1 Preface

Most of what I say in these talks concerns some joint work with Davide Gaiotto and Greg Moore. Original motivation from supersymmetric quantum field theory, but I won't have much to say about that. Turned out to be connected to work of various mathematicians, especially Hitchin, Fock-Goncharov, Kontsevich-Soibelman, Joyce-Song.

Main aim: to explain some new structure on moduli spaces $\mathcal{M}$ of solutions of Hitchin equations (character varieties). Roughly:

- A $\mathbb{C}^\times$-family of canonical local coordinate systems around every point.
- A collection of integer "invariants" counting certain networks of trajectories on C.

This new structure appears naturally in QFT, should have lots of uses. Concretely, one use is to get more explicit information about the hyperkähler metric on $\mathcal{M}$.

2 Hitchin's equations

Fix a compact complex curve C, group $G = U(K)$, and $R \in \mathbb{R}_+$. The *Hitchin equations* for a tuple (E, h, φ, D) — E a complex rank K vector bundle, h a Hermitian metric in E, φ a $(1,0)$-form with values in $End(E)$, D a h-unitary connection in E — are

$$F_D + R^2[\varphi, \varphi^{\dagger_h}] = 0, \qquad \bar\partial_D \varphi = 0.$$

Not obvious there are any solutions at all, but there are [Hitchin, Simpson]. Indeed there is a generically smooth moduli space

$$\mathcal{M} = \mathcal{M}(C, G, R)$$

parameterizing solutions modulo equivalence.

A small extension: we will sometimes fix some marked points on C and consider a moduli space parameterizing (φ, D) which have poles at these points, with some fixed singularity type. The most tractable examples are of this sort.

In any case, $\mathcal{M}$ carries a natural Riemannian metric: holding (E, h) fixed and varying (φ, D),

$$\|(\delta\varphi, \delta D)\|^2 = 2\mathrm{i}\int_C \mathrm{Tr}(\delta\varphi\delta\varphi^{\dagger_h} + R^{-2}\delta D^{\dagger_h}\delta D)$$

This turns out to be a hyperkähler metric. (Morally, by hyperkähler quotient.)

(Recall what this means: g is Kähler wrt 3 different complex structures J_1, J_2, J_3 obeying quaternion relations. Thus get 3 Kahler forms $\omega_1, \omega_2, \omega_3$. Also 3 hol symplectic forms $\varpi_3 = \omega_1 + \mathrm{i}\omega_2$ etc.)

In particular, $\nabla\varpi_i = 0$, so if we focus on J_i and hol. volume form ϖ_i^r ($\dim_{\mathbb{R}} \mathcal{M} = 4r$), we get a Calabi-Yau manifold. But, in a very implicit way!

We want to understand it more concretely.

3 Higgs bundles

Let's first focus on J_3.

A *Higgs bundle* (of rank K) over C is a tuple $(E, \bar\partial, \varphi)$ where

- $(E, \bar\partial)$ is a holomorphic vector bundle of rank K over C,

- φ is a $(1,0)$-form with values in $\mathrm{End}(E)$, with $\bar{\partial}\varphi = 0$.

Let $\mathcal{M}^{Higgs}(C,G)$ denote the moduli space of (semistable = polystable) such Higgs bundles over C, of degree zero, up to equivalence. It's naturally a complex symplectic space. (Some singularities (orbifold?) but think of it as a manifold.)

The "forgetful" map

$$(E,h,\varphi,D) \mapsto (E,\bar{\partial}_D,\varphi)$$

gives an isomorphism [Hitchin-Simpson]

$$\mathcal{M}(C,G,R) \simeq \mathcal{M}^{Higgs}(C,G),$$

In particular this induces a complex symplectic structure on $\mathcal{M}(C,G,R)$.

This gives a concrete picture of $\mathcal{M}$: given a Higgs bundle (E,φ) we can define

- *spectral curve*

  $$\Sigma = \{(z,\lambda) : z \in C, \lambda \in T^*_z C, \det(\varphi - \lambda \mathbf{1}) = 0\}.$$

  $\pi : \Sigma \to C$ (generically) smooth branched K-fold cover of C.

- *spectral sheaf* over T^*C,

  $$\mathcal{L} = \ker(\varphi - \lambda \mathbf{1}) \subset \pi^* E,$$

  supported on Σ, generically a line bundle.

