls emanate from locus where Σ_u is singular.



What is $\mathcal{D}$? Moral answer first: a neighborhood of zero section in T^*C is hyperkähler; imagine Σ_u lies in this neighborhood, then

$$\mathcal{D} = \{u : \text{there are } I_\zeta\text{-hol curves ending on } \Sigma_u\} \subset \mathcal{B}.$$

The real definition of $\mathcal{D}$ is a tropical version of this.

NB: really a *family* of scattering diagrams $\mathcal{D}_\zeta$, since $\zeta \in \mathbb{C}^\times$ can vary. Today, just focus on one — say, $\zeta = 1$.

Canonical coordinate

Given

$$(u, \zeta) \in (\mathcal{B} \times \mathbb{C}^\times) \setminus \mathcal{D}$$

there is (conjecturally — but proven for $K = 2$ and for some u with $K > 2$) a canonical holomorphic Darboux coordinate system on $(\mathcal{M}, I_\zeta)$ in some neighborhood U of $\pi^{-1}(u)$, meaning a map

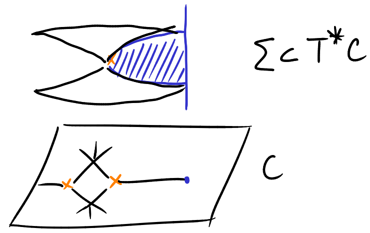
$$\Psi_u : U \to T = H^1(\Sigma_u, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^{2n}.$$

The coordinate systems Ψ_u are best understood as (local) maps on moduli spaces of flat connections,

$$\Psi_u : \mathcal{M}^\flat(GL(K,\mathbb{C}), C) \to \mathcal{M}^\flat(GL(1,\mathbb{C}), \Sigma_u).$$

To construct Ψ_u, first build another network of walls, $\mathcal{W}(u) \subset C$. Defined by (tropical version of):

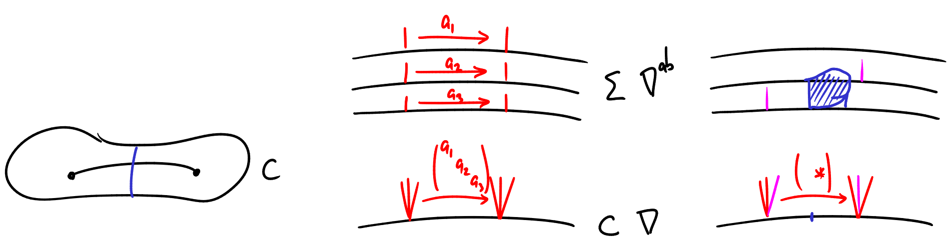
$$\mathcal{W}(u) = \{z : \text{there are } I_\zeta\text{-hol bigons ending on } \Sigma_u \text{ and } \pi^{-1}(z)\} \subset C.$$



Then

$$\Psi_u : \nabla \mapsto \nabla^{\mathrm{ab}}$$

such that there is an isomorphism $\iota : \nabla \simeq \pi_* \nabla^{\mathrm{ab}}$ on $C \setminus \mathcal{W}(u)$, and parallel transports of ∇ are those of $\pi_* \nabla^{\mathrm{ab}}$ (diagonal) plus corrections from holomorphic discs (nilpotent).



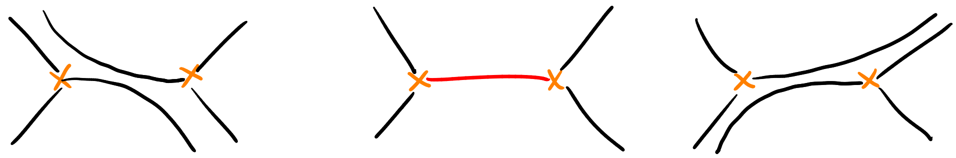
This is "almost-diagonalization" of ∇: expresses $GL(K)$-parallel transports of ∇ on C in terms of $GL(1)$-parallel transports of ∇^{ab} on Σ. e.g. for *closed* curves:

$$\mathrm{Tr}\, Hol_{\wp} \nabla = \sum_{\gamma \in H_1(\Sigma, \mathbb{Z})} \Omega(\wp, \gamma) Hol_{\gamma} \nabla^{\mathrm{ab}}$$

$\mathrm{Tr}\, Hol_{\wp} \nabla : \mathcal{M}^{\flat}(GL(K, \mathbb{C})) \to \mathbb{C}$ like "canonical theta function" in sense of [Gross-Hacking-Keel]; $\Omega(\wp, \gamma)$ like counts of broken lines (hopefully literally *equal* to those counts).

