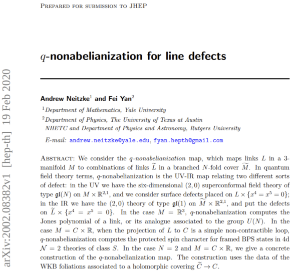
Protected spin characters, link invariants, and spectral networks

Andrew Neitzke
Yale University

String-Math 2020

Preface

I'll describe joint work with Fei Yan. Part is on arXiv ($N = 2$), part is in progress ($N > 2$).



PREPARED FOR SUBMISSION TO JHEP

q-nonabelianization for line defects

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ABSTRACT: We consider the *q-nonabelianization* map, which maps links L in a 3-manifold M to combinations of links $\widetilde{L}$ in a branched N-fold cover $\widetilde{M}$. In quantum field theory terms, q-nonabelianization is the UV-IR map relating two different sorts of defect: in the UV we have the six-dimensional $(2,0)$ superconformal field theory of type $\mathfrak{gl}(N)$ on $M \times \mathbb{R}^{2,1}$, and we consider surface defects placed on $L \times \{x^4 = x^5 = 0\}$; in the IR we have the $(2,0)$ theory of type $\mathfrak{gl}(1)$ on $\widetilde{M} \times \mathbb{R}^{2,1}$, and put the defects on $\widetilde{L} \times \{x^4 = x^5 = 0\}$. In the case $M = \mathbb{R}^3$, q-nonabelianization computes the Jones polynomial of a link, or its analogue associated to the group $U(N)$. In the case $M = C \times \mathbb{R}$, when the projection of L to C is a simple non-contractible loop, q-nonabelianization computes the protected spin character for framed BPS states in 4d $\mathcal{N} = 2$ theories of class S. In the case $N = 2$ and $M = C \times \mathbb{R}$, we give a concrete construction of the q-nonabelianization map. The construction uses the data of the WKB foliations associated to a holomorphic covering $\widetilde{C} \to C$.

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- Computing well-known link invariants for links in $M = \mathbb{R}^3$ in a new way,
- Constructing a (fairly explicit) embedding of a skein algebra into a quantized cluster algebra.

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We consider an $\mathcal{N} = 2$ supersymmetric field theory in $d = 4$, at a point of its Coulomb branch. IR description: $U(1)^r$ gauge theory.

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Question: what are the supersymmetric ground states of this system, with electromagnetic (+flavor) charge γ?

They are counted by a supersymmetric index: framed protected spin character (PSC)

$$\underline{\overline{\Omega}}(L, \gamma) := \mathrm{Tr}_{\mathcal{H}_{L,\gamma}} (-q)^{2J_3} q^{2I_3} \quad \in \quad \mathbb{Z}[q, q^{-1}].$$

(J_3 = Poincare, $I_3 = SU(2)_R$)

Framed protected spin characters

There is work on $\underline{\overline{\Omega}}(L, \gamma)$ from many points of view: semiclassical computation (for Lagrangian theories), quiver quantum mechanics, wall-crossing, spectral networks (for class S theories), localization, ...

[Gaiotto-Moore-N, Cordova-N, Moore-Royston-van den Bleeken]

[Gabella, Ito-Okuda-Taki, Galakhov-Longhi-Moore, ...]

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The framed PSC is a crude UV-IR map for line defects: think of UV line defect as decomposing into a sum of IR line defects,

$$L \quad \rightsquigarrow \quad F(L) := \sum_{\gamma} \underline{\overline{\Omega}}(L,\gamma) X_{\gamma}$$

with X_{γ} representing an IR Wilson-'t Hooft line of abelian electromagnetic (+ flavor) charge γ.

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- In the pure $\mathcal{N} = 2$ theory with gauge group $G = SU(2)$, at one point in weak coupling region of the Coulomb branch, writing charges $\gamma = (e, m)$, one has

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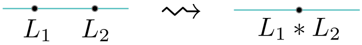
- In the Argyres-Douglas theory of type (A_1, A_3), charges $\gamma = (a, b, c)$, and there is a line defect L and a point of Coulomb branch for which

$$F(L) = X_{(1,0,0)} + X_{(1,1,0)} + X_{(0,0,-1)} + (-q - q^{-1})X_{(1,0,-1)} + X_{(0,-1,-1)} + X_{(1,-1,-1)} + X_{(1,1,-1)}.$$

Constraints from operator products

Reduce to 3 dimensions along S^1 with a twist by $(-q)^{2J_3} q^{2I_3}$; line defects wrapped around S^1 reduce to local operators, BPS when placed on the x^3-axis. These have a nonsingular, noncommutative, associative OPE $*$, in general complicated. [Gaiotto-Moore-N, Yagi, ...]



