Spectral Networks

Joint work w/ Gaiotto, Moore. In progress.

I'll introduce a new structure:

- $\mathfrak{g} = A_{K-1}$
- curve C (smooth compact)
- tuple $(\varphi_2, \dots, \varphi_K)$
  φ_i mero. section of $K_C^{\otimes i}$
- $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$

$\longrightarrow$ Spectral network

Spectral network $\longrightarrow$ integers $\Omega(\gamma)$
= 4d BPS degeneracies
= DT invts for local CY

integers $\Omega(\gamma)$ $\longrightarrow$ HK metric on Hitchin system $\mathcal{M}_H(G, C)$
= IR Lagrangian on $\mathbb{R}^3 \times S^1$

Spectral network $\longrightarrow$ Framed 2d-4d wall-crossing

Spectral network $\longrightarrow$ Cluster coordinate system on space of flat G-connections on $C \setminus \{s_a\}$ = vevs of line operators on $\mathbb{R}^3 \times S^1$

(We previously did this for $\mathfrak{g} = A_1$.)

Spectral trajectories

Consider spectral cover

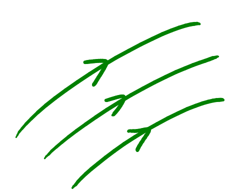
$$\Sigma = \left\{ \lambda^K - \sum_{i=2}^{K} \lambda^{K-i} \varphi_i = 0 \right\} \subset T^*C$$

Branched, K-sheeted covering. Assume (φ_i) generic $\Rightarrow$ only simple branch pts.

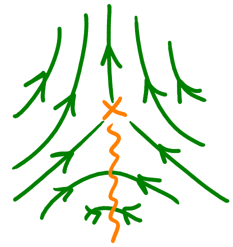
Call the roots (locally) $\lambda_1, \dots, \lambda_K$. Complex 1-forms.

Then define an ij-trajectory to be an oriented path in C along which $e^{i\vartheta}(\lambda_i - \lambda_j)$ is real and positive. $\left(\xrightarrow[ij]{} \; = \; \xleftarrow[ji]{} \right)$

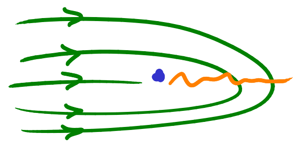
For fixed i, j these give a (local) foliation of C.

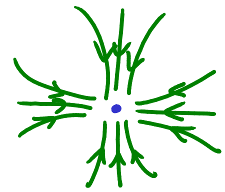


It is singular at pts. where $\lambda_i = \lambda_j$
(branch points)



(and at the poles of $\lambda_i - \lambda_j$
— less impt for what follows)

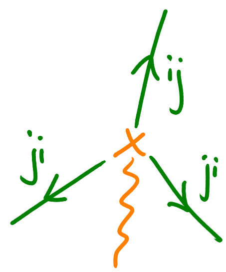




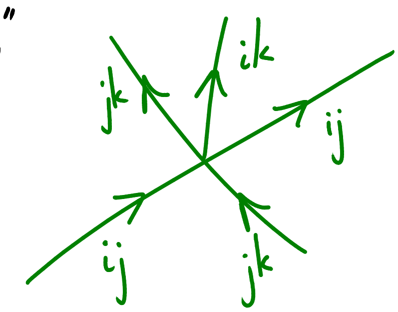
Spectral network

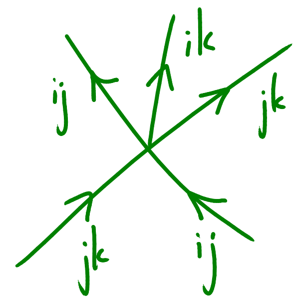
Network of spectral trajectories on C, built up as follows:

Begin at each branch pt. with



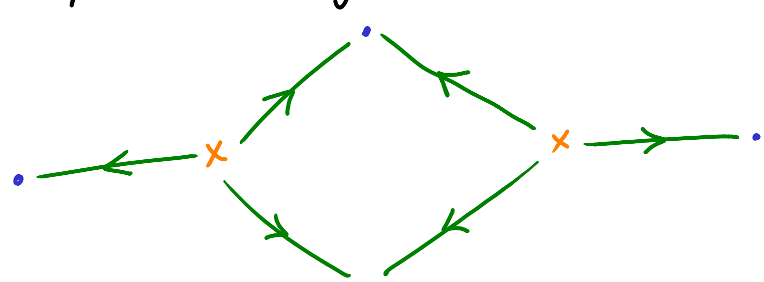
Evolve the trajectories for infinite time.
If two cross, they can "give birth" to a new one, which then also evolves for infinite time.





If we have "enough punctures" (e.g. one where φ_i has pole of order i) then all trajectories eventually asymptote to punctures. (So, the network is not *dense* on C.)

