Spectral networks in 2 and 3 dimensions

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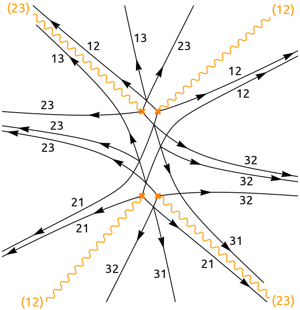
Preface

Last year in joint work with D. Gaiotto and G. Moore, studying $\mathcal{N} = 2$ supersymmetric field theories in 4 dimensions, we were led to the notion of spectral network.

A spectral network is a collection of curves drawn on a 2-manifold C, decorated by some extra discrete data, and obeying some local rules.



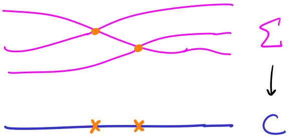
Preface

My aim today is to describe:

- what a spectral network is,
- how spectral networks are related to BPS states / DT invariants (and a recent surprise that came from them),
- how spectral networks allow one to “abelianize” flat $GL(K, \mathbb{C})$-connections on 2-manifolds (and hopefully 3-manifolds).

Our data

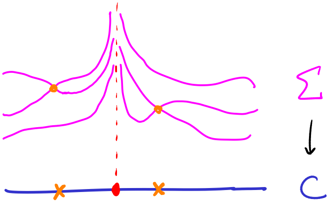
Suppose given a Riemann surface C and a K-fold branched covering $\Sigma \to C$, with $\Sigma \subset T^*C$.



Locally, Σ gives K holomorphic 1-forms λ_i on C.

Our data

Slight extension: allow C to have marked points (defects) where the covering Σ goes off to ∞.



Locally, Σ gives K meromorphic 1-forms λ_i on C.

C determines an $\mathcal{N} = 2$ supersymmetric QFT $S[A_{K-1}, C]$.
Σ determines a point on its Coulomb branch.

[Witten, Gaiotto-Moore-N, Gaiotto]

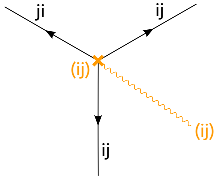
Defining spectral networks

Also fix a parameter $\vartheta \in S^1$.
Define a network $\mathcal{W}(\Sigma, \vartheta)$ of walls on C, as follows. [Gaiotto-Moore-N]

Each wall carries a label ij (locally defined on C) and obeys differential equation:

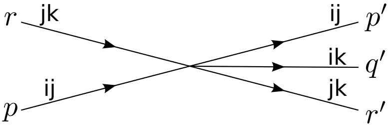
$$(\lambda_i - \lambda_j)\dot{z} = e^{i\vartheta}$$

Each branch point of $\Sigma \to C$ emits 3 walls



When walls of types ij, jk intersect they give birth to a new wall

[Berk-Nevins-Roberts, Cecotti-Vafa]



A sample spectral network

- $C = \mathbb{CP}^1$ with 3 defects
- $\Sigma \to C$ 3-fold cover with 6 branch points

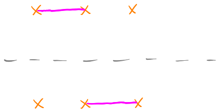
The case $K = 2$

In the case $K = 2$, the walls never cross: they are leaves of a global foliation on C.

In this case $\mathcal{W}(\Sigma, \vartheta)$ is a well-studied object: critical graph of a quadratic differential. [Strebel]

BPS counts via spectral networks

Fix the covering Σ and let ϑ vary. As we do so, the network $\mathcal{W}(\Sigma, \vartheta)$ sometimes jumps. The jumps occur when some of the walls hit each other head-on, i.e., the network degenerates.