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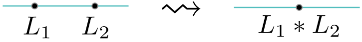

In the IR, the OPE is easy; charges and angular momenta just add, except for an extra contribution from Poynting vector: if $\langle , \rangle$ denotes Dirac-Schwinger-Zwanziger pairing,

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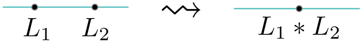

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(ie IR OPE algebra is quantum torus)
The UV-IR map is a homomorphism of OPE algebras:

$$F(L_1 *_{UV} L_2) = F(L_1) *_{IR} F(L_2)$$

The class S case

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(low energy limit of N fivebranes on $C \times \mathbb{R}^{3,1} \subset T^*C \times \mathbb{R}^{6,1}$)

A point of the Coulomb branch corresponds to a branched holomorphic N-fold covering $\widetilde{C} \to C$, $\widetilde{C} \subset T^*C$ (Seiberg-Witten curve).

The class S case

In this case, the OPE algebras (both UV and IR) can be described concretely as skein algebras.
[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]
[Drukker-Gomis-Okuda-Teschner, Gaiotto-Moore-N, Witten]
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To understand this, it is convenient to think in three dimensions: combine the Riemann surface C with the x^3-direction of spacetime, to make a 3-manifold

$$M = C \times \mathbb{R}$$

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If L is "flat," isotopic to a simple closed curve in $C \times \{x^3 = 0\}$, the line defect preserves $SO(3)$ and is 1/2-BPS.
[Drukker-Morrison-Okuda, ...]

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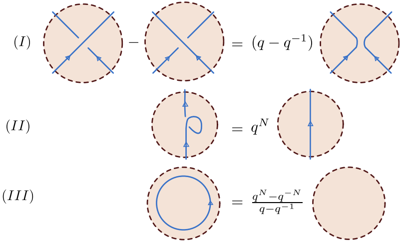

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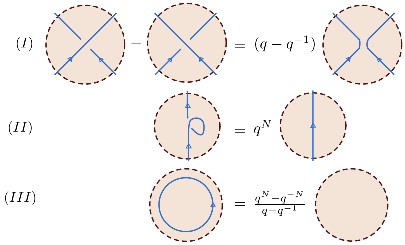
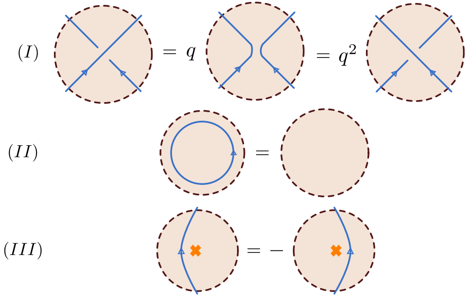


$L_1 *_{UV} L_2$ is defined by "stacking" L_1, L_2 in the $\mathbb{R}$ direction.
(to get the whole OPE algebra, for $N > 2$, one should also include networks)
[Sikora, Kuperberg, Xie, Bullimore, Tachikawa-Watanabe, Gabella, ...]

Skein algebras

The IR OPE algebra is (twisted) $gl(1)$ skein algebra of $\widetilde{C}$: the space of formal $\mathbb{Z}[q, q^{-1}]$-linear combinations of framed oriented links $\widetilde{L}$ in $\widetilde{M} = \widetilde{C} \times \mathbb{R}$, modulo relations



(in relation III the cross denotes 1-d branch locus of $\widetilde{M} \to M$)

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Then the skein relations immediately imply

$$X_{\gamma_1} *_{IR} X_{\gamma_2} = (-q)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

where now $\langle , \rangle$ is the intersection pairing on $H_1(\widetilde{C}, \mathbb{Z})$.

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$$F(L) = \sum_{\widetilde{L}} \alpha(\widetilde{L})\widetilde{L},$$
where $\alpha(\widetilde{L}) \in \mathbb{Z}[q, q^{-1}]$.
- F is a homomorphism from the $gl(N)$ skein algebra of M to the $gl(1)$ skein algebra of $\widetilde{M}$.