This gives *Hitchin fibration*

$$\rho : \mathcal{M}^{Higgs} \to \mathcal{B}$$

where

$$\begin{aligned}\mathcal{B} &= \{(\varphi_1, \ldots, \varphi_K) \in \bigoplus_{i=1}^{K} H^0(C, K_C^{\otimes i})\} \\ &= \{\text{branched } K - \text{fold covers } \Sigma \to C \text{ in } T^*C\}\end{aligned}$$

Use Σ_u for the cover corresponding to $u \in \mathcal{B}$. Let

$$\mathcal{B}' = \{u \in \mathcal{B} : \Sigma_u \text{ is smooth}\}$$

The fiber over $u \in \mathcal{B}'$ is a compact complex torus,

$$\rho^{-1}(u) = Jac(\Sigma_u)$$

(degree $K(1 - g_C) - (1 - g_\Sigma)$). (Can recover (E, φ) from $(\mathcal{L}, \lambda)$ by pushforward.)

Singular fibers over discriminant locus.

4 Semiflat metric

Let's describe an approximation to the hyperkähler metric on $\mathcal{M}$.

Have

$$\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$$

fitting into a local system Γ over $\mathcal{B}'$, and function $Z : \Gamma \to \mathbb{C}$ (local functions Z_γ)

$$Z_\gamma = \oint_\gamma \lambda$$

where λ is the tautological 1-form on $\Sigma \subset T^*C$. This gives a metric on $\mathcal{B}'$,

$$g_{\mathcal{B}'} = \frac{\mathrm{i}}{4\pi^2} \langle \mathrm{d}Z \otimes \mathrm{d}\bar{Z} \rangle$$

e.g. in 1-dim case set $a = Z_{\gamma_1}$, $\tau = \mathrm{d}Z_{\gamma_2}/\mathrm{d}Z_{\gamma_1}$, then

$$g_{\mathcal{B}'} = \frac{1}{4\pi^2}(\operatorname{Im}\tau)\,\mathrm{d}a \otimes \mathrm{d}\bar{a}.$$

The torus fibers can be identified by "Gauss-Manin" connection (using Hodge theory they are all $\mathrm{Hom}(\Gamma_u, U(1))$), so tangent space splits; and each fiber carries a (flat) Kähler metric, g_{fiber}. Then our approximation is

$$g^{\mathrm{sf}} = g_{\mathcal{B}'} + \frac{1}{R^2} g_{fiber}.$$

This is already a nice hyperkähler metric.

NB, g^{sf} is the *exact* metric if $K = 1$. In that case $\Sigma = C$ and $\mathcal{M}$ is simply $T^*Jac(C)$.

However for larger K it can't be the right one, e.g. because it doesn't extend over the singular fibers. The *actual* metric can be thought of as g^{sf} plus "quantum corrections." One crude formula:

$$g = g^{\mathrm{sf}} + O\left(\sum_{\gamma\in\Gamma_u} \Omega(\gamma; u) e^{-R|Z_\gamma|}\right)$$

where $\Omega(\gamma; u) \in \mathbb{Z}$.

Remarks:

- For example, if $K = 2$, $u \in \mathcal{B}$ is just a pair

  $$u = (\varphi_1, \varphi_2),$$

  define *trajectories* of φ_2 to be straight lines in the coordinate

  $$w = \int \sqrt{\varphi_2}$$

  then $\Omega(\gamma; u)$ is counting saddle connections (with weight 1) and closed trajectories (with weight -2). [Klemm-Lerche-Mayr-Vafa-Warner

- $\Omega(\gamma; u)$ actually *jump* as we move around on $\mathcal{B}$. When $K = 2$, they are DT invariants for some category [Bridgeland-Smith, Smith] — Fukaya category of a certain CY 3-fold [Diaconescu, Donagi, Pantev]. Jumping governed by wall-crossing formula [Kontsevich-Soibelman]. Nevertheless g is smooth.

- The corrections are exponentially suppressed as $R \to \infty$, away from the places where some $Z_\gamma \to 0$, i.e. where some 1-cycle on Σ_u collapses; these are the singular fibers. Thus up to exponentially small corrections the theory "abelianizes" away from the singular fibers.

- One expects a version of this picture for any CY manifold: fibration by special Lagrangian tori, and in some "large complex structure" limit, the fibers collapse onto the base. [Strominger-Yau-Zasl Kontsevich-Soibelman, Todorov] In our case fibers have $\omega_2 = \omega_3 = 0$, so are sLag if we consider complex structure J_2 or J_3. Our case is a particularly computable example of this story.