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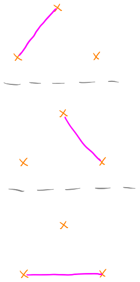

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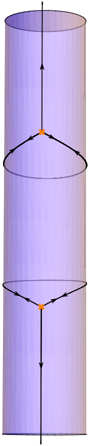

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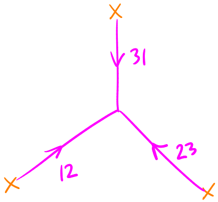


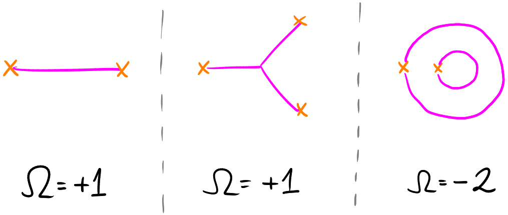
BPS counts via spectral networks

We define a map $\Omega_\Sigma : H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$ which counts the degenerations of $\mathcal{W}(\Sigma, \vartheta)$ (in an appropriate sense).

The $\Omega_\Sigma(\gamma)$ are BPS state counts in the theory $S[A_{K-1}, C]$.

[Klemm-Lerche-Mayr-Vafa-Warner, Gaiotto-Moore-N]

Some examples of the rules:

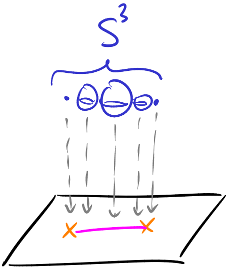


Donaldson-Thomas invariants

The $\Omega_\Sigma(\gamma)$ are BPS state counts; equivalently, they are expected to be generalized Donaldson-Thomas invariants for the Fukaya category of a certain noncompact Calabi-Yau threefold $X[A_{K-1}, C]$ (deformation of $\mathbb{C}^2/\mathbb{Z}_K$ singularity fibered over C).

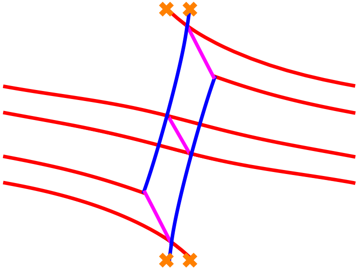
[Douglas, Bridgeland, Kontsevich-Soibelman, Joyce-Song, Diaconescu-Donagi-Pantev, Bridgeland-Smith]

The objects we counted on C should lift to special Lagrangian 3-cycles in $X[A_{K-1}, C]$.



Another degeneration

A more exotic degeneration of spectral network, which can occur for $K > 2$: [Galakhov-Longhi-Mainiero-Moore-N]



This degeneration leads to an infinite sequence of nonzero BPS counts, not just one:

$$\{\Omega_\Sigma(n\gamma)\} = 3, -6, 18, -84, 465, -2808, \ldots$$

They grow exponentially:

$$\Omega_\Sigma(n\gamma) \sim (-1)^{n+1} n^{-5/2} \, a \, e^{cn}$$

with $c = \log(\frac{256}{27})$, $a = \sqrt{\frac{3}{8\pi}}$.

Exponential growth in field theory

This degeneration occurs in a concrete example: take $C = \mathbb{CP}^1$, $\mathfrak{g} = A_2$, with (irregular) defects at $z = 0$ and $z = \infty$.

The corresponding theory is pure $\mathcal{N} = 2$ super Yang-Mills with $G = SU(3)$. Thus, this theory has exponential growth in its BPS spectrum:

$$\Omega(n\gamma) \sim e^{cn}$$

In a non-gravitational theory, this was unexpected for us. Nothing like this was seen in $SU(2)$ Yang-Mills. [Seiberg-Witten, Bilal-Ferrari]

It seems that going from $SU(2)$ to $SU(3)$ makes things drastically more interesting.

Exponential growth in supergravity

Exponential growth of the BPS counts would not be a surprise in supergravity theories. There one expects

$$\Omega(n\gamma) \sim e^{cn^2}$$

on grounds of black hole entropy.

[Bekenstein, Hawking, Strominger-Vafa, Maldacena-Strominger-Witten]

Exponential growth in field theory

The result bothered us, so we checked it a second way.

