1 Preface

A new construction of hyperkähler spaces. They are torus fibrations (with degenerate fibers) over special Kähler manifolds.

The starting point for this construction is a rather complicated set of data. All these data would be summarized for physicists as "the infrared limit and BPS degeneracies of an $\mathcal{N} = 2$ supersymmetric field theory." A crucial part of the data is a set of integer "invariants" which obey a rather complicated and exotic-looking wall-crossing formula, described by Kontsevich and Soibelman. This construction of hyperkähler manifolds gives a purely geometric perspective on why the wall-crossing formula should be true.

The starring role in this construction is played by a set of piecewise-analytic functions $\mathcal{X}_\gamma : \mathcal{M} \times \mathbb{C}^\times \to \mathbb{C}^\times$. Holomorphic Darboux coordinates. Very roughly, we may say that we are gluing $\mathcal{M}$ out of fiducial pieces which look like $(\mathbb{C}^\times)^{2r}$, with the gluing maps given by the symplectomorphisms of KS.

I'll also describe an interesting set of examples where the moduli space you get is one which has been encountered before: moduli space of Higgs bundles (with singularities) over a Riemann surface C. Only for $SU(2)$. In these examples the holomorphic Darboux coordinates turn out to be something easy to understand, and their jumps have a simple geometric meaning in terms of a certain foliation on C.

2 SUSY gauge theory data

Begin with data:

- A complex manifold $\mathcal{B}$, of complex dimension r.
- A divisor $D \subset \mathcal{B}$. Let $\mathcal{B}' = \mathcal{B} \setminus D$.

- A local system of lattices Γ over $\mathcal{B}'$, with antisymmetric pairings $\langle , \rangle$, such that $\Gamma_g = \Gamma / rad \langle , \rangle$ has rank $2r$.
- A "central charge" homomorphism $Z : \Gamma \to \mathbb{C}$, varying holomorphically over $\mathcal{B}$, constant on $rad\langle , \rangle$.
- Integer invariants $\Omega : \Gamma \to \mathbb{Z}$.

Subject to conditions:

- Transversality: $\langle dZ, dZ \rangle = 0$.
- Nondegeneracy: $\langle dZ, d\bar{Z} \rangle > 0$.
- Monodromy of Γ around each component $D_i \subset D$ is

$$\gamma' \to \gamma' + \sum_{\gamma : Z_\gamma(u) \to 0 \text{ as } u \to D_i} \Omega(\gamma; u) \langle \gamma, \gamma' \rangle \gamma.$$

- Wall-crossing formula for Ω: Define *walls of marginal stability*, $W_\gamma \subset \mathcal{B}'$, by

$$W_\gamma = \{ u : \exists \gamma_1, \gamma_2 \in \Gamma_u, \text{linearly independent}, \\ \gamma_1 + \gamma_2 = \gamma, Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+ \}.$$

  Then $\Omega(\gamma; u)$ is locally constant over $\mathcal{B}' \setminus W_\gamma$. The jumping behavior of $\Omega(\gamma; u)$ is given by Kontsevich-Soibelman formula. Consider an algebraic symplectic torus $T_u := \Gamma_u^* \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Each γ canonically gives a function $X_\gamma : T \to \mathbb{C}^\times$, with $X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}$. Define an element

$$\mathcal{K}_\gamma : X_{\gamma'} \to X_{\gamma'} (1 - \sigma(\gamma) X_\gamma)^{\langle \gamma', \gamma \rangle}.$$

  (σ is a refinement of the antisymmetric pairing: $\sigma(\gamma) = \pm 1$.) Then choose a convex sector $S \subset \mathbb{C}$ and form a product in clockwise order,

$$A_S(u) = \left(\overrightarrow{\prod}_{\gamma : Z(\gamma;u)\in S} \mathcal{K}_\gamma^{\Omega(\gamma;u)}\right) \in G$$

$A_S(u)$ is then invariant under deformation of u in which no $Z(\gamma; u)$ enters or leaves S. This is strong enough to determine $\Omega(\gamma; u)$ from any $\Omega(\gamma; u_0)$.

