Wall-Crossing (3)

Last time: WCF for any $N=2$, $d=4$ field theory.

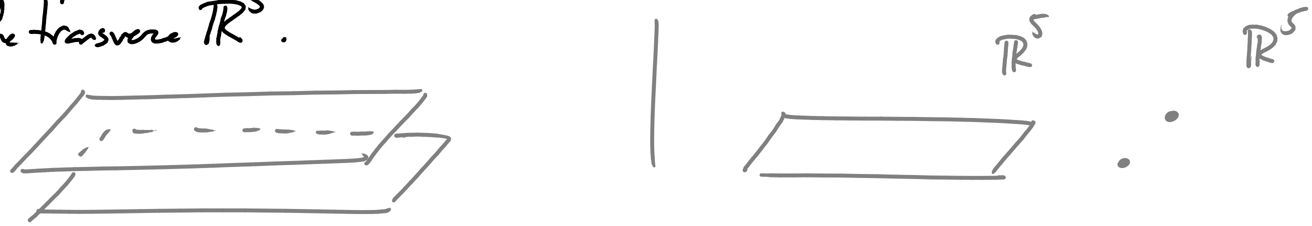
Now let's consider some theories where BPS states can be understood geometrically.

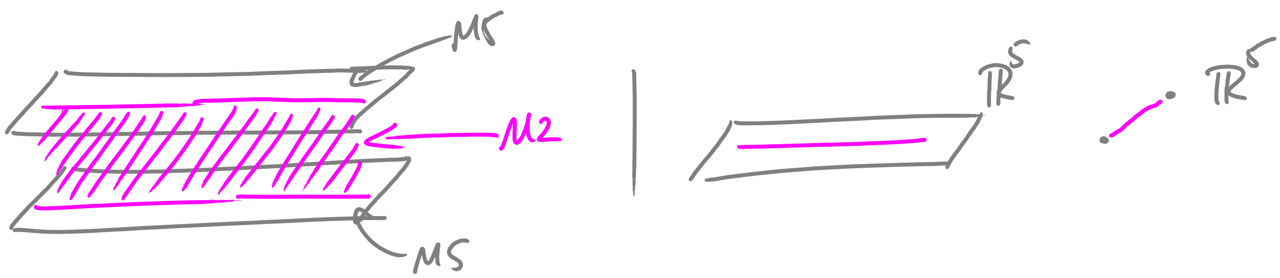
The idea: There is a rather mysterious SCFT in $\underline{d=6}$ with $(2,0)$ SUSY.
More exactly, a family of them: one for each ADE Dynkin diagram.
No known Lagrangian description: only get info rather indirectly!

Construction from M-theory: (for A_{K-1} theories)

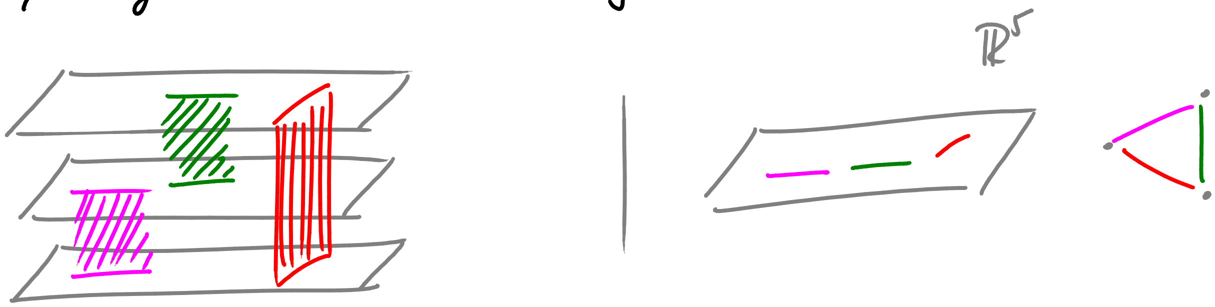
In M-theory have the M5-brane. Consider K M5-branes, look at the low energy dynamics.

The theory has a "Coulomb branch" where the K branes are separated in the transverse $\mathbb{R}^5$.





