References

Gaiotto-Moore-N "Four-dimensional wall crossing via three-dimensional field theory"

"Wall-crossing, Hitchin systems, and the WKB approximation"

"Spectral networks"

Gaiotto "Opers and TBA"

Iwaki-Nakanishi "Exact WKB analysis and cluster algebras"

Allegretti-Bridgeland "The monodromy of meromorphic projective structures"

Hollands-N "Exact WKB and abelianization for the T_3 equation"

N "Integral iterations for harmonic maps"

Bridgeland-Smith "Quadratic differentials as stability conditions"

Fock-Goncharov "Moduli spaces of local systems and higher Teichmülle theory"

Kontsevich-Soibelman "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations"

Last time:

$$\begin{bmatrix} C \text{ compact R.s.} \\ u=(\phi_1,\dots,\phi_N) \\ \vartheta\in\mathbb{R}/2\pi\mathbb{Z}\end{bmatrix} \rightsquigarrow \begin{bmatrix} \text{spectral cover } \Sigma_u \xrightarrow{\pi} C \\ \text{spectral network } \mathcal{W}=\mathcal{W}(u,\vartheta)\end{bmatrix} \rightsquigarrow \begin{bmatrix} \text{path lifting / "nonabelianization"} \\ \hat{F}_{\mathcal{W}}: \widetilde{\mathcal{C}}_{\text{conn}}(\Sigma_u, GL(1)) \to \mathcal{C}_{\text{conn}}(C, GL(N)) \\ \nabla^{ab} \mapsto \nabla \end{bmatrix}$$

① Spectral coordinates and DT invariants

Say we fix (generic) conjugacy classes of monodromy around poles of u and suppose poles of ϕ_i have order i, generic residues.

1) Then $\hat{F}_{\mathcal{W}}$ gives map of moduli spaces

$$\hat{F}_{\mathcal{W}}: \widetilde{\mathcal{M}}(\Sigma_u, GL(1)) \to \mathcal{M}(C, GL(N))$$

2) $\hat{F}_{\mathcal{W}}$ is local symplectomorphism
$\Rightarrow$ locally invertible:

$$\begin{array}{ccccc} U_{\mathcal{W}} & \xrightarrow{\hat{F}_{\mathcal{W}}^{-1}} & \widetilde{\mathcal{M}}(\Sigma, GL(1)) & \xrightarrow{\mathcal{X}_\gamma} & \mathbb{C}^\times \\ \cap & & & & \\ \mathcal{M}(C, GL(N)) & & & & \end{array} \qquad \mathcal{X}^{\mathcal{W}}_\gamma : U_{\mathcal{W}} \to \mathbb{C}^\times$$

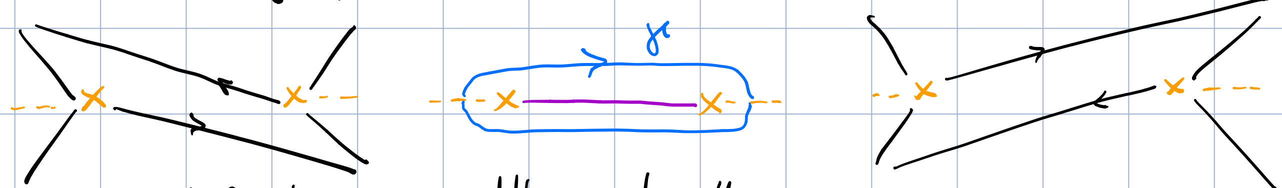
$[\gamma \in H_1(\Sigma, \mathbb{Z})]$ $\quad \mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \pm \mathcal{X}_{\gamma+\gamma'}$

$\mathcal{X}_\gamma(\nabla^{ab}) = \mathrm{Hol}_\gamma \nabla^{ab} \in \mathbb{C}^\times$

The $\mathcal{X}^{\mathcal{W}}_\gamma$ give local coordinates on $\mathcal{M}(C, GL(N))$

- For $N=2$, ϕ_2 generic, $\mathcal{X}^{\mathcal{W}}_\gamma$ are Fock-Goncharov coords.
- For $N=3$, $\phi_3 \ll \phi_2$, ϕ_2 generic, $\mathcal{X}^{\mathcal{W}}_\gamma$ are higher-rank Fock-Goncharov coords.

