1) H is sol^n set of $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$x_1 = -2x_4$
$x_2 = -x_3$
x_3, x_4 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

So a) $\dim H = 2$

b) $\left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for H

2)

$$\begin{bmatrix} 3 & 3 & 0 & 6 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a) Rows in RREF containing pivots: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

b) $x_1 = -x_2$
$x_3 = 0$
$x_4 = 0$
x_2, x_5 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$$

so basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) Columns in A containing pivots: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} \right\}$

③ a) $\dim \operatorname{Ker} T + \dim \operatorname{Ran} T = \dim W$

$$0 + \dim \operatorname{Ran} T = 3$$

so $\dim \operatorname{Ran} T = 3$

b) Since $\dim \operatorname{Ran} T = 3$ but $\dim V = 4$ it is impossible for $\operatorname{Ran} T$ to be V

④ a) $$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} = (6-\lambda)(2-\lambda) + 4$$

$$= 12 - 8\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 8\lambda + 16$$

$$= (\lambda - 4)^2$$

the only eigenvalue of A is 4

b) $\lambda = 4$: $(A - 4I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = \frac{1}{2} x_2 \\ x_2 \text{ free} \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Picking say $x_2 = 2$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with $\lambda = 4$

And the $\lambda = 4$ eigenspace is $\operatorname{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}$

c) A has only a single 1-dim eigenspace, $1 < 2$

so A is not diagonalizable

(5) a)
$$\begin{vmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -(1 \cdot 1 \cdot 4 \cdot 2) = -8$$

b) Since $\det A \neq 0$, A is invertible so $5A$ is also invertible: rank $5A = 4$

c) Since $\det A \neq 0$, A is invertible so A^3 is also invertible: rank $A^3 = 4$

d) Since $\det A \neq 0$, A is invertible so $\operatorname{Ran} T = \operatorname{Col} A = \mathbb{R}^4$

(6) a) $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ could have dim 0, 1, or 2

(0 if $\vec{v}_1 = \vec{0}$ and $\vec{v}_2 = \vec{0}$; 1 if $\vec{v}_1$ is multiple of $\vec{v}_2 \neq \vec{0}$; 2 if $\vec{v}_1, \vec{v}_2$ lin indep)

b) $\vec{v}_1 - \vec{v}_2 = -(\vec{v}_1 + \vec{v}_2) + 2\vec{v}_1$ so $\vec{v}_1 - \vec{v}_2$ is a lin comb of $\vec{v}_1$ and $\vec{v}_1 - \vec{v}_2$

so $\operatorname{Span}\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$

which could have dim 0, 1, or 2 (just like previous part)

(7) a)
$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

b) $T(p) = \vec{0}$ means $p(0) = 0$, $p(1) = 0$, $p(2) = 0$

so $p(t) = t(t-1)(t-2)(at^3 + bt^2 + ct + d)$

$= a \cdot t^4(t-1)(t-2) + b \cdot t^3(t-1)(t-2) + c \cdot t^2(t-1)(t-2) + d \cdot t(t-1)(t-2)$

so $\{t^4(t-1)(t-2),\ t^3(t-1)(t-2),\ t^2(t-1)(t-2),\ t(t-1)(t-2)\}$

is a basis for $\operatorname{Ker} T$

c) $\dim \operatorname{Ker} T + \dim \operatorname{Ran} T = \dim W$

$4 + \dim \operatorname{Ran} T = 7$

$\dim \operatorname{Ran} T = 3$

8

a) FALSE (it can only be between 0 and 4)

b) TRUE

c) FALSE (can also be a point or the whole $\mathbb{R}^3$)

d) TRUE ($A\vec{v} = \lambda\vec{v}$ $B\vec{v} = \mu\vec{v}$ $\Rightarrow$ $(A+B)\vec{v} = (\lambda+\mu)\vec{v}$)

e) FALSE

f) FALSE

g) TRUE

h) FALSE

i) FALSE

j) TRUE

$\left.\begin{array}{l}\text{k) TRUE}\\ \text{l) TRUE}\end{array}\right\}$ (these two are really the <u>same</u> question)