Preface

Aim in one sentence: describe connection between "quantum curves" and Hitchin equations.

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Data

Fix a compact Riemann surface C and a quadratic differential $\varphi_2 \in H^0(C, K_C^2)$.

Two different things we could do with this data:

Schrodinger operators

In local coordinates write

$$\varphi_2 = q_2(z)\mathrm{d}z^2$$

and then consider the operator

$$D_\hbar = \partial_z^2 - q_2/\hbar^2$$

(for $\hbar \in \mathbb{C}^\times$). $D_\hbar$ is globally well defined provided we do two things:

- let it act as $D_\hbar : K_C^{-1/2} \to K_C^{3/2}$ rather than on functions,
- restrict to a coordinate atlas where transition functions are Mobius transformations, i.e. fix a *complex projective structure.*

Fix once and for all the complex projective structure coming from *Fuchsian uniformization.*

We can also interpret D as a flat $SL(2,\mathbb{C})$-connection:

$$D_\hbar f = 0 \leftrightarrow \nabla^S_\hbar \begin{pmatrix} f \\ -\hbar f' \end{pmatrix} = 0$$

where in local coordinates

$$\nabla^S_\hbar = \mathrm{d} + \frac{1}{\hbar}\begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix}$$

Then the change of coordinates

$$z \mapsto \frac{az+b}{cz+d}$$

is accompanied by a gluing map:

$$\pm \begin{pmatrix} (cz+d)^{-1} & 0 \\ \hbar c & cz+d \end{pmatrix}$$

These gluings build globally a rank 2 holomorphic vector bundle

$$0 \to K_C^{\frac{1}{2}} \to E_\hbar \to K_C^{-\frac{1}{2}} \to 0$$

$E_\hbar$ a deformation of $E_0 = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$ ("indigenous bundle"), all isomorphic for $\hbar \neq 0$. $\nabla^S_\hbar$ is a connection in $E_\hbar$.

Remark: these *global* objects $\nabla^S_\hbar$ are candidates for the "quantum curve" coming from topological recursion applied to the Hitchin spectral curve [Eynard-Orantin,Dumitrescu-Mulase]

$$\Sigma = \{x^2 + \varphi_2 = 0\} \subset T^*C$$

Hitchin's equations

Begin again with E_0. Make it into a *Higgs bundle* by writing Higgs field $\varphi \in H^0(\mathrm{End}\, E_0)$:

$$\varphi = \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix} \mathrm{d}z$$

This is globally defined on its own, no need for projective structure. It is *not* a connection though.

Key theorem [Hitchin,Simpson]: there exists a unique Hermitian metric h in E, such that if D is the Chern connection in E we get

$$F_{D_h} + [\varphi, \varphi_h^\dagger] = 0$$

Thus there is a 1-parameter family of flat connections, for $\zeta \in \mathbb{C}^\times$,

$$\nabla_\zeta^H = \frac{\varphi}{\zeta} + D_h + \varphi^{\dagger_h}\zeta$$

This looks a lot like $\nabla_\hbar^S$. But they are not the same. Here we have a sort of symmetry (real structure) relating $\zeta \leftrightarrow -1/\bar{\zeta}$.

The relation

Analogy: two connections over a Frobenius manifold — $\nabla_\hbar^S$ is like "Dubrovin connection", ∇_ζ^H like "tt^* connection." [Cecotti-Vafa, Dubrovin] This analogy becomes sharp in some special cases.

What is the relation between these two?

Introduce a new parameter $R \in \mathbb{R}_+$ and rescale $\varphi \to R\varphi$ in ∇_ζ^H. Then have harmonic metrics $h = h(R)$, and flat connections

$$\nabla_{R,\zeta}^H = \frac{R\varphi}{\zeta} + D_{h(R)} + R\zeta\varphi^{\dagger_{h(R)}}$$

Now fix $\hbar \in \mathbb{C}^\times$ and take scaling limit:

$$\zeta = R\hbar, \qquad R \to 0$$

Theorem (DFKMMN, confirming a conjecture of [Gaiotto]): in this limit $\nabla_{R,\zeta}^H \to \nabla_\hbar^S$.

