M340L Second Midterm Exam, April 8, 2010

1. The matrices $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 12 \\ 1 & 0 & -2 & -1 & -3 \\ 0 & 2 & 6 & 5 & 37 \\ 2 & 1 & -1 & 2 & 23 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ are row-equivalent.

a) Find a basis for $Col(A)$. What is $dim(Col(A))$?

Since there are pivots in the 1st, 2nd and 4th columns of $A_{rref} = B$, our basis is the first, second and fourth columns of A, namely $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 5 \\ 2 \end{pmatrix} \right\}$, and the dimension is 3.

b) Find a basis for $Nul(A)$. What is $dim(Nul(A))$?

From B we see that our equations reduce to

$$\begin{aligned} x_1 &= 2x_3 - 4x_5 \\ x_2 &= -3x_3 - x_5 \\ x_3 &= x_3 \\ x_4 &= - 7x_5 \\ x_5 &= x_5, \end{aligned}$$

so our basis is $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -1 \\ 0 \\ -7 \\ 1 \end{pmatrix} \right\}$, and our space is 2 dimensional.

c) Find a basis for $Row(A)$. What is $dim(Row(A))$?

The nonzero rows of B, namely $\{(1, 0, -2, 0, 4), (0, 1, 3, 0, 1), (0, 0, 0, 1, 7)\}$ form a basis. The dimension is 3.

d) $M_{2,2}$ is the space of 2×2 matrices. Let V be the subspace of $M_{2,2}$ spanned by $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 6 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} 12 & -3 \\ 37 & 23 \end{pmatrix} \right\}$. Find a basis for V.

This is almost the same problem as (a). If you rewrite everything in terms of the standard basis for $M_{2,2}$, the coordinates of our five vectors are just the columns of A, of which only the first, second and fourth are linearly independent. So our basis is $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix} \right\}$. Note, however, that the basis elements are 2×2 matrices, (that is, elements of V), not columns. The answer to (a) gives the *coordinates* of the answer to (d).

2. On R^3, let $\mathcal{E}$ be the standard basis and let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$.

Let $\mathbf{v} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

a) Compute the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$

$$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \text{ while } P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 5 & -3 & -1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}.$$

b) Compute $[\mathbf{v}]_{\mathcal{B}}$.

$$[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 5 & -3 & -1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 28 \\ -7 \\ -9 \end{pmatrix}.$$

c) In P_2, let $\mathcal{C} = \{1+t+t^2, 2+3t+t^2, 1+t+2t^2\}$, and let $\mathbf{w} = 5-2t+3t^2$. Find $[\mathbf{w}]_{\mathcal{C}}$. (Justify your answer!)

Let $\mathcal{E}' = \{1, t, t^2\}$ be the standard basis for P_2. Note that $P_{\mathcal{E}'\mathcal{C}}$ is the same as the matrix $P_{\mathcal{E}\mathcal{B}}$ that you computed above, and that $[\mathbf{w}]_{\mathcal{E}'} = [\mathbf{v}]_{\mathcal{E}}$.

$$[\mathbf{w}]_{\mathcal{C}} = P_{\mathcal{C}\mathcal{E}'}[\mathbf{v}]_{\mathcal{E}'} = \begin{pmatrix} 5 & -3 & -1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 28 \\ -7 \\ -9 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 6 & 5 \\ -5 & 0 \end{pmatrix}$ and let $B = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}$.

a) Find the characteristic equation of A.

$0 = det \begin{pmatrix} 6-\lambda & 5 \\ -5 & -\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 25$. Note that $\lambda^2 - 6\lambda + 25$ by itself is not a correct answer. $\lambda^2 - 6\lambda + 25$ is the characteristic *polynomial.* The characteristic *equation* is $\lambda^2 - 6\lambda + 25 = 0$.

(b) Find the eigenvalues of A (you do not need to find the eigenvectors).

Solving this equation gives $\lambda = 3 \pm 4i$.

c) The eigenvalues of B are 1, 2 and 3. Find the corresponding eigenvectors.

(Note: you may get some *simple* fractions in your calculations, but if you get any truly ugly denominators, you've made a mistake.)

These are obtained by row-reducing $B - I$, $B - 2I$, and $B - 3I$, to get vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, respectively, or possibly multiples of these vectors.

4. a) Find a 2×2 matrix with eigenvalues 1 and 3, and with corresponding eigenvectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

$$A = PDP^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} -17 & 12 \\ -30 & 21 \end{pmatrix}.$$

A number of people set up the equations $A \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $A \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and tried to solve the resulting system of equations for the elements of A. That isn't wrong, but it sure is doing this problem the hard way!

(b) Is $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$ diagonalizable? Why or why not?

No. The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, which has only one root: $\lambda = 1$. However, $A - I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ has rank 1, not zero, so the eigenvalue 1 has *geometric* multiplicity $2 - 1 = 1$, even as it has *algebraic* multiplicity 2.

A lot of people wrote that it's not diagonalizable because it only has one eigenvalue. That's not enough, as it's possible for an eigenvalue to have geometric multiplicity two! It just doesn't happen in this case. (E.g., the identity matrix also has only one eigenvalue, but it's diagonalizable.)

Finally, a number of people computed the determinant, saw that it was nonzero, and said that proves the matrix is diagonalizable. $det \neq 0$ implies that a matrix is *invertible*, which doesn't say anything, one way or the other, about whether it's diagonalizable.

5. True/false. Just mark each statement with a T (or TRUE) or an F (or FALSE). You do not need to justify your answers, and partial credit will not be given.

a) If a square matrix has determinant zero, then its null space is at least 1-dimensional.

True. There is at least one free variable.

b) The plane $x_1 + 2x_2 + 3x_3 = 6$ is a subspace of R^3.

False. The plane does not contain the origin. (It also isn't closed under addition or scalar multiplication.)

c) If an $m \times n$ matrix has rank k, then its null space has dimension $m - k$.

False. The dimension is $n - k$.

d) If A is a 4×7 matrix, then the dimension of $Col(A)$ equals the dimension of $Row(A)$.

True. Being 4×7 is irrelevant. The dimension of the column space and row space are always equal to the rank (and to each other).

e) If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are bases for a vector space V, then $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{C}\mathcal{D}} P_{\mathcal{B}\mathcal{C}}$.

False. The correct formula is $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{B}\mathcal{C}} P_{\mathcal{C}\mathcal{D}}$.

f) The geometric multiplicity of an eigenvalue is at least as big as the algebraic multiplicity of that eigenvalue.

False. The geometric multiplicity is at *most* the algebraic multiplicity.

g) If the characteristic equation of a square matrix A is $(\lambda - 1)^3(\lambda + 2) = 0$, then $\lambda = 1$ is an eigenvalue with algebraic multiplicity 3.

True. It's a triple root.

h) If the characteristic equation of a real matrix A has complex roots, then there is no basis of R^n consisting of eigenvectors of A.

True. There may be a basis for C^n, but not for R^n, since some of the eigenvectors will be complex.

i) If $\mathcal{B}$ is a basis consisting of eigenvectors of A, then $[A]_{\mathcal{B}}$ is diagonal.

True. That's why finding eigenvalues and eigenvectors is called *diagonalization.*

j) If A is a 5×5 matrix with eigenvalues 11, 25, 32 and 47, and if the geometric multiplicity of $\lambda = 11$ is 2, then A is diagonalizable.

True. We can find 5 linearly independent eigenvectors, 2 for 11, one for 25 one for 32 and one for 47.