3) $\hat{F}_{\mathcal{W}}$ jumps when $\mathcal{W}$ jumps. Simplest example: $(N=2)$



$\mathcal{W}=\mathcal{W}(u,\vartheta_c-\varepsilon)$ $\qquad$ saddle connection with charge γ, $\arg Z_\gamma = \vartheta_c$ $\qquad$ $\mathcal{W}'=\mathcal{W}(u,\vartheta_c+\varepsilon)$

In this example one computes $\hat{F}_{\mathcal{W}} = \hat{F}_{\mathcal{W}'} \circ \mathcal{K}_\gamma$ where $\mathcal{K}_\gamma: \widetilde{\mathcal{M}}(\Sigma, GL(1)) \circlearrowleft$

$$\mathcal{K}_\gamma^* \mathcal{X}_\mu = \mathcal{X}_\mu (1 \pm \mathcal{X}_\gamma)^{\langle \mu, \gamma\rangle}$$

sorry

Slogan: a saddle connection of charge γ induces the transformation $\mathcal{K}_\gamma$

Similarly a 3-string web of charge γ ———————— $\mathcal{K}_\gamma$

ring domain ———————— $\mathcal{K}_\gamma^{-2}$

More generally, at any ϑ_c where $\mathcal{W}(u,\vartheta)$ jumps, $\hat{F}_{\mathcal{W}}$ changes by a xform $T_{\vartheta_c,u} : \widetilde{\mathcal{M}}(\Sigma, GL(1)) \circlearrowleft$

Def u is not on a wall of marginal stability if $\arg Z_\gamma(u) = \arg Z_{\gamma'}(u) \iff \mathbb{Q}\gamma = \mathbb{Q}\gamma'$.

Fact If u is not on a wall of marginal stability, then

$\exists\, \Omega(\gamma,u) \in \mathbb{Z}$ such that $T_{\vartheta_c,u} = \prod_{\gamma :\, \arg Z_\gamma = \vartheta_c} K_\gamma^{\Omega(\gamma,u)}$ with K_γ defined as above.

There's an algorithm computing the $\Omega(\gamma,u)$ from "degenerate spectral network" $\mathcal{W}(u,\vartheta_c)$.

Call the $\Omega(\gamma,u)$ "DT-like invariants." (For $N=2$ and ϕ_2 meromorphic w/ poles order ≥ 2 they are DT invariants)

Remark

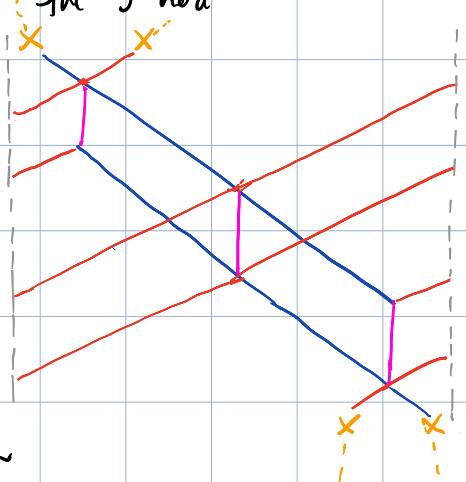
In simple cases $\Omega(\gamma,u)$ "count" finite webs with charge γ.

eg for $N=2$, $\Omega(\gamma,u) = $ (# saddle connections of ϕ_2 with charge γ)
-2 (# ring domains of ϕ_2 with charge γ)

But in general there can be lots of overlapping finite webs, and then it can be hard to disentangle their individual contributions to $\Omega(\gamma,u)$.