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Many ingredients in the scheme have appeared before:

- If $M = C \times \mathbb{R}$, $q = 1$ (ie no spin info): abelianization/spectral networks [Gaiotto-Moore-N]
- If $M = \mathbb{R}^3$, $N = 2$: vertex model for Jones polynomial as reinterpreted using (2,0) theory [Witten, Gaiotto-Witten]

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Also the answer has appeared before in some cases, obtained by a different method:

- If $M = C \times \mathbb{R}$, $N = 2$: quantum trace [Bonahon-Wong]
- If $M = C \times \mathbb{R}$, general N, special loci in Coulomb branch: spectral networks plus R-matrices [Gabella]

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$F(L)$ is a trace over the Hilbert space of the (2,0) theory on $M \times \mathbb{R}^{2,1}$ with surface defect on $L \times \mathbb{R}^{0,1}$. To compute it, one can compactify time on S^1, including the twist by $(-q)^{2J_3} q^{2I_3}$, and compute a partition function.

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This reduces to partition function in (a twisted and Ω-deformed version of) 5-dimensional $\mathcal{N} = 2$ super Yang-Mills with $G = U(N)$, on $M \times \mathbb{R}^2$, with symmetry breaking determined by the covering $\widetilde{M} \to M$, and with a Wilson line defect in the fundamental representation inserted along L.

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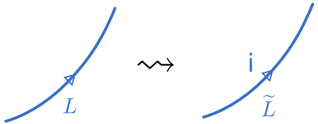

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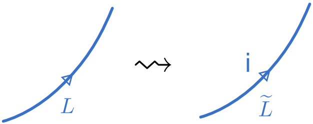

This zeroth-order guess is not right, as one quickly sees by looking at examples.

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W-bosons are in the adjoint, so labeled by two sheets i, j of $\widetilde{M} \subset T^*M \simeq TM$. Each sheet is described locally by a vector field v_i on M. To be mutually BPS with the defect, the W-bosons must travel along BPS ij-trajectories determined by $v_i - v_j$.

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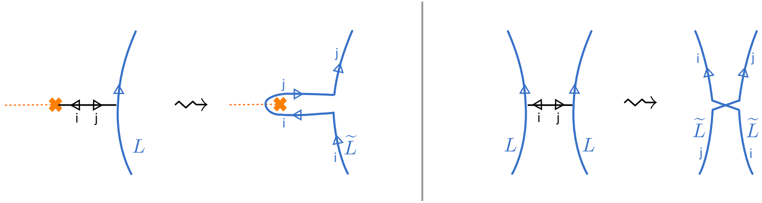
The W-bosons can transport charge between different arcs of the link L, and mediate interactions with locus of M where symmetry is partially restored (branch locus of $\widetilde{M} \to M$):



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Each $\widetilde{L}$ comes equipped with a framing (I will spare you the rules for this), and a factor $\alpha(\widetilde{L}) \in \mathbb{Z}[q, q^{-1}]$ (next slide).

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- A factor $\pm(q - q^{-1})$ for each W-boson trajectory contributing to $\widetilde{L}$, with both ends on L.

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- A factor $\pm(q - q^{-1})$ for each W-boson trajectory contributing to $\widetilde{L}$, with both ends on L.
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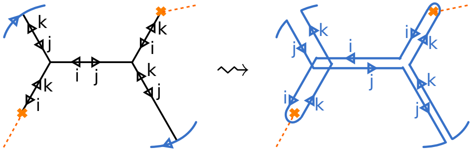
We determined these factors by bootstrap, using the constraint that UV-IR map obeys skein relations and isotopy invariance.

It would be great to derive them in a more fundamental way in the Ω-deformed 5-dimensional Yang-Mills. (The factor $q - q^{-1}$ was explained by Gaiotto-Witten when $M = \mathbb{R}^3$, in terms of 2 fermion zero modes from broken supercharges; maybe a similar explanation applies in our case.)