Simplest example of this is

$$\mathcal{K}_{0,1}\mathcal{K}_{1,0} = \mathcal{K}_{1,0}\mathcal{K}_{1,1}\mathcal{K}_{0,1}.$$

3 Example

A concrete example (pure $SU(2)$ gauge theory): $\mathcal{B}$ is the complex plane, parameterized by u. D consists of two points $u = \pm\Lambda^2$, for some constant $\Lambda > 0$. $\mathcal{B}$ is the base of a family of elliptic curves Σ_u,

$$\Sigma_u = \{y^2 = (x^2 - u)^2 - \Lambda^4\} \subset \mathbb{C}^2_{x,y},$$

equipped with the one-form

$$\lambda = \frac{1}{\pi\sqrt{2}} \frac{x^2}{y} dx,$$

charge lattice $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$,

$$Z_\gamma(u) = \oint_\gamma \lambda.$$

(Cuts run from $\pm\sqrt{u - \Lambda^2}$ to $\pm\sqrt{u + \Lambda^2}$.)

Draw the wall of marginal stability.

WCF in this case is

$$\mathcal{K}_{2,-1}\mathcal{K}_{0,1} = \mathcal{K}_{0,1}\mathcal{K}_{2,1}\mathcal{K}_{4,1}\mathcal{K}_{6,1}\cdots\mathcal{K}_{2,0}^{-2}\cdots\mathcal{K}_{8,-1}\mathcal{K}_{6,-1}\mathcal{K}_{4,-1}\mathcal{K}_{2,-1}.$$

4 The construction

So now assume we have all this $\mathcal{N} = 2$ gauge theory data.

Define $\mathcal{M}' \to \mathcal{B}'$ to be the total space of the real torus fibration $\Gamma_g^* \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi\mathbb{Z})$. Any $\gamma \in \Gamma$ induces a coordinate function θ_γ on the fibers. (More precisely, twist the torus fibration by the set of quadratic refinements; let's overlook this.)

Complete $\mathcal{M}'$ by including singular fibers appropriately, to a C^∞ manifold (or sometimes orbifold) $\mathcal{M}$. We'll construct a family of hyperkähler metrics $(\mathcal{M}, g)$ depending on a parameter $R \in \mathbb{R}_+$.

Begin with an approximation to the desired metric (exact in the $R \to \infty$ limit). To describe it, use a complex structure on $\mathcal{M}'$ in which

$$\pi R dZ_\gamma + i d\theta_\gamma \tag{4.1}$$

is a $(1,0)$ form. Then write Kähler potential

$$K = \frac{iR}{16\pi} \langle Z, \bar{Z} \rangle. \tag{4.2}$$

(Translation invariant along the fibers.) The torus fibers are flat, there are isometries which shift the angles; call this metric "semi-flat", g^{sf}. It is actually hyperkähler. It lives only on $\mathcal{M}'$, no extension over D!

At large R, the exact metric we are after is close to this one:

$$g = g^{\mathrm{sf}} + \mathcal{O}\left(\sum_\gamma \Omega(\gamma; u) e^{-2\pi R |Z(\gamma; u)|} \right). \tag{4.3}$$

The corrections will smooth out the singularities over D.

5 Twistorial construction

How are we to describe the exact metric efficiently? Basic idea: $\mathcal{M}$ has three standard symplectic forms ω_i and complex structures J_i. In fact a $\mathbb{CP}^1$ worth; parameterize by ζ.

$$J^{(\zeta)} = \frac{i(-\zeta + \bar{\zeta})J_1 - (\zeta + \bar{\zeta})J_2 + (1 - |\zeta|^2)J_3}{1 + |\zeta|^2}.$$

For any ζ, $(\mathcal{M}, J^{(\zeta)})$ carries a holomorphic symplectic form,

$$\varpi = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-.$$

Morally, knowing the holomorphic symplectic form for all ζ is enough to reconstruct the hyperkähler metric. (A variant of *twistor space* construction.)

We'll construct ϖ by locally giving "holomorphic Darboux coordinates"

$$\mathcal{X}_\gamma : \mathcal{M} \times \mathbb{C}^\times \to \mathbb{C}^\times$$

where $\mathcal{X}_\gamma(\cdot, \zeta)$ is constant on $\mathcal{M}$ for each $\gamma \in \mathrm{rad}\langle , \rangle$ and

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'},$$

and then defining

$$\varpi = \frac{1}{4\pi^2 R} \langle d\log\mathcal{X}, d\log\mathcal{X}\rangle .$$

Then patch together at the end.

