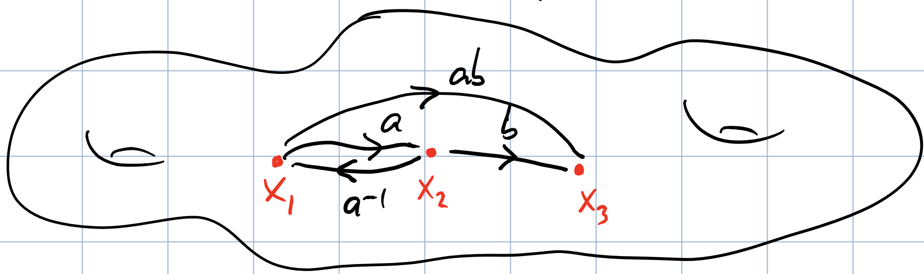
Def For S top. space, $D \subset S$,

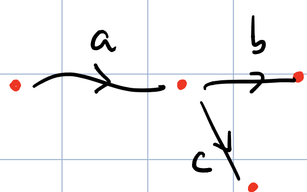
$\pi_{\leq 1}(S,D)$ = fundamental groupoid of (S,D)

this has: objects = D

morphisms(x_1,x_2) = $\{$paths from x_1 to $x_2\}$/homotopy



Def $\mathbb{Z}[\pi_{\leq 1}(S,D)]$ = <u>groupoid ring</u> of $\pi_{\leq 1}(S,D)$

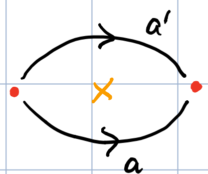


$$(2a+b)(a-c) = -2ac$$

Now take $\begin{array}{c}\Sigma \\ \downarrow \pi \\ C\end{array}$ branched over Δ, as before, and $z \in C$.

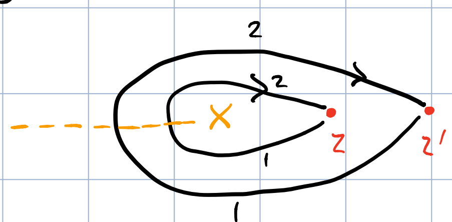
Def $A(\Sigma, z) = \mathbb{Z}[\pi_{\leq 1}(\Sigma \setminus \pi^{-1}(\Delta), \pi^{-1}(z))]/\sim$

where $a_1 \sim -a_2$ if there's a homotopy in Σ from a_1 to a_2 crossing Δ once.



$$a' = -a$$

<u>Remark</u> • The rings $A(\Sigma, z)$ form a <u>local system</u> of rings over $C \setminus \Delta$.



• $A(\Sigma, z) = \bigoplus_{i,j} \underbrace{A_{ij}(\Sigma, z)}_{\text{paths from } z^{(i)} \text{ to } z^{(j)}}$

• Similarly define rings $A(\Sigma, \{z_1, \dots, z_n\})$

Path lifting

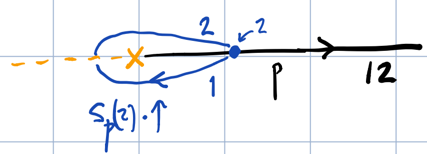
① Solitons

Each ij-trajectory p in $W = W(\vartheta, u)$ carries extra datum

$S_p(z) \in \Lambda_{ij}(\Sigma, z) \otimes_{\mathbb{Z}_2}$ co-orientations(p) (section of local system over p)

determined inductively:

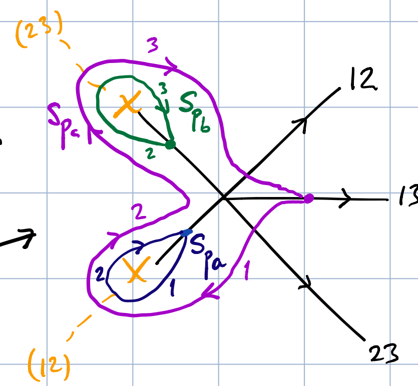
a) for p born from branch point, S_p given by



b) for p_c born from intersection of p_a and p_b,

$S_{p_c} = S_{p_a} S_{p_b}$

(all with co-or ↑)



Fact $\arg\left(-\int_{S_p}\lambda\right) = \vartheta$. (so can measure "length" of $s_p(z)$ as $\left|\int_{S_p}\lambda\right|$, monotonically increasing)

② Lifts

From now on assume W is finite.