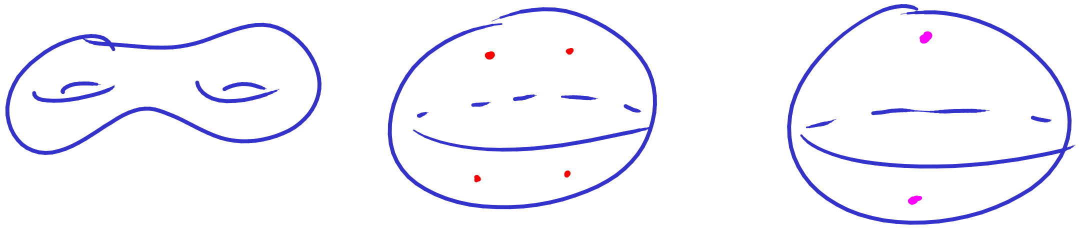
If $K>2$, many different flavors of strings:



Compactify the $(2,0)$ A_K theory on a Riemann surface C

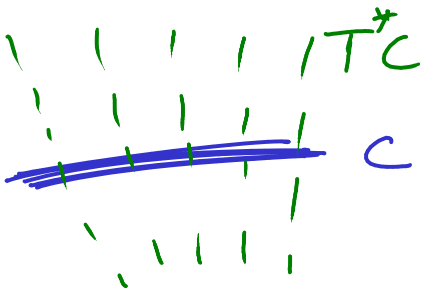
(carrying "defect operators" inserted at points).

With an appropriate topological twist, this leads to an $\mathcal{N}=2$ theory in 4 dimensions.

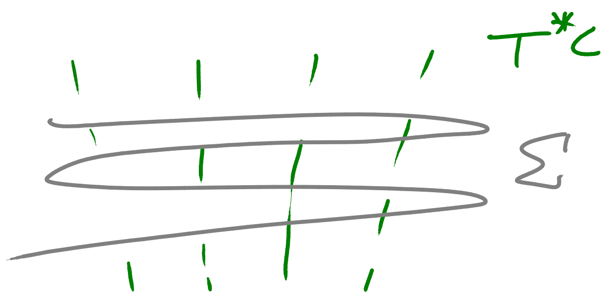


In M-theory, can think of this as studying

$$\begin{array}{lccc} \text{M-theory in} & T^*C & \times & \mathbb{R}^{6,1} \\ & \cup & & \cup \\ \text{M5-branes on} & C & \times & \mathbb{R}^{3,1} \end{array}$$



To go out on Coulomb branch, try to separate the M5-branes:



Best we can do is separate K branes on $C \rightsquigarrow 1$ brane on Σ, a K-fold cover of C.

Σ is given by an equation of the form

$$\lambda^K + \varphi_2 \lambda^{K-2} + \varphi_3 \lambda^{K-3} + \cdots + \varphi_K = 0 \qquad \lambda \in T^*C$$

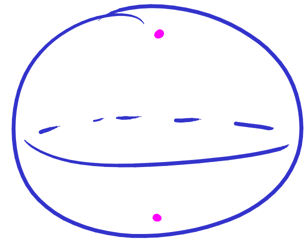
φ_n is a meromorphic n-differential on C.

Different choices of the $\varphi_n \longleftrightarrow$ different "shapes" for Σ

$\longleftrightarrow$ different points of Coulomb branch $\mathcal{B}$.

Ex $K=2$

$C = \mathbb{CP}^1$ w/ 2 punctures

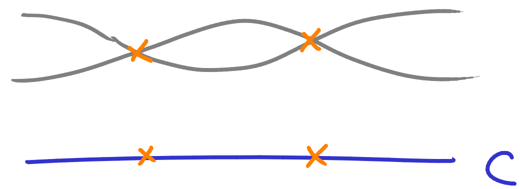


Σ given by $\lambda^2 = \varphi_2$

where $\varphi_2(z) = \left(\frac{\Lambda^2}{z} + 2u + \Lambda^2 z\right)\left(\frac{dz}{z}\right)^{\otimes 2}$

So, in local coordinates ($\lambda = x\, dz$)