5 Flat connections

How to construct the actual metric on $\mathcal{M}$? Use "twistor" perspective.

Idea: being hyperkähler, $\mathcal{M}$ carries not just 3 complex structures but a whole *family* of complex structures J_ζ, and holomorphic symplectic forms ϖ_ζ, for $\zeta \in \mathbb{CP}^1$. Given in terms of the three symplectic forms ω_i, by

$$\varpi_\zeta = \omega_+/\zeta + \omega_3 + \omega_- \zeta$$

with

$$\omega_\pm = \omega_1 \pm \mathrm{i}\omega_2$$

Knowing the ω_i would be enough to recover the hyperkähler metric. So, knowing ϖ_ζ is certainly enough.

Fix $\zeta \in \mathbb{C}^\times$. How to think about (J_ζ, ϖ_ζ)? Given (E, h, φ, D) obeying Hitchin's equations,

$$\nabla(\zeta) = R\zeta^{-1}\varphi + D + R\zeta\varphi^{\dagger_h}$$

is a *flat* connection in E. Let $\mathcal{M}^{flat}(G_\mathbb{C}, C)$ be the space of (reductive, i.e. completely reducible) flat connections in rank K complex bundles, up to equivalence. The "forgetful" map

$$(E, h, \varphi, D) \mapsto (E, \nabla(\zeta))$$

identifies [Donaldson-Corlette]

$$\mathcal{M} \simeq \mathcal{M}^{flat}(G_\mathbb{C}, C)$$

We get many different identifications, one for each $\zeta \in \mathbb{C}^\times$. $\mathcal{M}^{flat}$ is naturally complex symplectic manifold [Goldman, Atiyah-Bott] so pullback gives (J_ζ, ϖ_ζ) on $\mathcal{M}$.

6 WKB

We have two problems. One is to understand complex symplectic structure on $\mathcal{M}^{flat}$, the other is to understand the map which is pulling that back to $\mathcal{M}$.

Both problems get solved at once, using (a version of) "exact WKB method." [Ecalle, Voros, ..., Iwaki-Nakanishi]

Simplified example: Airy equation

$$f''(z) - \frac{z}{\zeta^2} f(z) = 0$$

Rewrite it as 1st-order system

$$\left[\partial + \frac{1}{\zeta}\varphi\right]\psi = 0, \qquad \psi = \begin{pmatrix} f \\ \zeta f' \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \mathrm{d}z$$

For any fixed ζ, this is a flat connection,

$$\nabla(\zeta) = \mathrm{d} + \frac{1}{\zeta}\varphi$$

What are the solutions like? Locally, try *diagonalizing* φ, i.e. writing

$$\varphi = g(z) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} g(z)^{-1}$$

where each

$$\lambda_i = \pm\sqrt{z}\,\mathrm{d}z$$

In terms of

$$\widetilde{\psi}(z) = g(z)^{-1}\psi(z)$$

the equation becomes

$$\left[\mathrm{d} + \frac{1}{\zeta}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - g^{-1}\mathrm{d}g\right]\widetilde{\psi}(z) = 0$$

Then a first approximation to a solution, in the $\zeta \to 0$ limit, would be

$$\psi_i(z,\zeta) = \exp\left[-\frac{1}{\zeta}\int_0^z \lambda_i(z)\right] e_i(z)$$

where

$$\varphi(z)e_i(z) = \lambda_i(z)e_i(z)$$

Indeed, can build formal power series solutions, in the form

$$\psi_i^F(z,\zeta) = \exp\left[-\frac{1}{\zeta}\int_0^z \lambda_i^F(z,\zeta)\right] e_i^F(z,\zeta)$$

But, these solutions can't really exist; if they did, they would be exchanged by monodromy around $z = 0$, but the true equation is regular there, thus has no such monodromy! The only way out: the power series have zero radius of convergence in ζ.

Despite this, if we fix $\vartheta \in S^1$, then in a neighborhood of *generic* z, there exist actual solutions (given by Borel summation),

$$\psi_i^\vartheta(z,\zeta) = \exp\left[-\frac{1}{\zeta}\int_0^z \lambda_i^\vartheta(z,\zeta)\right] e_i^\vartheta(z,\zeta)$$

such that

1. We have asymptotics

   $$\psi^\vartheta(z,\zeta) \sim \psi^F(z,\zeta) \qquad \zeta \to 0 \text{ in } \ell_\vartheta$$

   (more precisely, $C \in Aut\, E_z$ with $\psi_i^\vartheta = C\psi_i^{F,leading}$ is finite as $\zeta \to 0$; similar statement for higher orders),

2. $\psi_i^\vartheta(z,\zeta)$ depends only on $z^{\frac{3}{2}}/\zeta$.