[Galakhov-Longhi-Mainiero-Moore-N]

The idea: as Σ is varied, Ω_Σ changes according to wall-crossing formula. [Denef-Moore, Kontsevich-Soibelman, Joyce-Song, Gaiotto-Moore-N., Manschot-Pioline-Sen, ...]

Begin from Σ in "strong coupling" chamber, where we know $\Omega_\Sigma(\gamma) = 1$ for $\gamma \in \{\gamma_1, \ldots, \gamma_{12}\}$, else $\Omega_\Sigma(\gamma) = 0$.

Then vary Σ across a few walls, and use wall-crossing to deduce the exponentially-growing spectrum.

Algebraic equations governing field theory spectra

So, we've found that field theory contains much crazier stuff than we thought.

$$\{\Omega_\Sigma(n\gamma)\} = 3, -6, 18, -84, 465, \dots$$

But, there is also some new structure. To see it, define generating function

$$P(z) = \prod_{n=1}^{\infty} (1 - z^n)^{n\Omega(n\gamma)/3}.$$

It obeys

$$P(z) = 1 - zP(z)^4.$$

So these BPS counts are governed by algebraic equation!

[Kontsevich, Gross-Pandharipande-Siebert]

Spectral network algorithm for computing the BPS spectrum gives a natural explanation of this equation. We believe BPS spectra in all theories of class S will have similar equations, for similar reasons.

Quivers

There is another approach to BPS counts, via quiver representations.

[Denef, Alim-Cecotti-Cordova-Espahbodi-Rastogi-Vafa, Cecotti-del Zotto, Cecotti-N-Vafa, ...]

Spectral networks give the same result as quivers, in cases that have been studied by both methods.

The exponentially growing BPS counts in the $SU(3)$ theory which we just discussed are related to the Kronecker 3-quiver.

(The $SU(3)$ theory also contains spectra related to Kronecker m-quiver for any m!)

(Non-)Abelianization

A second application of spectral networks:

Let $\mathcal{M}(C, GL(K))$ be moduli of flat $GL(K, \mathbb{C})$-connections over C, with singularities at the marked points.

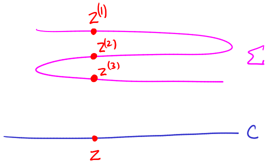
Let $\mathcal{M}(\Sigma, GL(1))$ be moduli of [almost] flat $GL(1, \mathbb{C})$-connections over Σ, with singularities at preimages of marked points.

How are the two kinds of connection related?

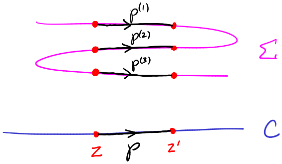
Pushing forward

A naive (wrong) guess: pushforward — given $(\mathcal{L}, \nabla^{\mathrm{ab}}) \in \mathcal{M}(\Sigma, GL(1))$ define $(\pi_* \mathcal{L}, \pi_* \nabla^{\mathrm{ab}}) \in \mathcal{M}(C', GL(K))$:

$$(\pi_* \mathcal{L})_z = \bigoplus_{i=1}^{K} \mathcal{L}_{z^{(i)}}$$



Parallel transport of $\pi_* \nabla^{\mathrm{ab}}$ given "sheetwise" by ∇^{ab}.

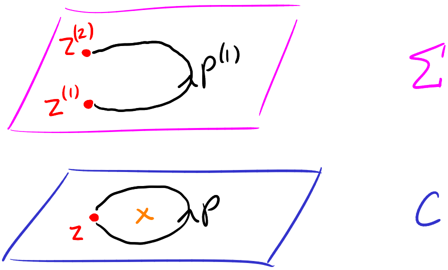


Pushing forward

Pushforward $\pi_* \nabla^{\mathrm{ab}}$ is a flat $GL(K)$-connection, but only over

$$C' = C \setminus \{\text{branch points of } \Sigma \to C\}.$$

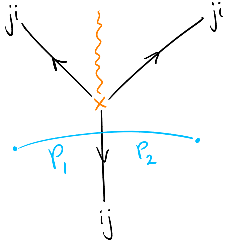
Can't extend $\pi_* \nabla^{\mathrm{ab}}$ to the whole C, because of monodromy around branch points!



