M340L Fall 2010: Exam 1 Practice Problems

Problem 1. Consider the system of 3 linear equations in 3 variables:

$$\begin{aligned} 5x_1 - 3x_2 + 8x_3 &= 4 \\ x_1 + x_2 &= 0 \\ 6x_1 - 2x_2 + 8x_3 &= 4 \end{aligned}$$

a) Express this system as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$. What are A and $\mathbf{b}$? How is $\mathbf{x}$ related to the variables x_1, x_2, x_3?

b) What is the solution set S_1 of this system? Describe it in parametric form. (Recall that "parametric form" means giving a formula for the solutions such as $\mathbf{x} = t\mathbf{v} + \mathbf{p}$ where t is arbitrary, or $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \mathbf{p}$ where t_1 and t_2 are arbitrary.)

c) What is the solution set S_2 of the system $A\mathbf{x} = \mathbf{0}$? Describe it in parametric form.

d) Describe S_1 and S_2 geometrically as subsets of $\mathbb{R}^3$: is each one an empty set, a point, a line, a plane, or a three-dimensional space? How are they related geometrically to one another?

Problem 2. Consider the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -2 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix}.$$

a) For each of the products AB, BC, and AC, either calculate the product or write "not defined" if it is not defined.

b) Are the columns of A linearly independent? Are the columns of B linearly independent?

c) Do the columns of A span $\mathbb{R}^3$? Do the columns of C span $\mathbb{R}^2$?

Problem 3. Consider the matrix and vector

$$A = \begin{bmatrix} 4 & 7 & 12 & -3 & 6 \\ 0 & 3 & 11 & -1 & 7 \\ 0 & 0 & 2 & 4 & -9 \\ 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 7 \\ 4 \end{bmatrix}.$$

a) What is the reduced row echelon form of A?

b) Is A invertible?

c) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for $\mathbf{x}$? If so, is the solution unique?

Problem 4. Consider the matrices

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 6 \\ 0 & 0 & 14 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 2 \end{bmatrix}.$$

a) Is A invertible? If it is, find its inverse.

b) Is B invertible? If it is, find its inverse.
c) Is BC invertible?

Problem 5. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 13 \\ 14 \\ 15 \\ 16 \end{bmatrix}, \qquad \mathbf{v}_5 = \begin{bmatrix} 17 \\ 18 \\ 19 \\ 20 \end{bmatrix}.$$

a) Is the set $\{\mathbf{v}_1\}$ linearly independent?
b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent?
c) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent?
d) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? If not, write a nontrivial solution to the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$. (Timesaving hint: Look at $\mathbf{v}_3 - \mathbf{v}_2$ and $\mathbf{v}_2 - \mathbf{v}_1$.)
e) Is the set $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent? (Timesaving hint: Look at $\mathbf{v}_5 - \mathbf{v}_4$ and $\mathbf{v}_4 - \mathbf{v}_2$.)

Problem 6. Consider the discrete dynamical system

$$\mathbf{x}_{n+1} = A\mathbf{x}_n$$

where each $\mathbf{x}_n \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}.$$

a) If we know that $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is $\mathbf{x}_2$?
b) Write a formula for $\mathbf{x}_{100}$ in terms of $\mathbf{x}_0$ and A.
c) Is it possible to determine $\mathbf{x}_3$ uniquely given $\mathbf{x}_4$? If so, write a formula for $\mathbf{x}_3$ in terms of $\mathbf{x}_4$.

Problem 7. True or False. If a statement is sometimes true and sometimes false, write "false". You do not have to justify your answers. There will be no partial credit.

a) If two matrices A and B are both invertible then AB is also invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.
b) If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then it is consistent for every $\mathbf{b}$.
c) A homogeneous linear system is always consistent.
d) A consistent linear system either has exactly one solution or infinitely many.
e) If a linear system has more variables than equations, then it is consistent.
f) If T is a linear transformation, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is also linearly dependent.
g) If T is a linear transformation, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is also linearly independent.
h) If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then it has infinitely many nontrivial solutions.
i) If the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is not 1-1, then the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

j) The transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by $T(\mathbf{x}) = -4\mathbf{x}$ is a linear transformation.

k) The transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3 \\ 3x_2 \end{bmatrix}$ is a linear transformation.

l) If a 4×6 matrix (4 rows, 6 columns) has 4 pivots, then its columns span $\mathbb{R}^4$.

m) If a 5×2 matrix (5 rows, 2 columns) has 2 pivots, then its columns are linearly dependent.