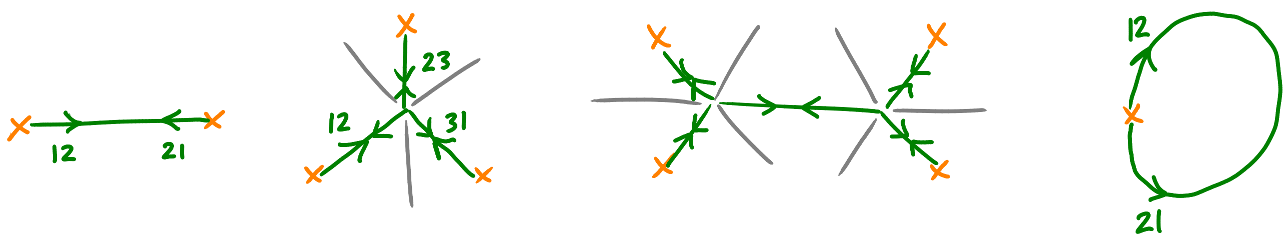
In the special case $\mathfrak{g} = A_1$ this leads to a simple structure: trajectories cannot *cross* so get a *cell decomposition* of C.
($\rightsquigarrow$ an *ideal triangulation*.)



But more generally these networks may be rather intricate.
(Show some examples, varying with ϑ.)

BPS degeneracies

At some special values of ϑ, spectral network jumps discontinuously.
At these ϑ, there exist collisions – sub-networks where traj. are meeting head-on:



Whenever we have such a sub-network, it can be canonically lifted to a class $\gamma \in H_1(\Sigma, \mathbb{Z})$.
For any $\gamma \in H_1(\Sigma, \mathbb{Z})$, define $\Omega(\gamma) \in \mathbb{Z}$ to be the # of such collisions w/ lift γ.
[weighted by $(-1)^{\# \text{loops}}$]

Conj 1) $\Omega(\gamma)$ are BPS degeneracies in the 4d $\mathcal{N}=2$ theory $S[A_{K-1}, C]$ obtained by compactifying $(2,0)$ theory on C. (String networks)

2) $\Omega(\gamma)$ depend on $(\varphi_2, \dots, \varphi_K)$ in a way governed by Kontsevich–Soibelman WCF.

3) $\Omega(\gamma)$ are gen DT invariants attached to Fukaya category of a local CY:
[Diaconescu–Donagi–Pantev, Bridgeland–Smith for A_1]
A_{K-1} singularity fibered over C.

Framed wall-xing

The spectral network consists of walls of marginal stability:

In $S[A_{K-1}, C]$ we have BPS surface operator $\mathbb{S}_z$ for $z \in C$
A path from z to z' gives an interface between $\mathbb{S}_z$ and $\mathbb{S}_{z'}$.
BPS states of this interface can appear/decay when z or z' cross the spectral network.

(This is actually the main tool we use...)

Cluster coordinates (roughly)

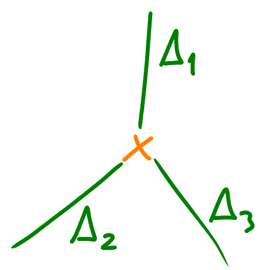
$\mathcal{M} = \{$flat G-conn. on C w/ invariant flag @ each $s_a\}$,

$\mathcal{M}_{ab} = \{\mathbb{C}^\times$-conn. on triv. l.b. $\mathcal{L} \to \Sigma\}$

We'll give a map $\mathcal{M}_{ab} \to \mathcal{M}$ which we conjecture is $\simeq$ onto open dense subset ("big cell"). ie $(\mathbb{C}^\times)^{2g} \to \mathcal{M}$. [For A_1, shear coords.]

Idea: $\pi_*(\mathcal{L})$ has induced diagonal connection away from branch pts. But it has monodromy around branch pts.

Modify it by cutting and gluing along the edges of the spectral network: along each ij-edge we have a canonical $\Delta: \mathcal{L}_j \to \mathcal{L}_i$ and we glue by $\mathbb{1} + \Delta$



Resulting conn. is no longer diagonal, extends over branch pts.

$$\left[\text{essentially b/c} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$$

so gives a point of $\mathcal{M}$.

- The coordinate functions on $\mathcal{M}$ so obtained are vevs of "IR line operators" in $S[C, A_{K-1}]$ on $\mathbb{R}^3 \times S^1$.

- When the spectral network jumps, the corresp. coordinate xform is expected to be a simple "cluster-like" map: of the form

$$X_\gamma \longrightarrow X_\gamma (1 - X_{\gamma'})^{\langle \gamma, \gamma' \rangle}$$