Weight factors

For $gl(N)$ the story is more interesting: we have to include contributions from webs of W-boson trajectories



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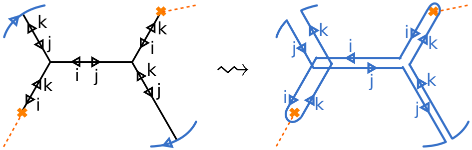

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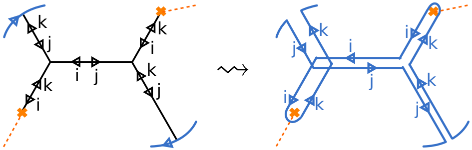

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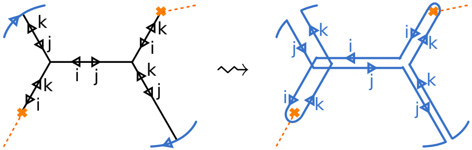

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We also have to include a factor q^w where w is sum of winding numbers in $\binom{N}{2}$ different projections, and factors related to framing as before.

The case $M = \mathbb{R}^3$

When $M = \mathbb{R}^3$ we can take the symmetry breaking vectors to be constant $v_i \in \mathbb{R}^3$. Then $\widetilde{M}$ is a trivial cover,

$$\widetilde{M} = \bigsqcup_{i=1}^{N} \mathbb{R}^3$$

The physics here is the (2,0) theory in flat $\mathbb{R}^{5,1}$, with simple symmetry breaking turned on, and a surface defect on $L \times \mathbb{R}^{0,1}$. [Gaiotto-Witten]

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The $gl(1)$ skein algebra in this case is just $\mathbb{Z}[q, q^{-1}]$.

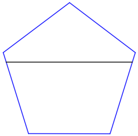
In this case $F(L)$ is a familiar link invariant,

$$F(L) = q^{Nw(L)} P_{HOMFLY}(L, a = q^N, z = q - q^{-1})$$

where $w(L)$ is the self-linking number of L. (For $N = 2$ this gives Jones polynomial.)

An example in $M = \mathbb{R}^3$

For example: take $N = 2$, with $v_1 - v_2$ pointing along x-axis, and L a specific (polygonal) unknot. Projection in xy-plane:



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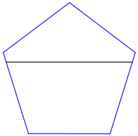
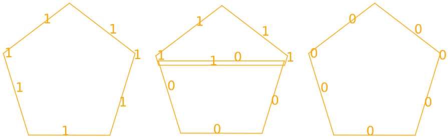


There is one possible W-boson trajectory, traveling in the x-direction, connecting points of L with the same y and z coordinates. 3 lifts contribute to $F(L)$:



Summing their contributions:

$$F(L) = (q^{-1}) + (q - q^{-1}) + (q^{-1}) \;=\; q + q^{-1}$$

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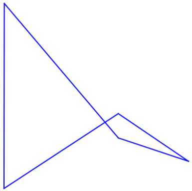
Plotting the same example in the yz-plane:



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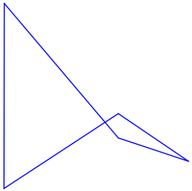

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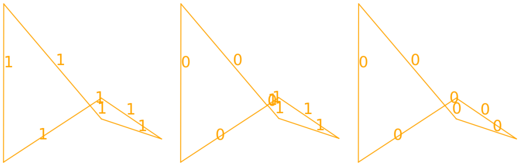

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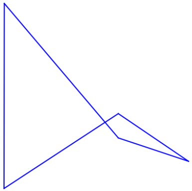

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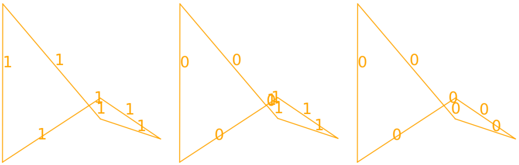

Using this projection, our description of F reduces to the "vertex model" for Jones polynomial. [Kauffman, ..., Gaiotto-Witten]

More examples in $M = \mathbb{R}^3$

For $N > 2$, if we take $v_1, \ldots, v_N \in \mathbb{R}^3$ nearly collinear, we get an analog of the vertex model, familiar from theory of quantum groups.