Do these properties determine ψ_i^ϑ uniquely? Number 2 and $\nabla\psi = 0$ determine them up to ambiguity

$$\psi_i' = \sum_{j=1}^{2} \psi_j M_{ij}$$

with M_{ij} constants. Now look at leading behavior:

$$\|\psi_i^F\| \sim \exp\left(\pm\mathrm{Re}\left(\frac{2}{3}z^{\frac{3}{2}}/\zeta\right)\right)$$

For generic (z,ϑ), either $\|\psi_1^F\| \gg \|\psi_2^F\|$ as $\zeta \to 0$ along ℓ_ϑ or vice versa, depending on $\mathrm{Re}(z^{\frac{3}{2}}/\zeta)$. Say $\|\psi_1^F\| \gg \|\psi_2^F\|$ along ℓ_ϑ. Then ψ_2^ϑ is unique but ψ_1^ϑ is ambiguous — can shift by constant multiple of ψ_2^ϑ. So M has to be *unipotent*.

For generic (z,ϑ), we can make both unique by requiring instead

$$\psi^\vartheta(z,\zeta) \sim \psi^F(z,\zeta) \qquad \zeta \to 0 \text{ in } H_\vartheta$$

with H_ϑ the open half-plane. Indeed, generically H_ϑ contains a "Stokes ray" where $z^{\frac{3}{2}}/\zeta \in \mathrm{i}\mathbb{R}$, so M would have to be both upper and lower triangular, i.e. $M = 1$. But if

$$z^{\frac{3}{2}}/e^{\mathrm{i}\vartheta} \in \mathbb{R}$$

this uniqueness fails, and then ψ_i^ϑ really *can* jump by a unipotent matrix.

So, what we have found: for each fixed ϑ, the z-plane divided into three sectors (call the walls $\mathcal{W}(\vartheta)$). Let Σ be the spectral curve

$$\Sigma = \{\det(\lambda - \varphi) = 0\} \subset T^*C,$$

$\mathcal{L} \to \Sigma$ the spectral line bundle

$$\mathcal{L}_\lambda = \ker(\lambda - \varphi).$$

The operation of building an actual solution with given formal asymptotics, $e_i(z) \mapsto e_i^\vartheta(z,\zeta)$, gives an isomorphism in each sector

$$\iota(\zeta) : \pi_*\mathcal{L} \to E$$

taking $\nabla(\zeta)$ to $\pi_*\nabla^{\mathrm{ab}}(\zeta)$, where $\nabla^{\mathrm{ab}}(\zeta)$ is connection in $\mathcal{L}$, of the asymptotic shape

$$\nabla^{\mathrm{ab}}(\zeta) \sim \lambda/\zeta + D^{\mathrm{ab}}.$$

ι jumps by constant unipotent transformations when we cross a wall. Thus ∇^{ab} extends over walls. *Almost* extends over branch point but not quite: holonomy -1. So call ∇^{ab} an *almost-flat* connection.

7 $\mathcal{W}$-pairs

Let us axiomatize this story a bit.

Fix a branched K-fold covering $\Sigma \to C$ and a network $\mathcal{W}$ of "walls" inside C, such that each wall $w \subset \mathcal{W}$ is labeled with two sheets ij of $\pi^{-1}(w)$.

A *$\mathcal{W}$-pair* is a tuple $(E, \nabla, \mathcal{L}, \nabla^{\mathrm{ab}}, \iota)$:

- ∇ a flat connection in rank K bundle E over C,
- ∇^{ab} an almost-flat connection in rank 1 bundle $\mathcal{L}$ over Σ,
- $\iota : \pi_* \mathcal{L} \to E$ an isomorphism away from the walls of $\mathcal{W}$ and branch points of Σ, carrying ∇^{ab} to ∇,

such that at a wall labeled ij, ι jumps by a *unipotent* automorphism of $\pi_* \mathcal{L}$,

$$\iota_+ = (\mathbf{1} + S) \circ \iota_-$$

with $S : \mathcal{L}_j \to \mathcal{L}_i$ ($\pi_* \mathcal{L} = \oplus_i \mathcal{L}_i$).