6 Semi-flat twistor construction

The metric g^{sf} arises from the choice

$$\mathcal{X}^{\mathrm{sf}}_\gamma(\zeta) := \exp\left[\pi R\zeta^{-1} Z(\gamma; u) + i\theta_\gamma + \pi R\zeta \bar{Z}(\gamma; u)\right].$$

Note the essential singularity at $\zeta = 0$. Characteristic of solution to a differential equation with irregular singularities.

7 Instanton corrections

Now how to get the corrections? Simplest example (incomplete): $\mathcal{B} = $ disc $|a| < |\Lambda|$, $D = \{a = 0\}$, monodromy

$$(\gamma_e, \gamma_m) \to (\gamma_e, \gamma_m + \gamma_e),$$

central charges

$$Z(\gamma_e) = a, \quad Z(\gamma_m) = \frac{1}{2\pi i}\left(a \log\frac{a}{\Lambda} - a\right).$$

Exact metric on $\mathcal{M}$ is known in this case: a "$U(1)$ bundle" over $\mathbb{R}^3$ with one degenerate fiber — defining $\vec{x}$ by

$$a = x^1 + ix^2, \quad \theta_e = 2\pi R x^3,$$

the metric is

$$g = V(\vec{x})^{-1}\left(\frac{d\theta_m}{2\pi} + A(\vec{x})\right)^2 + V(\vec{x})d\vec{x}^2,$$

where

$$V = V^{\mathrm{sf}} + V^{\mathrm{inst}},$$

with

$$V^{\mathrm{sf}} = -\frac{R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right),$$
$$V^{\mathrm{inst}} = \frac{R}{2\pi}\sum_{n\neq 0} e^{in\theta_e} K_0(2\pi R|na|),$$

and an appropriate A obeying $dA = \star dV$.

The instanton corrections V^{inst} resolve the singularity at $a = 0$; break translation invariance in θ_e; and are exponentially small at large R. They have a neat description in terms of $\mathcal{X}$:

$$\mathcal{X}_e = \mathcal{X}_e^{\text{sf}},$$
$$\mathcal{X}_m = \mathcal{X}_m^{\text{sf}} \exp\left[\frac{i}{4\pi}\int_{\ell_+} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')] - \frac{i}{4\pi}\int_{\ell_-} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')^{-1}]\right],$$

where we choose the contours $\ell_\pm$ as

$$\ell_\pm = \left\{\zeta : \pm\frac{a}{\zeta} \in \mathbb{R}_-\right\}.$$

Then calculate

$$\varpi = \frac{1}{4\pi^2 R}\frac{d\mathcal{X}_m^{\text{sf}}}{\mathcal{X}_m^{\text{sf}}} \wedge \frac{d\mathcal{X}_e^{\text{sf}}}{\mathcal{X}_e^{\text{sf}}}$$

directly: that shows we got the right $\mathcal{X}$. $\mathcal{X}_m$ is discontinuous along $\ell_\pm$:

$$(\mathcal{X}_m)^{cw}_{\ell_+} = (\mathcal{X}_m)^{ccw}_{\ell_+}(1-\mathcal{X}_e^q)^{-q},$$
$$(\mathcal{X}_m)^{cw}_{\ell_-} = (\mathcal{X}_m)^{ccw}_{\ell_-}(1-\mathcal{X}_e^{-q})^{q}.$$

The discontinuity is just the KS factor:

$$\mathcal{X}^{cw} = \mathcal{X}^{ccw}\mathcal{K}_{0,\pm 1}.$$

Because it's a symplectomorphism, it doesn't lead to a discontinuity in ϖ. Also, because of the placement of $\ell_\pm$, the discontinuity in $\mathcal{X}_m$ vanishes in the limit $\zeta \to 0$ or $\zeta \to \infty$.

8 Riemann-Hilbert problem

Proposal: for each fixed u, obtain the desired functions $\mathcal{X}_\gamma$ as the solution to an infinite-dimensional "Riemann-Hilbert problem."

We require:

- $\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$,
- Each $\mathcal{X}_\gamma^{-1}(-1/\bar{\zeta}) = \overline{\mathcal{X}}_\gamma(\zeta)$,
- Each $\mathcal{X}_\gamma$ is continuous on $\mathcal{M}$ and analytic in ζ, except that the collection $\mathcal{X} = \{\mathcal{X}_\gamma\}$ jumps by $\mathcal{K}_\gamma^{\Omega(\gamma;u)}$ along the ray

$$\ell_\gamma = \{(u,\zeta) : Z(\gamma;u)/\zeta \in \mathbb{R}_-\} \subset \mathcal{M} \times \mathbb{C}^\times$$

- Each $\mathcal{X}_\gamma (\mathcal{X}_\gamma^{\rm sf})^{-1}$ is finite as $\zeta \to 0, \infty$.