Proof: Study directly how $h(R)$ behaves as $R \to 0$, by perturbing around $q_2 = 0$: find

$$h(R) = \begin{pmatrix} \lambda/R & 0 \\ 0 & R/\lambda \end{pmatrix}$$

where $\lambda = \lambda_0 + O(R^4)$, and $\lambda_0^2 \mathrm{d}z \mathrm{d}\bar{z}$ is the hyperbolic metric. Plugging this in directly,

$$\nabla^H_{R,\zeta} \to \mathrm{d} + \frac{1}{\hbar} \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix} \mathrm{d}z + \begin{pmatrix} \partial_z \log \lambda_0 & 0 \\ 0 & -\partial_z \log \lambda_0 \end{pmatrix} \mathrm{d}z + \hbar \begin{pmatrix} 0 & 0 \\ \lambda_0^2 & 0 \end{pmatrix} \mathrm{d}\bar{z}.$$

Gauge transformation by

$$\begin{pmatrix} 1 & 0 \\ -\hbar \partial_z \log \lambda_0 & 1 \end{pmatrix}$$

gives $\nabla^S_\hbar$.

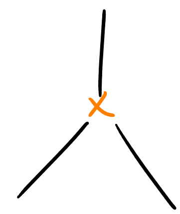
Stokes phenomena

For $\nabla^S_\hbar$ and ∇^H_ζ people have studied the WKB expansion ($\hbar \to 0$ or $\zeta \to 0$) and the question of how to go beyond it to construct *actual* solutions.[Voros, Ecalle, ..., Iwaki-Nakanishi, Gaiotto-Moore-N] Here meet *Stokes phenomenon*: solutions with WKB asymptotics inevitably *jump* at some *Stokes curves*.

In both cases it's believed that the Stokes curves are *critical graphs*

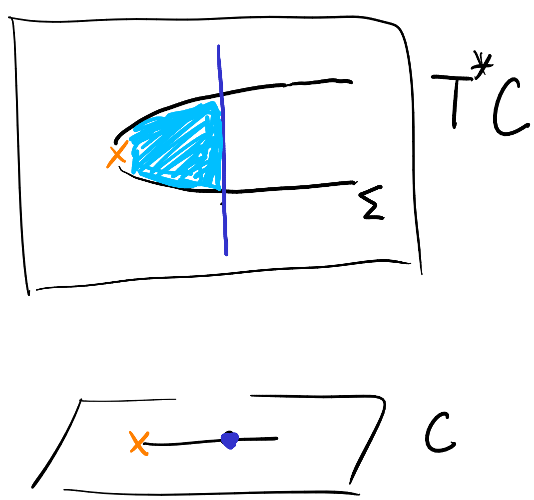
$$CG(\varphi_2, \vartheta = \arg \zeta)$$

Trajectories are paths with $e^{-\mathrm{i}\vartheta}\sqrt{\varphi_2}$ real, emanating from zeroes of φ_2. This limiting process presumably is part of the explanation of why.



Stokes phenomena vs topological string

For topological recursion, might be better to interpret the critical graph in terms of spectral curve: a neighborhood of zero section in T^*C is hyperkähler; claim: $CG(\varphi_2, \vartheta)$ is locus of $z \in C$ for which there exists a special Lagrangian bigon in T^*C of phase ϑ, with boundary on Σ and $T^*_z C$.



Topological recursion for WKB gives *perturbative* B model open partition function for $CY3$

$$X = \{uv + x^2 + \varphi_2(z) = 0\} \subset (T^*C)^{\oplus 3} \to C,$$

$\Psi(x, \hbar)$, with x a point of Σ. sLag bigons, A model branes, appear when we try to find a *nonperturbative* completion, the sections $\Psi(z, \hbar)$ of the oper; this picture seems to involve a brane on the whole fiber $T^*_z C$. [Nadler-Zaslow]