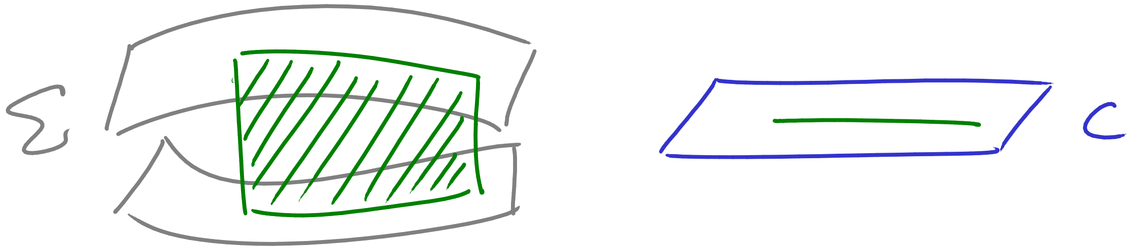
Σ is $\left\{x^2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}\right\} \subset \mathbb{C}^\times \times \mathbb{C}^\times$



The $\mathcal{N}=2$ theory that we get from this construction is the pure $\mathcal{N}=2$ SYM with gauge group $G = SU(2)$!

And Σ is the "Seiberg-Witten curve."

BPS state: M2-branes stretched between the sheets of Σ.



Tension depends on z: it's ~ the distance between branes, $|\lambda| = |x\, dz|$

So the total mass of the state from 4d POV is

$$M = \int |\lambda| = \int |x\, dz|$$

while the central charge is $\quad Z_\gamma = \oint_\gamma \lambda = \oint_\gamma x\, dz$

BPS bound: $M \geq |Z|$

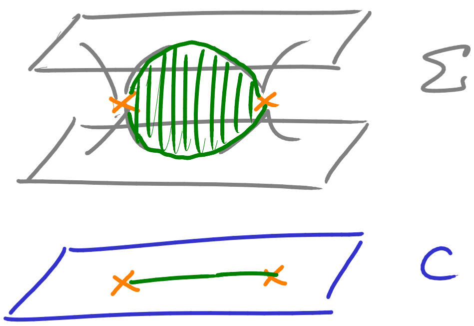
here just says $\int |x\, dz| \geq \left| \int x\, dz \right|$

So the string gives a BPS state only if $M = |Z|$ i.e. $\left| \int x\, dz \right| = \int |x\, dz|$

i.e. $x\, dz$ has <u>constant phase</u> along the string.

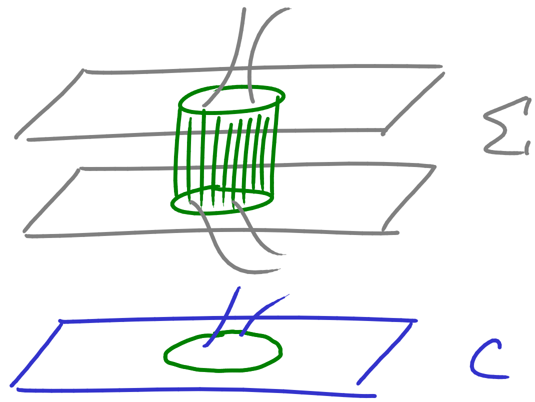
To make a BPS state, we need a string w/ <u>finite</u> <u>total mass</u>.

Two ways to get one:



BPS hypermultiplet

$(S_0,\ \Omega = +1)$



BPS vectormultiplet

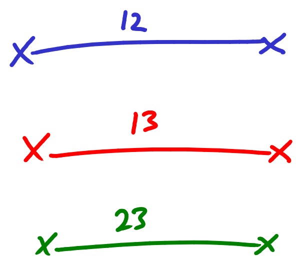
$(S_{1/2},\ \Omega = -2)$

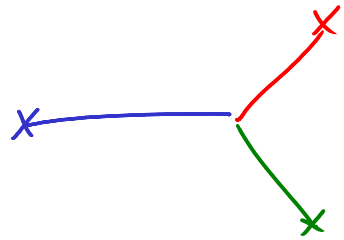
The <u>boundary</u> of the M2-brane determines a cycle γ on Σ

$$Z_\gamma = \oint_\gamma \lambda \quad \text{— Seiberg-Witten formula}$$

In higher-K theories we expect more complicated possibilities..

$K=3$





So the appearance/dis. of these objects as we vary moduli should be governed by the WCF...