Like "diagonalizing" the connection ∇ away from $\mathcal{W}$.

Forgetting about WKB, could study this problem in its own right: given (E, ∇) and $\mathcal{W}$, can we find a $\mathcal{W}$-pair $(E, \nabla, \mathcal{L}, \nabla^{\mathrm{ab}}, \iota)$? Often you can!

8 Fock-Goncharov

For example: say $K = 2$, C is a surface with n punctures. Fix an *ideal triangulation* of C. Build a double covering with one branch point in each triangle, maximal choice of branch cuts. Inscribe a spectral network $\mathcal{W}$, "tripod" in each triangle, with all walls ending on the same puncture carrying the same label.

Theorem [retelling of Fock-Goncharov]: if (E, ∇) is *generic* flat $GL(2)$-connection over C, then it can be extended to a $\mathcal{W}$-pair ("abelianization"), in 2^n ways up to isomorphism. Conversely, given almost-flat $GL(1)$-connection $(\mathcal{L}, \nabla^{\text{ab}})$ over Σ, there is a unique way to extend it to a $\mathcal{W}$-pair ("nonabelianization").

Idea of existence and uniqueness for abelianization: need to decompose $E = \iota(\mathcal{L}_1) \oplus \iota(\mathcal{L}_2)$ in each domain. The line $\iota(\mathcal{L}_1)$ in each domain is eigenspace of the monodromy around the nearest puncture. Gluing together of $\mathcal{L}$ along walls is determined by requiring that the connection is diagonal afterward:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} = \mathbf{1}$$

which determines

$$t_1 t_2 = -1, \quad a = -t_1, \quad b = -t_2, \quad c = -t_1.$$

So also get ∇^{ab} almost-flat.

9 Coordinates

Space of $\mathcal{W}$-pairs gives a Lagrangian correspondence which is a covering map onto open dense subset,

$$F_{\mathcal{W}} : \mathcal{M}^{flat}(\Sigma, GL(1)) \to \mathcal{M}^{flat}(C, GL(2))$$

But $\mathcal{M}^{flat}(\Sigma, GL(1))$ is $(\mathbb{C}^\times)^m$. (After suppressing the twisting.) So, we get *local coordinates* on $\mathcal{M}^{flat}(C, GL(2))$. Moreover $F_{\mathcal{W}}$ is a local *symplectomorphism*: follows that we actually get *Darboux* coordinates!

Easy to describe them concretely: look at a loop around two neighboring branch points, corresponding coordinate is Fock-Goncharov or complexified shear coordinate.

Note it's not a unique coordinate system, it really *depends* on the triangulation, or equivalently on the network $\mathcal{W}$ up to isotopy. A *flip* of the triangulation leads to a "cluster mutation," of the form

$$\mathcal{X}'_\mu = \mathcal{X}_\mu (1 + \mathcal{X}_\gamma)^{\pm\langle\gamma,\mu\rangle}.$$

10 WKB spectral networks

Fix a generic (u, ϑ). Build a network $\mathcal{W}(u, \vartheta)$:

- Each branch point of type (ij) emits three walls carrying labels ij or ji, obeying the differential equation $e^{\mathrm{i}\vartheta}(\lambda_i - \lambda_j)$ real, generalizing the trajectories of quadratic differentials.
- When an ij trajectory meets a jk trajectory, create a new ik trajectory born at their intersection. (This is necessary for consistency of $\mathcal{W}$-pair!) [Berk-Nevins-Roberts] Evolve this trajectory also for infinite time. If it meets another kl trajectory then it will give birth to a new il trajectory, and so on.

In general, this leads to a very complicated picture, since the trajectories run around C forever. To make it simpler, use Higgs bundles / connections with *singularities*. [show pictures]