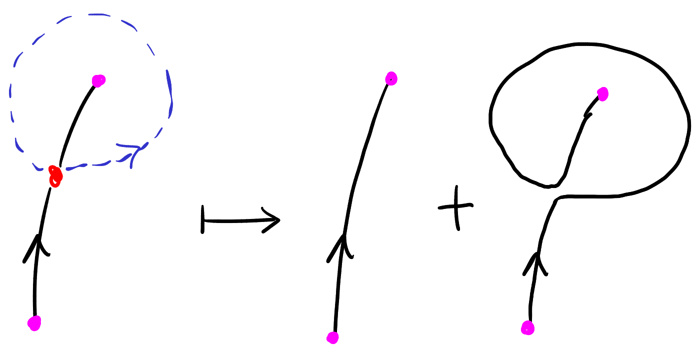
So what we've said: $\mathcal{W}(u)$ produces a map of *path groupoid algebras*

$$F_u : \mathbb{Z}[\pi_{\leq 1}(C)] \to \mathbb{Z}[\pi_{\leq 1}(\Sigma)]$$



When u crosses a wall in $\mathcal{D}$, the topology of $\mathcal{W}(u)$ changes.

Then F_u jumps by an automorphism of $\mathbb{Z}[\pi_{\leq 1}(\Sigma)]$ *determined by* the holomorphic curves ending on Σ_u. e.g. for a single disc, it's:



If we're only interested in the holonomies of $GL(1)$-connections ∇^{ab}, enough to pass to the subquotient $\mathbb{Z}[H_1(\Sigma)]$. Then this automorphism becomes $\mathfrak{X}$ version of cluster transformation:

$$X_\gamma \to X_\gamma (1 + X_\mu)^{\langle \gamma, \mu \rangle}$$

But on the full $\mathbb{Z}[\pi_{\leq 1}(\Sigma)]$ it's some *noncommutative version* of cluster transformation. Similar ideas discussed by [Goncharov-Kontsevich].

Clusters

In some cases, the Ψ_u are objects known in the cluster world.

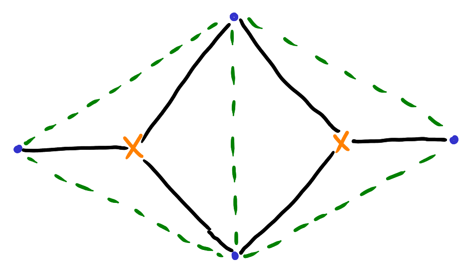
Best-known example is a slight variant of what I said: take $G = SU(2)$ instead of $G = U(2)$, and fix a Riemann surface C with $n \geq 1$ punctures. Then by the same construction, we have local coordinates

$$\Psi_u : \mathcal{M}^\flat(SL(2,\mathbb{C}), C) \to \mathcal{M}^\flat(GL(1,\mathbb{C}), \Sigma_u)^{odd}.$$

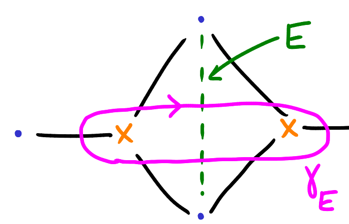
On the other hand it's known that (a cover of) $\mathcal{M}^\flat(PSL(2,\mathbb{C}), C)$ is $\mathfrak{X}$-cluster variety [Fock-Goncharov, Fomin-Shapiro-Thurston]:

- One seed for every *ideal triangulation* T of C.
- Cluster coordinates X_E^T associated to the *edges* of the triangulations.
- Mutations for *flips* of triangulations.

Each $u \in \mathcal{B}$ gives an ideal triangulation $T(u)$ of C — determined by the spectral network $\mathcal{W}(u)$, via picture like



The cluster coordinates of ∇ are among the holonomies of $\Psi_u(\nabla)$:



$$X_E^{T(u)} = Hol_{\gamma_E} \Psi_u(\nabla)$$

Recently [Berenstein-Retakh] introduced a noncommutative version of the corresponding cluster algebra [Fomin-Shapiro-Thurston]. Preliminarily, it looks like it fits into the above structure. Namely: fix u and consider a subalgebra of $\mathbb{Z}[\pi_{\leq 1}(\Sigma_u)]$, consisting of *lifts* of the edges in the triangulation $T(u)$. This gives four generators for each edge, corresponding to the $x_{ij}, x_{ji}, x_{ij}^{-1}, x_{ji}^{-1}$ in the Berenstein-Retakh algebra. Their "triangle relation" comes from relation in $\mathbb{Z}[\pi_{\leq 1}(\Sigma_u)]$. Applying the noncommutative cluster transformation above when we change triangulations then gives the "noncommutative Plucker relations" used by Berenstein-Retakh relating different noncommutative clusters.

There is also an extension to higher rank K: for some special u, Ψ_u gives cluster coordinates introduced in [Fock-Goncharov]. (Unlike $K = 2$, these u are not an open dense subset of $\mathcal{B}$.) A noncommutative version of this hasn't been discussed anywhere yet as far as I know; seems natural to try again taking paths which begin and end at preimages of punctures.