Cutting and gluing

To get an honest flat connection ∇ over the whole C, use a spectral network $\mathcal{W}(\Sigma, \vartheta)$.

Parallel transport of ∇ along a path $\wp \subset C$ is that of $\pi_* \nabla^{\mathrm{ab}}$, except that we splice in a unipotent matrix whenever $\wp$ crosses $\mathcal{W}$:



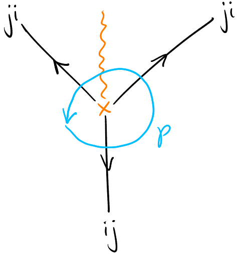
$$P_{\nabla,\wp} = P_{\pi_* \nabla^{\mathrm{ab}},\wp_1} (1 + e) P_{\pi_* \nabla^{\mathrm{ab}},\wp_2}$$

Here e is an "elementary matrix whose only nonzero entry is in the *ij* position", or more invariantly, $e : \mathcal{L}_{z^{(i)}} \to \mathcal{L}_{z^{(j)}}$.

Flatness constraint

Requiring flatness for the resulting ∇ determines the unipotent "correction" matrices.

Flatness around branch point:



$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

forces

$$a = 1/D_2, \quad c = 1/D_2, \quad b = D_2$$

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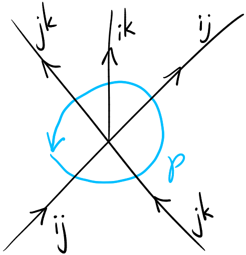
Flatness around intersection of walls:



$$\begin{pmatrix}1 & a & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & b\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -a' & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -c'\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & -b'\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}D_1 & 0 & 0\\0 & D_2 & 0\\0 & 0 & D_3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

forces

[Cecotti-Vafa]

$$a' = a, \quad b' = b, \quad c' = ab$$

Nonabelianization map

So, by this "cutting and gluing along walls of $\mathcal{W}$", we have defined a map

$$\Psi_{\mathcal{W}} : \mathcal{M}(\Sigma, GL(1)) \to \mathcal{M}(C, GL(K))$$

Both spaces are Poisson manifolds, and $\Psi_{\mathcal{W}}$ is a local leafwise symplectomorphism. Moreover $\mathcal{M}(\Sigma, GL(1)) \simeq (\mathbb{C}^\times)^n$.

Thus $\Psi_{\mathcal{W}}$ gives local Darboux coordinate system on $\mathcal{M}(C, GL(K))$.

Spectral coordinates

For each spectral network $\mathcal{W}$ on C, we get a local Darboux coordinate system $\Psi_{\mathcal{W}}$ on $\mathcal{M}(C, GL(K))$.

These coordinate systems are in some sense canonical and have some applications:

- Physically, expanding holonomy of ∇ in terms of these coordinates gives the UV-IR relation for line defects, i.e. counting of framed BPS states.
- If $\mathcal{W}$ and $\mathcal{W}'$ are related by a degeneration as discussed earlier, can compute the coordinate transformations relating $\Psi_{\mathcal{W}}$ to $\Psi_{\mathcal{W}'}$. This is the key in the computation of BPS counts mentioned earlier, and in the proof that they obey the expected wall-crossing formula. [Kontsevich-Soibelman]
- They are convenient for the purpose of studying the hyperkähler metric on $\mathcal{M}$.

(Non-)Abelianization for connections over 3-manifolds

It seems that there is also a version for 3-manifolds.

