Exam 1 Practice Problem Answers

① a) $A = \begin{bmatrix} 5 & -3 & 8 \\ 1 & 1 & 0 \\ 6 & -2 & 8 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b) To solve $A\vec{x} = \vec{b}$: $\left[\begin{array}{ccc|c} 5 & -3 & 8 & 4 \\ 1 & 1 & 0 & 0 \\ 6 & -2 & 8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 5 & -3 & 8 & 4 \\ 6 & -2 & 8 & 4 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -8 & 8 & 4 \\ 0 & -8 & 8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$

So: $x_1 = \frac{1}{2} - x_3$ $\quad$ ie $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$x_2 = -\frac{1}{2} + x_3$

$$\boxed{\vec{x} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

c) Same reduction for $A\vec{x} = \vec{0}$ would give $\boxed{\vec{x} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$

d) Both S_1 and S_2 are lines. They are parallel to one another, related by translation by the vector

$$\vec{p} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}.$$

(2) a) $AB = \begin{bmatrix} 1 & 3 \\ 4 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & 12 & 3 \\ 0 & -8 & 12 \\ 2 & 4 & 0 \end{bmatrix}}$

$BC = \boxed{\text{not defined}}$

$AC = \begin{bmatrix} 1 & 3 \\ 4 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 13 & 2 \\ 10 & 8 \\ 3 & 0 \end{bmatrix}}$

b) Cols. of A <u>are</u> lin indep (2 vectors that aren't proportional)
Cols of B <u>are not</u> lin indep (3 vectors in $\mathbb{R}^2$, $3>2$)

c) Cols. of A <u>do not</u> span $\mathbb{R}^3$ (2 vectors in $\mathbb{R}^3$, $2<3$)
Cols. of C <u>do</u> span $\mathbb{R}^2$ (2 pivots)

(3) a) RREF of $A = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$

b) $\boxed{\text{Yes}}$, A is invertible (it has 5 pivots)

c) $A\vec{x} = \vec{b}$ is equivalent to $\vec{x} = A^{-1}\vec{b}$
so $\boxed{\text{Yes}}$, it has a $\boxed{\text{unique solution}}$

④ a) $\boxed{\text{Yes}}$, $A^{-1} = \frac{1}{12-6}\begin{bmatrix} 4 & -6 \\ -1 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix}}$

b) $\boxed{\text{No}}$, B has only 2 pivots

c) $\boxed{\text{No}}$, BC is not invertible (because B is not invertible)

⑤ a) $\boxed{\text{Yes}}$ (a set of 1 nonzero vector is always lin indep)

b) $\boxed{\text{Yes}}$ ($\vec{v}_1$ and $\vec{v}_2$ are not proportional)

c) $\boxed{\text{No}}$ (5 vectors in $\mathbb{R}^3$, $5>3$)

d) $\vec{v}_3 - \vec{v}_2 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$ $\vec{v}_2 - \vec{v}_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$ so $\vec{v}_3 - \vec{v}_2 = \vec{v}_2 - \vec{v}_1$

so $\boxed{\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}, \text{ not lin indep}}$

e) $\boxed{\text{No}}$ $\left(2(\vec{v}_5 - \vec{v}_4) = \vec{v}_4 - \vec{v}_2 \text{ so } \vec{v}_2 - 3\vec{v}_4 + 2\vec{v}_5 = \vec{0}\right)$

⑥ a) $\vec{x}_1 = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\vec{x}_0 = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\vec{x}_1 = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ 6 \end{bmatrix}}$

b) $\boxed{\vec{x}_{100} = A^{100}\vec{x}_0}$

c) $A^{-1} = \frac{1}{8-7}\begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$

$\vec{x}_4 = A\vec{x}_3$

so $\vec{x}_3 = A^{-1}\vec{x}_4$

$$= \boxed{\begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \vec{x}_4 \quad (\text{yes})}$$

7 a) $\boxed{\text{False}}$ $\left((AB)^{-1} = B^{-1}A^{-1}\right)$

b) $\boxed{\text{False}}$ (e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$ is consistent for $\vec{b} = \vec{0}$ but not for any other $\vec{b}$)

c) $\boxed{\text{True}}$ (always has the trivial solution)

d) $\boxed{\text{True}}$ (exactly one sol^n if no free variables, otherwise infinitely many sol^n's)

e) $\boxed{\text{False}}$ (e.g. $\begin{matrix} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{matrix}$)

f) $\boxed{\text{True}}$ (if $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ then also $x_1 T(\vec{v}_1) + \cdots + x_n T(\vec{v}_n) = \vec{0}$)

g) $\boxed{\text{False}}$ (e.g. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$: $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ lin indep but $\left\{T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right\} = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ isn't)

h) $\boxed{\text{True}}$ (if it has nontriv. sol. then it has a free variable)

i) $\boxed{\text{True}}$ (proved in lecture)

j) $\boxed{\text{True}}$ ($-4(c\vec{x}) = c(-4\vec{x})$, and $-4(\vec{x}+\vec{y}) = -4\vec{x} + -4\vec{y}$)

k) $\boxed{\text{False}}$ $(T(\vec{0}) \neq \vec{0})$

l) $\boxed{\text{True}}$ (there's a pivot in every row)

m) $\boxed{\text{False}}$ (there's a pivot in every column)