11 WKB asymptotics

Recall a point (u, β) of $\mathcal{M}(C, G, R)$ gives a family of flat connections $\nabla(\zeta)$. Fix ϑ. Then we also have a spectral network $\mathcal{W}(u, \vartheta)$. Claim: for large enough R, there is a family of $\mathcal{W}(u, \vartheta)$-pairs,

$$(E, \nabla(\zeta), \mathcal{L}, \nabla^{\mathrm{ab},\vartheta}(\zeta), \iota(\zeta))$$

such that if we let

$$\mathcal{X}_\gamma^\vartheta(\zeta) = Hol_\gamma \nabla^{\mathrm{ab},\vartheta}(\zeta)$$

then as $\zeta \to 0$ in H_ϑ,

$$\mathcal{X}_\gamma^\vartheta(\zeta) \sim c_\gamma e^{RZ_\gamma/\zeta + \mathrm{i}\beta_\gamma}$$

and as $\zeta \to \infty$ in H_ϑ,

$$\mathcal{X}_\gamma^\vartheta(\zeta) \sim c_\gamma e^{R\bar{Z}_\gamma \zeta + \mathrm{i}\beta_\gamma}$$

12 Riemann-Hilbert problem

Now let's set

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\vartheta = \arg\zeta}(\zeta)$$

still holding fixed our point $(u, \beta) \in \mathcal{M}$ i.e. our family $\nabla(\zeta)$.

These functions have, for some real constants c_γ and β_γ,

1. $\mathcal{X}_\gamma(\zeta) \sim c_\gamma e^{RZ_\gamma/\zeta + \mathrm{i}\beta_\gamma}$ as $\zeta \to 0$ from *any* direction
2. $\mathcal{X}_\gamma(\zeta) \sim c_\gamma^{-1} e^{R\bar{Z}_\gamma \zeta + \mathrm{i}\beta_\gamma}$ as $\zeta \to \infty$ from *any* direction
3. $\mathcal{X}_\gamma(\zeta)$ depends on ζ in a piecewise holomorphic way: jumps when $\mathcal{W}(u, \vartheta = \arg\zeta)$ jumps. This happens at the rays $Z_\gamma/\zeta \in \mathbb{R}_-$,

by an automorphism of the form

$$\mathcal{X}_\mu \to \mathcal{X}_\mu \prod_{n=1}^{\infty} (1 - \mathcal{X}_{n\gamma})^{n\Omega(n\gamma)\langle\mu,\gamma\rangle}$$

(Note, exponentially small correction as $\zeta \to 0$, thus consistent with the asymptotics.)

These conditions are enough to *determine* the functions $\mathcal{X}_\gamma(\zeta)$. Since $\mathcal{X}_\gamma(\zeta)$ are Darboux coordinates, in principle this also determines the holomorphic symplectic form ϖ_ζ.

Simple example: "pentagon" — $K = 2$, consider Higgs bundles on $\mathbb{CP}^1$ with irregular singularity at $z = \infty$,

$$\varphi_1 = 0,$$

$$\varphi_2 = (z^3 + \Lambda z + u)\mathrm{d}z^2$$

Draw the Hitchin base $\mathcal{B}$. `[show pictures]`

For points in strong coupling region, get

$$\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = 1$$

while in weak coupling region,

$$\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = \Omega(\pm(\gamma_1 + \gamma_2)) = 1$$

Draw the corresponding pictures in the ζ-plane.

Next simplest example: "cylinder" — $K = 2$, Higgs bundles on $\mathbb{CP}^1$ with irregular singularities at $z = 0$ and $z = \infty$,

$$\varphi_1 = 0,$$

$$\varphi_2 = \left(\frac{1}{z} + u + z\right)\left(\frac{\mathrm{d}z^2}{z^2}\right)$$

Here we meet in weak coupling region

$$\Omega(n\gamma_1+(n+1)\gamma_2) = 1, \quad \Omega((n+1)\gamma_1+n\gamma_2) = 1, \quad \Omega(\gamma_1+\gamma_2) = -2.$$

In general what are the $\Omega(\gamma)$? Fix $K = 2$. Look at the family of spectral networks $\mathcal{W}(u, \vartheta)$ — in this case, just critical graphs of quadratic differentials $e^{2i\vartheta}\varphi_2$. Define the notion of a saddle connection or closed family with charge γ. They can occur at $\vartheta = \arg Z_\gamma$. Then let

$$\Omega(\gamma; u) = \#\{\text{saddle conn with charge } \gamma\} - 2\#\{\text{closed traj with charge } \gamma\}$$

In more general cases, for $K > 2$, we can meet much wilder invariants. `[Show three-pronged network.]`

An interesting counting problem, not much explored!