Theorem: if all these conditions hold, there exists an hyperkähler metric g such that $\mathcal{X}_\gamma(u,\theta;\zeta)$ is holomorphic on $(\mathcal{M}, J^{(\zeta)})$ for fixed ζ, and ϖ is the holomorphic symplectic form. Proof: construct the twistor space $\mathcal{Z}(\mathcal{M})$.

Moreover, argue that the desired $\mathcal{X}_\gamma$ indeed exist, for sufficiently large R, by formulating the solution at each fixed u in terms of an integral equation:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\rm sf}(\zeta) \exp\left[-\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma';u) \langle \gamma, \gamma' \rangle \times \right.$$
$$\left. \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')) \right]$$

This equation encodes the jump and asymptotic conditions at any fixed u. But the jump conditions change when we cross a wall in

$\mathcal{B}$. Consider the composite jump across some group of rays ℓ_γ which come together at a wall — see the $\mathcal{X}_\gamma$ are continuous in u only if the WCF is satisfied!

9 Comments about g

- Does the solution exist for small R? Physically, we expect yes (certainly $\mathcal{M}$ does). But no mathematical proof. It might only work when the $\Omega(\gamma; u)$ come from a physical theory!
- Expanding around large R, find the 1-instanton corrections in a form closely analogous to what we got in the abelian theory, but also higher multi-instanton corrections. Labeled by rooted trees. They're generally suppressed relative to the 1-instanton piece at large R, *except* at the wall. Here, e.g. for the simplest decay the discontinuity of the 2-instanton piece cancels the disappearing 1-instanton piece, and so on. No microscopic understanding yet.
- Behavior at the singular divisor D: one expects physically that the singularities in g^{sf} will be resolved or at least reduced in g. For the simplest kind of singularity, where a single $Z(\gamma; u) \to 0$ with $\Omega(\gamma; u) = 1$, this is basically what we showed above. Behavior at loci where components of D meet remains to be understood, probably interesting.
- This is only a small part of the full story that would be relevant for compact CY3: that story should involve quaternionic-Kähler metrics, not hyperkähler.

10 Higgs bundles

An important special case of our construction: take $\mathcal{M}$ to be the moduli space of solutions of Hitchin equations on a curve C (with ramification).

Consider an $SU(2)$-connection D on a complex rank 2 bundle V over C, and $\varphi \in \Omega^{1,0}(sl(V))$. Hitchin's equations:

$$\bar{\partial}_D \varphi = 0$$
$$R^2[\varphi, \varphi^*] = F_D$$

$\mathcal{M}$ is the space of solutions, modulo gauge equivalence.

To be precise, consider case where D and φ have simple poles at finitely many points z_i, with fixed residues.

Recall that $\mathcal{M}_{\zeta=0}$ is moduli space of Higgs bundles: pairs (E, φ) where

- E is a holomorphic rank 2 vector bundle over C, with fixed determinant,
- $\varphi \in H^0(\mathrm{End}\, E \otimes K)$.

Given φ define the *spectral curve* to be the double covering given by the 2 eigenvalues of φ:

$$\Sigma = (z, x) : \det(x - \varphi(z)) = 0 \subset T^*C$$

In our case $\det(x - \varphi(z)) = x^2 - u$ where u is a quadratic differential on C, with double poles at the z_i.

This moduli space will arise from our construction. What is the data?

- $\mathcal{B}$ is the space of holomorphic quadratic differentials u on C, with double poles at each z_i, fixed residues.

- Define $\Sigma_u = \{(z,x) : x^2 - u = 0\} \subset T^*C$. Then
$$\Gamma_u = \{\gamma \in H_1(\Sigma_u, \mathbb{Z}) : \sigma\gamma = -\gamma\}.$$
$\langle , \rangle$ is the intersection pairing.
- There is a canonical 1-form
$$\lambda = x dz$$
on Σ; the central charge map is
$$Z(\gamma) = \frac{1}{\pi} \oint_\gamma \lambda.$$
- Given u and an angle ϑ there is a real foliation on C: define the leaves ("WKB curves") to be ones for which $\lambda \in e^{i\vartheta}\mathbb{R}$. For generic ϑ all leaves meet a singular point in at least one direction. The integers $\Omega(\gamma; u)$ are counting how many leaves of finite extent appear as ϑ is varied. Two types. Saddle connections: define γ to be the difference of two lifts to Σ, then these contribute $\Omega(\gamma; u) = +1$. Closed loops: define γ to be the difference of two lifts to Σ, then these contribute $\Omega(\gamma; u) = -2$.