[Freed-N **in progress**]

Given:

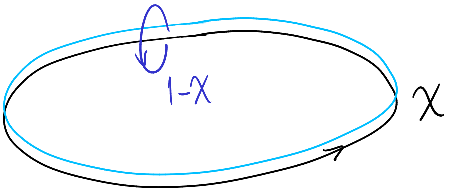
- 3-manifold M
- K-fold branched cover $X \to M$
- 3d spectral network $\mathcal{W}$ on M

get a map between $GL(1)$-connections ∇^{ab} over X and $GL(K)$-connections ∇ over M.

(Non-)Abelianization for connections over 3-manifolds

$GL(1)$-connection ∇^{ab} over X is not quite flat: it has delta-function curvature at some codimension-2 loci $S \subset X$ (scars), where *ij* and *ji* walls collide.

Simplest case: $S \subset X$ a circle (framed), ∇^{ab} has holonomy $\mathcal{X}$ around S, $1 - \mathcal{X}$ around circle linking S.



∇^{ab} with this kind of holonomy correspond to solutions of some algebraic equations in $\mathbb{C}^\times$-valued variables.

(Non-)Abelianization for connections over 3-manifolds

Given such a $GL(1)$-connection ∇^{ab} over X, we build a $GL(K)$-connection ∇ on M: start with $\pi_* \nabla^{\mathrm{ab}}$ and splice in unipotent matrices at the walls of $\mathcal{W}$.

Parallel to the 2-d case discussed before.

Triangulated hyperbolic 3-manifolds

Suppose M is a triangulated hyperbolic 3-manifold, with cusps.

It is known that $SL(2)$-connections ∇ over M correspond to solutions of algebraic equations: one shape variable $\mathcal{X}_i \in \mathbb{C}^\times$ for each tetrahedron, one gluing equation for each edge.

[W. Thurston]

(There is a similar picture for $SL(K)$-connections, $K > 2$.)

[Dimofte-Gabella-Goncharov, Garoufalidis-D. Thurston-Zickert]

We believe our 3d nonabelianization map recovers these equations, at least for $K = 2$. Expect there is a double cover $X \to M$, and canonical associated 3d spectral network, with one scar $S_i \subset X$ for each tetrahedron. [Cecotti-Cordova-Vafa]

The shape variables are the holonomies of ∇^{ab} around components S_i. The gluing equations come from relations in $H^1(X \setminus S, \mathbb{Z})$.

Abelianization for Chern-Simons theory

The classical $SL(2)$ Chern-Simons action evaluated on ∇ is (very roughly) $\sum_i \mathrm{Li}_2(\mathcal{X}_i)$. [Neumann, Dupont, Zickert, ...]

We expect that this is part of a more general story: whenever ∇^{ab} and ∇ are related by nonabelianization (with only simple scars),

$GL(K)$ Chern-Simons action of ∇ is equal to
$GL(1)$ Chern-Simons action for ∇^{ab} plus $\sum_i \mathrm{Li}_2(\mathcal{X}_i)$.

[Witten, Ooguri-Vafa, Cecotti-Vafa]

(Lack of gauge invariance in the first term compensated by branch choice in the second term, so that the sum is really canonically defined.)

A similar equivalence for quantum Chern-Simons has been proposed before. [Cecotti-Cordova-Vafa]

Physics of 3d spectral networks

This construction raises a question in physics.

2d spectral network has a clear meaning in terms of BPS states living on the canonical surface defect in a theory of class $S[A_{K-1}]$. [Gaiotto, Alday-Gaiotto-Gukov-Tachikawa-Verlinde, Gaiotto-Moore-N]

What does the 3d spectral network mean? Does it have a natural interpretation in a theory of class $R[A_{K-1}]$?

[Dimofte-Gaiotto-Gukov, Cecotti-Cordova-Vafa]

Summation

- Spectral networks are a new geometric structure naturally associated to a branched cover $\Sigma \to C$, $\Sigma \subset T^*C$.
- They can be used to compute BPS counts of theories of class S / DT invariants of CY 3-folds.
- They can also be used to (non)abelianize flat connections over 2-manifolds, and (hopefully) 3-manifolds.