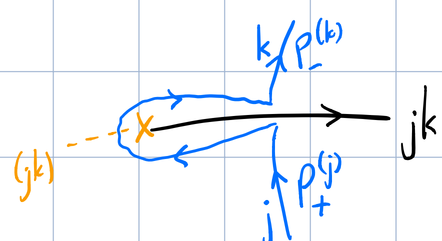
For $z_1, z_2 \notin W$ define $F = F_W : \pi_{\le 1}(C, \{z_1, z_2\}) \longrightarrow \mathcal{A}(\Sigma, \{z_1, z_2\})$

- For a path P not meeting W

$$F(P) = \sum_{i=1}^{N} P^{(i)}$$

- For a path P meeting W once in interior,

$$F(P) = \sum_{i=1}^{N} P^{(i)} + P_+^{(j)} S_p(z) P_-^{(k)}$$



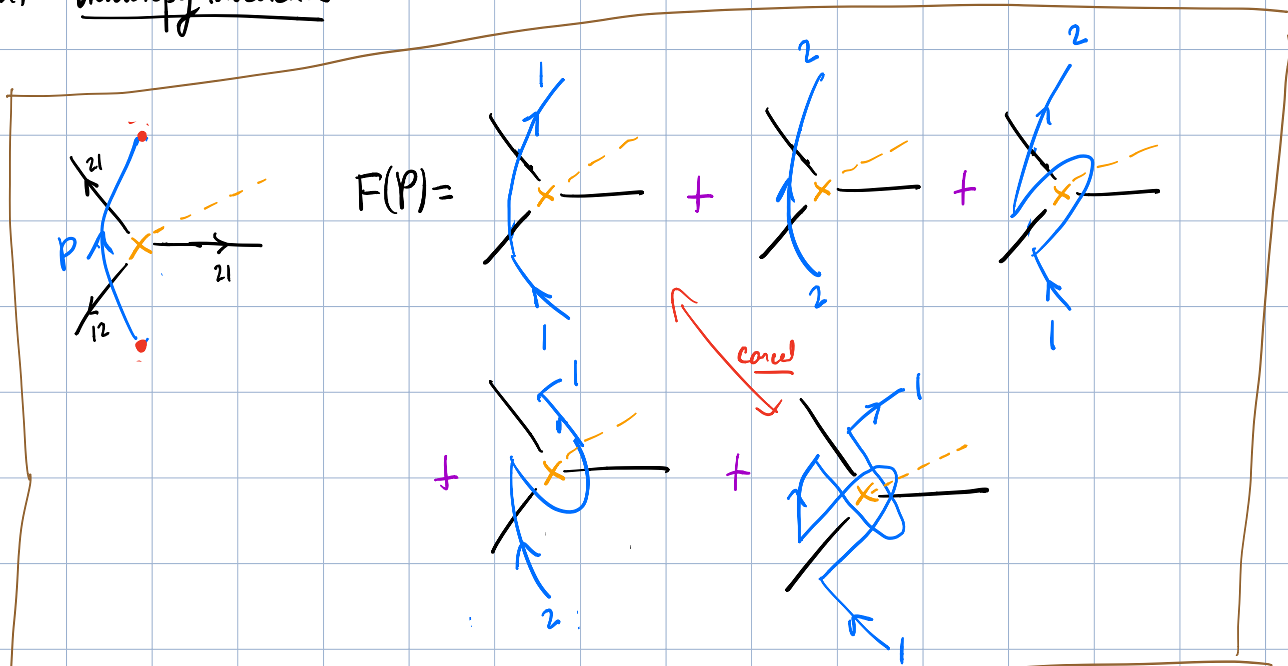
- For general P on $C \setminus \Delta$, ends not on W, break into pieces and require