A "bad" example: the "3-herd"

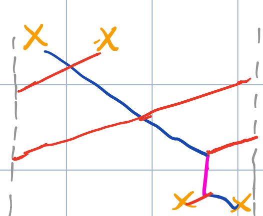
(sheet labels: 12 13 23)



If 3-herd appears at some ϑ_c the corresponding DT invariants are:

$$\left(\Omega(n\gamma)\right)_{n=1}^{\infty} = 3, -6, 18, -84, 465, -2808, \ldots$$

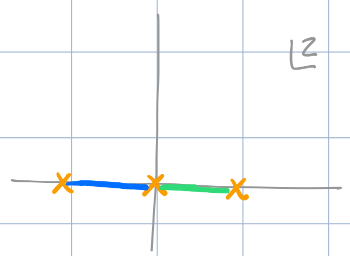
where γ = lift of the finite web



② Chambers and wall crossing

Let's take $N=2$, $C=\mathbb{CP}^1$, $\phi_2=(z^3-z+u)\,dz^2$. Finitely many saddle connections for any u.

at $u=u_1\approx 0$, $\phi_2=(z^3-z)\,dz^2$

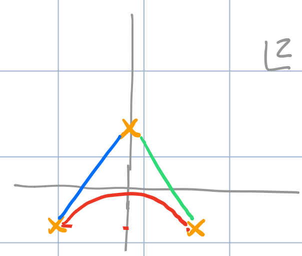


$\Omega(\pm\gamma_1,u_1)=1$

$\Omega(\pm\gamma_2,u_1)=1$

all other $\Omega(\gamma,u_1)=0$

at $u=u_2\approx\frac{i}{2}$, $\phi_2=(z^3-z+\frac{i}{2})\,dz^2$



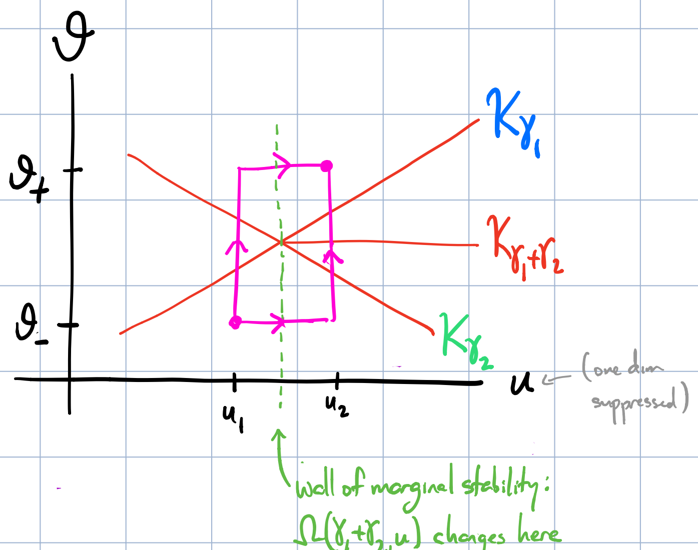
$\Omega(\pm\gamma_1,u_2)=1$

$\Omega(\pm\gamma_2,u_2)=1$

$\Omega(\pm(\gamma_1+\gamma_2),u_2)=1$

all other $\Omega(\gamma,u_2)=0$

Now plot (ϑ,u) space with a wall at every place that $W(\vartheta,u)$ jumps.



Each wall is carrying an automorphism of $\widetilde{\mathcal{M}}(\Sigma, GL(1))$. But $\hat{F}_{W(\vartheta,u)}$ depends only on (ϑ,u) so considering the two paths shown gives the identity:

$$K_{\gamma_2}\circ K_{\gamma_1}=K_{\gamma_1}\circ K_{\gamma_1+\gamma_2}\circ K_{\gamma_2}$$

The same property should hold much more generally for our $\Omega(\gamma,u)$, [even in $N>2$ cases]:

$$\prod^{\rightarrow}_{\substack{\gamma:\ \arg Z_\gamma\\ \in\\ (\vartheta_-,\vartheta_+)}} K_\gamma^{\Omega(\gamma,u_1)}=\prod^{\rightarrow}_{\substack{\gamma:\ \arg Z_\gamma\\ \in\\ (\vartheta_-,\vartheta_+)}} K_\gamma^{\Omega(\gamma,u_2)}$$

This is enough to determine the $\Omega(\gamma,u_2)$ from the $\Omega(\gamma,u_1)$ — this is Kontsevich-Soibelman wall-crossing formula.