13 Integral equation

Re-summarizing: given (C, G, R) we have Hitchin base $\mathcal{B}$, smooth locus $\mathcal{B}' \subset \mathcal{B}$, local system $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$, $\mathcal{M}$ fibered over $\mathcal{B}$ with generic fibers $H^1(\Sigma_u, U(1))$, coordinatized by $\beta_\gamma : \mathcal{M} \to U(1)$. Periods $Z_\gamma = \oint_\gamma \lambda$. We seek local functions

$$\mathcal{X}_\gamma : \mathcal{M} \times \mathbb{C}^\times \to \mathbb{C}^\times$$

which will be Darboux coordinates for holomorphic symplectic structures ϖ.

To find $\mathcal{X}_\gamma$, formulate integral equation (cf [Cecotti-Vafa, Dubrovin]). Fix $(u, \beta) \in \mathcal{M}$ generic. Note that if there were *no* jumps we would simply get

$$\mathcal{X}^{\mathrm{sf}}_\gamma(\zeta) = \exp\left(R\zeta^{-1}Z_\gamma + i\beta_\gamma + R\zeta\bar{Z}_\gamma\right)$$

In general, we require

$$\mathcal{X}_\mu(\zeta) = \mathcal{X}^{\rm sf}_\mu(\zeta)\exp\left[-\frac{1}{4\pi \mathrm{i}}\sum_\gamma \Omega(\gamma;u)\langle\mu,\gamma\rangle\int_{\mathbb{R}_- Z_\gamma}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_\gamma(\zeta'))\right].$$

14 Ooguri-Vafa metric

A "baby" example: take $\mathcal{B}'$ to be the punctured disc $\{0 < |u| < 1\}$, Γ generated by γ_e, γ_m,

$$Z_e(u) = u, \tag{14.1}$$

$$Z_m(u) = \frac{1}{2\pi i}(u\log u - u). \tag{14.2}$$

This is everything that was needed to specify the metric $g^{\rm sf}$ on a torus bundle over $\mathcal{B}'$, where the monodromy around $u = 0$ takes $\beta_m \to \beta_m + \beta_e$, matching the fact that $Z_m \to Z_m + Z_e$. We have $\tau = \frac{1}{2\pi i}\log u$ i.e. $q = e^{2\pi i\tau} = u$.

Now define

$$\mathcal{X}_e = \mathcal{X}_e^{\rm sf},$$

$$\mathcal{X}_m = \mathcal{X}_m^{\rm sf}\exp\Bigg[\frac{i}{4\pi}\int_{u\mathbb{R}_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')] - \frac{i}{4\pi}\int_{u\mathbb{R}_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')^{-1}]\Bigg],$$

$\mathcal{X}_m$ is discontinuous as a function of (u,ζ): on crossing the locus $\zeta \in u\mathbb{R}_+$ it jumps by $\mathcal{X}_m \to \mathcal{X}_m(1-\mathcal{X}_e)$, on crossing $\zeta \in u\mathbb{R}_-$ it jumps by $\mathcal{X}_m \to \mathcal{X}_m(1-\mathcal{X}_e^{-1})^{-1}$. On the other hand $\mathcal{X}_m \sim \exp(\pi R Z_m/\zeta)$ up to a *finite* correction as $\zeta \to 0$ from any direction.

So it's discontinuous, but its asymptotics are continuous. Stokes phenomenon.

Nevertheless,

$$\varpi = \frac{1}{4\pi^2 R} d\log \mathcal{X}_e \wedge d\log \mathcal{X}_m$$

is perfectly smooth, still has the 3-term expansion, defines a hyperkähler metric, which can be written explicitly and *does* extend to the nodal fiber at $u = 0$. (The fact that it extends is related to the fact that composing the two discontinuities reproduces the monodromy.)

This explicit metric ("Ooguri-Vafa metric") is of the form

$$g = g^{\mathrm{sf}} + O(e^{-R|u|})$$

as $R \to \infty$. Corrections are a sum of Bessel functions:

$$e^{\mathrm{i}n\beta_e} K_{0,1}(nR|u|)$$

15 Solving in general

Can solve it by iteration (for large enough R). Can also write a formal series expansion for the solution: explicitly, for rooted tree $\mathcal{T}$, define weight of $\mathcal{T}$

$$wt(\mathcal{T}) = \frac{1}{|\mathrm{Aut}(\mathcal{T})|} \prod_{i \in \mathrm{Nodes}(\mathcal{T})} c(\gamma_i) \prod_{(i,j) \in \mathrm{Edges}(\mathcal{T})} \langle \gamma_i, \gamma_j \rangle.$$