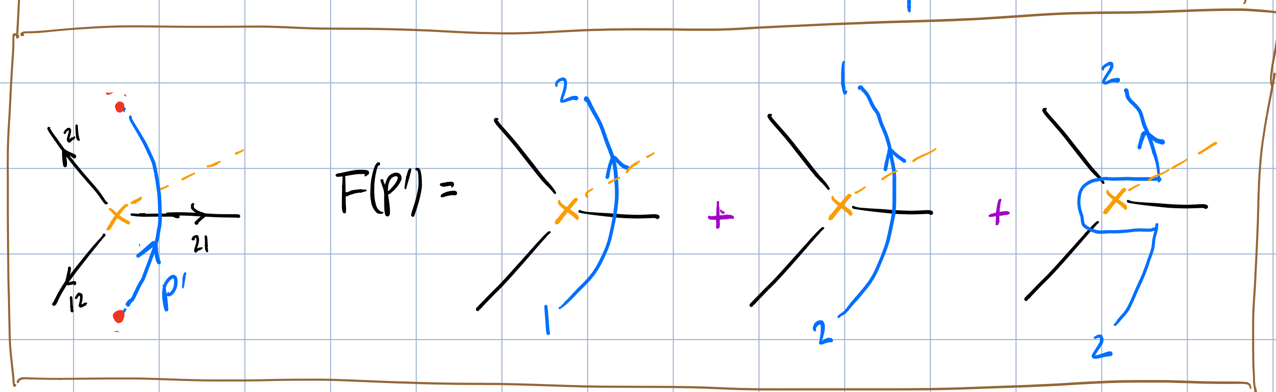
$$F(P_1 P_2) = F(P_1) F(P_2)$$

Prop F is well defined.

Key point: homotopy invariance

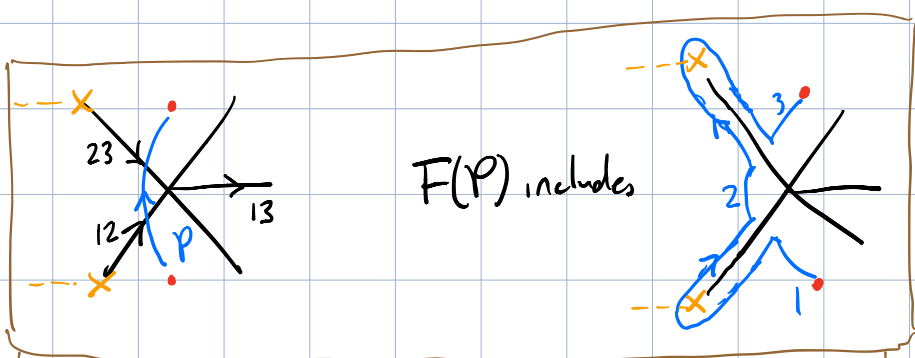
eg

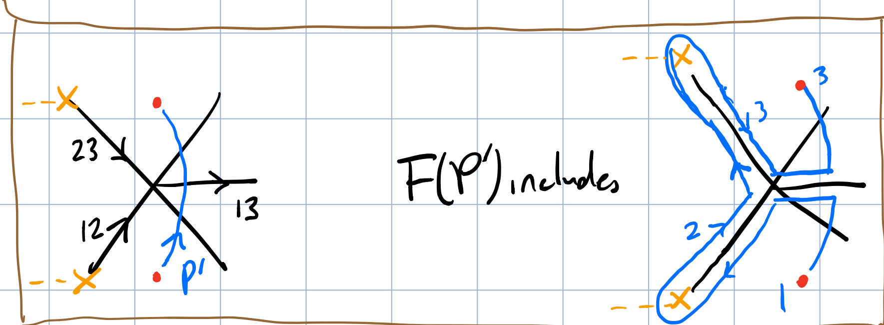




so $F(P) = F(P')$

Similarly





again, $F(P) = F(P')$

③ Nonabelianization

$$(\mathbb{C}^\times)^k$$
$$\cong$$

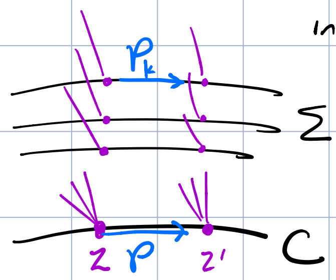
[holonomy -1 around $\pi^{-1}(\Delta)$]

"Dual" to F is a map (functor) $\hat{F} : \widetilde{\mathcal{M}}(\Sigma, GL(1)) =$ [almost flat $GL(1)$-conn. over Σ]

$$\downarrow$$

$\mathcal{M}(C, GL(N)) =$ [flat $GL(N)$-conn over $C \setminus$ punctures]

Idea of $\hat{F}$: say ∇ flat $GL(1)$-conn. in $\mathcal{L} \to \Sigma$



$$E_z = \bigoplus_{i=1}^{N} \mathcal{L}_{z^{(i)}},$$

if $F(P) = \sum_k{}' n_k P_k$, $\quad \mathrm{Hol}_{\hat{F}(\nabla)}(P) = \sum_k{}' n_k \mathrm{Hol}_\nabla(P_k)$