Can also consider irregular singularities. The simple example we started with corresponds to 2 irregular singularities of the weakest possible kind.

11 Darboux coordinates

Now to see how our description of the metric goes, need to construct the $\mathcal{X}_\gamma$. What does $\mathcal{M}_\zeta$ look like for $\zeta \in \mathbb{C}^\times$? Consider the (complex) connection:

$$\nabla_z = R\zeta^{-1}\varphi_z + D_z$$
$$\nabla_{\bar{z}} = R\zeta\bar{\varphi}_{\bar{z}} + D_{\bar{z}}$$

Hitchin's equations imply ∇ is flat. This identifies $\mathcal{M}_\zeta$ with a moduli space of flat connections.

So holomorphic functions on $\mathcal{M}_\zeta$ are gauge-invariant quantities built out of ∇. $\mathcal{X}_\gamma$ on $\mathcal{M}$ will be obtained this way.

Construct *canonical triangulation* $T(\vartheta, u)$ from the WKB foliation. One branch point in each face. Then to any quadrilateral associate a homology cycle:

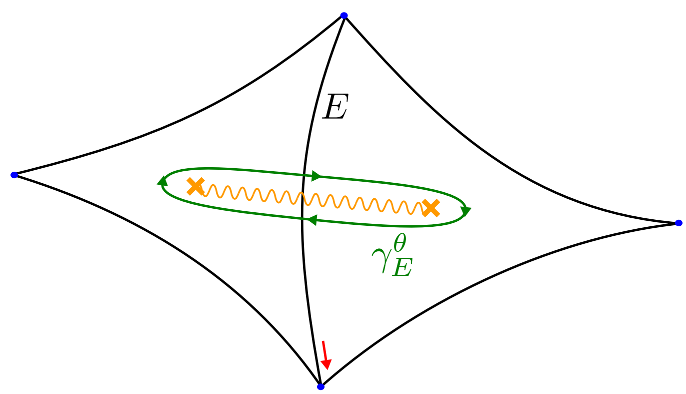

Figure 1: The construction of $\gamma_E^\vartheta \in H_1(\Sigma, \mathbb{Z})$.

Fock-Goncharov defined some coordinates on moduli spaces of flat connections, depending on:

- A triangulation of C. Take $T(\vartheta = \arg \zeta, u)$.
- A monodromy eigenvector at each singularity. Take the one which is exponentially smaller in norm along the edges.

Fock-Goncharov define the coordinate by

$$\mathcal{X}_\gamma := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}.$$

Now want to check $\mathcal{X}_\gamma$ have the properties we claimed.

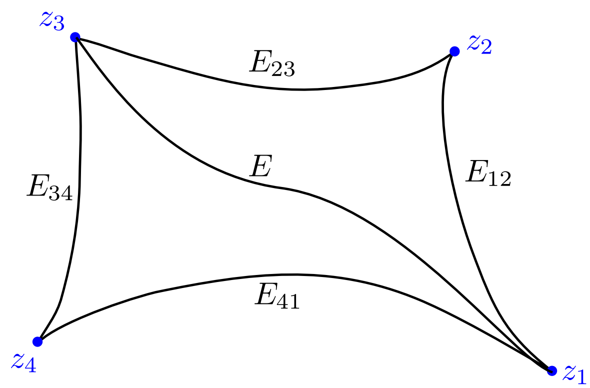

Figure 2: The quadrilateral Q_E associated to an edge E of T.

12 Jumps

As ϑ varies the foliation jumps in a specific way. The simplest jump comes from a saddle connection and gives a flip of the triangulation. Fock-Goncharov already studied the transformation of the triangulation under flips: it gives exactly the jump $\mathcal{K}_{\gamma_{BPS}}$ we needed for the WCF with $\Omega(\gamma) = 1$.