Let $\gamma_{\mathcal{T}}$ be decoration at root of $\mathcal{T}$. Define $\mathcal{G}_{\mathcal{T}}(\zeta)$ inductively: deleting root from $\mathcal{T}$ leaves trees $\mathcal{T}_a$, and

$$\mathcal{G}_{\mathcal{T}}(\zeta) = \frac{1}{4\pi \mathrm{i}} \int_{\mathbb{R}_{-} Z_{\gamma_{\mathcal{T}}}} \frac{\mathrm{d}\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \mathcal{X}^{\mathrm{sf}}_{\gamma_{\mathcal{T}}}(\zeta') \prod_a \mathcal{G}_{\mathcal{T}_a}(\zeta').$$

Saddle point method:

$$\mathcal{G}_{\mathcal{T}}(\zeta) \sim \exp\left(-R \sum_{i \in \mathrm{Nodes}(\mathcal{T})} |Z_{\gamma_i}|\right)$$

Then formal solution is

$$\mathcal{X}_\gamma(x,\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(x,\zeta) \exp\left[\sum_{\mathcal{T}} \langle \gamma, \gamma_{\mathcal{T}} \rangle wt(\mathcal{T}) \mathcal{G}_{\mathcal{T}}(x,\zeta)\right].$$

Leading contribution is from trees with one node: this gives

$$\log \mathcal{X}_\mu = \log \mathcal{X}_\mu^{\mathrm{sf}} + O\left(e^{-RM}\right)$$

where

$$M = \min\{|Z_\gamma| : \Omega(\gamma) \neq 0, \langle \mu, \gamma \rangle \neq 0\}.$$

Thus similarly for ϖ_ζ.

16 Wall-crossing formula

The invariants $\Omega(\gamma; u)$ can *vary* as we move around in the base $\mathcal{B}$. They are not totally arbitrary though: in particular, the $\mathcal{X}_\gamma$ have to exist. Draw picture of the jumping locus: "BPS rays" where $Z_\gamma/\zeta \in \mathbb{R}_-$ and $\Omega(\gamma) \neq 0$. Then follow a loop in parameter space: composition of the jump endomorphisms must be 1. This gives wall-crossing formula [Kontsevich-Soibelman].

17 Other interpretations of $\Omega(\gamma)$

- Dimensions of spaces of BPS states in 4-dimensional quantum field theory.

- DT invariants for a CY3 category. For $K = 2$ [Bridgeland-Smith, Smith]: take 3-fold

$$x^2 + y^2 + w^2 = \varphi_2(z),$$

look at its Fukaya category.)

- Counting Euler characteristics of moduli of quiver representations.

18 Mirror symmetry

What does it have to do with mirror symmetry?

Fix some ζ with $|\zeta| = 1$. (Concretely, say $\zeta = 1$.)

What we've said: $(\mathcal{M}, J_1)$ is divided up into domains, on each domain we have a preferred "best" holomorphic coordinate system. (Draw the picture in the pentagon theory — five different triangulations of pentagon.) Loosely speaking, we might say that $(\mathcal{M}, J_1)$ is being glued together from these pieces.

Mirror symmetry provides a very similar picture [Gross-Siebert, Gross-Hacking-Keel, Auroux]. There we would begin with $(\mathcal{M}, \omega_2)$ considered as a *real symplectic* space. Convenient to tame it with complex structure J_2.

In the mirror picture, gluing automorphisms between domains get associated to holomorphic discs in $(\mathcal{M}, \omega_2)$. Wall is the locus

$$\{u \in \mathcal{B} : \mathcal{M}_u \text{ has a disc ending on it}\}$$

We can see this in our framework: the lattice

$$\Gamma_u = H_2(\mathcal{M}, \mathcal{M}_u; \mathbb{Z})$$

measures topology of these discs. $Z_\gamma = \int_\gamma \varpi_3 = \int_\gamma \omega_1 + \mathrm{i}\omega_2$. Existence of a holomorphic disc implies in particular

$$\int_\gamma \varpi_2 = \int_\gamma \omega_1 + \mathrm{i}\omega_3 = 0$$

and thus

$$Z_\gamma \in \mathrm{i}\mathbb{R}$$

which is indeed where our walls are! So, propose that the walls give a tropical picture of the desired discs.

In some cases, can explicitly exhibit the desired discs [Lin].

Similar story for general ζ, "analytic continuation" of the usual mirror symmetry, involves turning on B field [Kontsevich-Soibelman].