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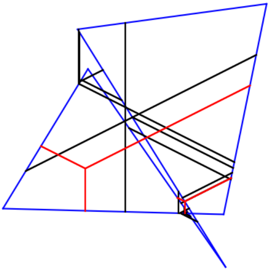

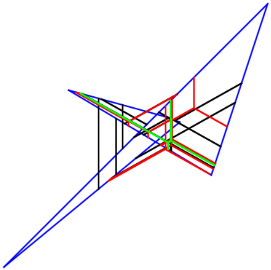

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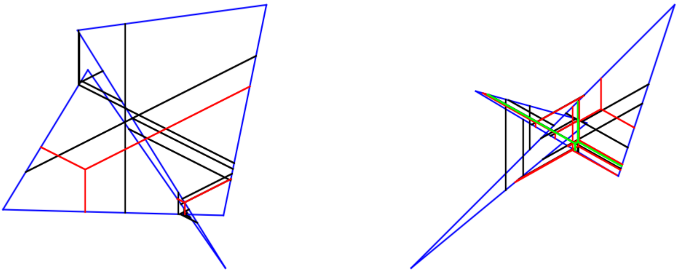

(left: an unknot, $N = 3$, sum 30 lifts to get $q^2 + 1 + q^{-2}$; right: a trefoil, $N = 3$, sum 47 lifts to get $q^7 + q^5 + 2q^3 + q - q^{-3} - q^{-5}$)

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Take C to be the 4-punctured sphere and $N = 2$ (so class S theory is $SU(2)$ gauge theory with 4 fundamental hypers.)

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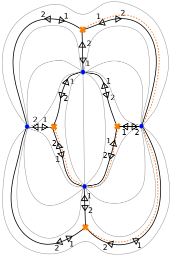
Turn on hypermultiplet masses, go to a point of Coulomb branch where Seiberg-Witten curve is

$$\widetilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4 - 1)^2}\, dz^2.$$



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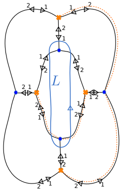

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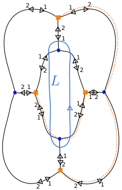

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Applying our scheme one gets

$$F(L) = X_{-\gamma_2-\mu_2+\mu_3} + X_{-\gamma_2-\mu_1-\mu_4} + X_{\gamma_1+\mu_1-\mu_4} + X_{-\gamma_1-\mu_1+\mu_4} + \\ + X_{\gamma_1-\gamma_2+\mu_1-\mu_4} + X_{\gamma_1-\gamma_2-\mu_2+\mu_3-2\mu_4} + X_{\gamma_1-2\gamma_2-\mu_2+\mu_3-2\mu_4}$$

i.e. 7 BPS states of various charges, spin zero. [Gaiotto-Moore-N]

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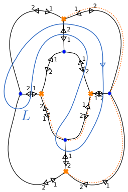

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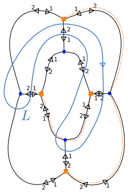

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i.e. a bunch of BPS states with various charges and spins.

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This transformation law says the various F for different points of Coulomb branch assemble into a single map from the skein algebra to a quantized cluster algebra.
[Fock-Goncharov, Muller, Goncharov-Shen, Schrader-Shapiro, ...]

Summary so far

I described joint work with Fei Yan — partly in progress — where we compute:

- Protected spin character counting BPS ground states with spin for line defects in $\mathcal{N} = 2$ theories of class S,
- Link "invariants" for links in $M = (surface) \times \mathbb{R}$ (with wall-crossing behavior),
- Well-known link invariants for links in $M = \mathbb{R}^3$ in a new way,
- A (fairly explicit) embedding of a skein algebra into a quantized cluster algebra.

Summary so far

In the last few moments I'll try to sketch some potential extensions and applications.

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UV-IR map here should go from $gl(N)$ skein module of M to twisted $gl(1)$ skein module of $\widetilde{M}$, describe how line defects behave under this flow.

The 3-dimensional case

Our description of the UV-IR map is 3-dimensionally covariant. It seems to apply for more general M — but with some new phenomena. [Freed-N, N-Yan in progress]

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Conjecture: these representations are spaces of ground states of the class S theory, formulated on the "Melvin" space $\mathbb{R}^2 \times_q S^1$; they form a hyperholomorphic (BBB) brane over moduli of flat connections, mirror to canonical coisotropic BAA brane with ℓ units of curvature; constructible by Riemann-Hilbert methods. [Gukov; Moore-N-Yan in progress]

Thanks

Thanks and my very best wishes to everyone in the String-Math community. I hope we can meet again in person before too long.