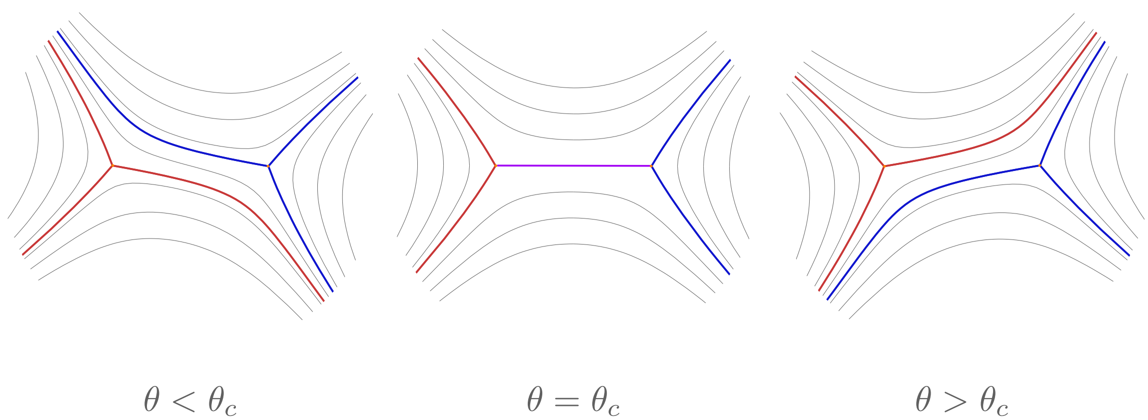

Figure 3:

There is also a more intricate jump coming from the appearance of a family of closed curves. This one was not studied by Fock-Goncharov. It gives a jump of the $\mathcal{X}_\gamma$ by $\mathcal{K}^{-2}_{\gamma_{BPS}}$, exactly as we needed with $\Omega(\gamma) = -2$.

Show the animations.

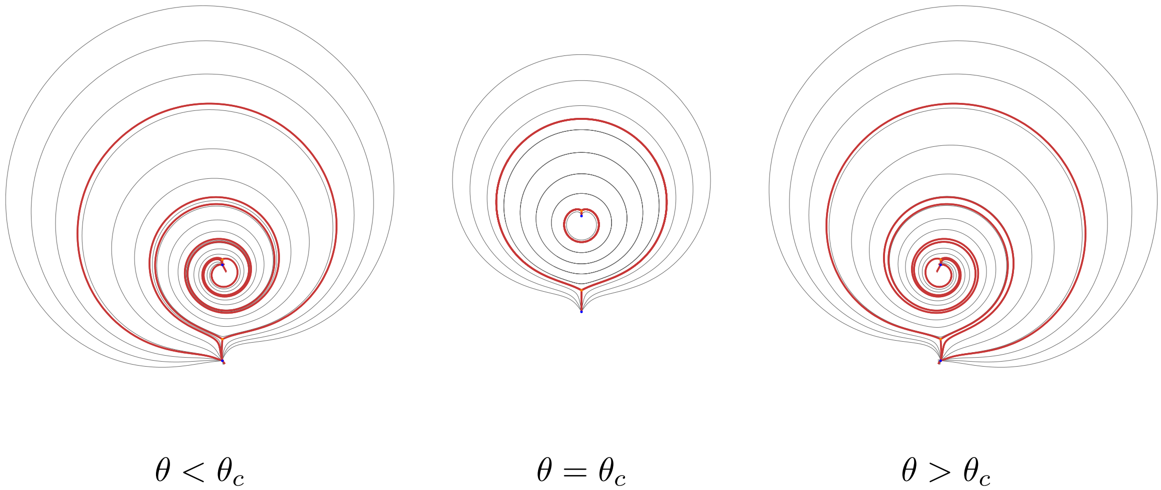

Figure 4:

13 Asymptotics

WKB approximation verifies the asymptotics of the coordinates $\mathcal{X}_\gamma(\zeta)$ as $\zeta \to 0$. Namely, in this approximation evaluating the parallel transport just reduces to integrating the Higgs piece φ/ζ. That's not exact, but you can neglect the error, because the contamination is exponentially smaller than the piece you are keeping.

One obtains as $\zeta \to 0$

$$\mathcal{X}_\gamma \sim c_\gamma \exp\left[\frac{1}{\pi}\oint_\gamma \lambda\right].$$

This is what we needed in order to "cap off" the twistor space.