Problem 1.

Let H be the set of all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ obeying the equations

$$\begin{aligned} x_1 + x_2 + x_3 + 2x_4 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

H is a subspace of $\mathbb{R}^4$.

(a) What is the dimension of H?

(b) Find a basis for H.

Problem 2.

Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & 0 & 6 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \end{bmatrix}.$$

(a) Find a basis for $\operatorname{Row} A$.

(b) Find a basis for $\operatorname{Nul} A$.

(c) Find a basis for $\operatorname{Col} A$.

Problem 3.

Let V be a vector space of dimension 4. Let W be a vector space of dimension 3. Consider a linear transformation T from W to V, with $\dim \operatorname{Ker} T = 0$.

(a) What is the dimension of $\operatorname{Ran} T$?

(b) Can $\operatorname{Ran} T$ be the whole V? If so, give an example. If not, why not?

Problem 4.

Consider the matrix

$$A = \begin{bmatrix} 6 & -1 \\ 4 & 2 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

(b) For each eigenvalue, find a corresponding eigenvector, and also describe the corresponding eigenspace.

(c) Is there a matrix P such that $PAP^{-1} = D$ with D diagonal? If so, write P and D.

Problem 5.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

(a) Calculate $\det A$.

(b) What is the rank of $5A$?

(c) What is the rank of A^3?

(d) What is the range of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$?

Problem 6.

Suppose V is a vector space with $\dim V = 3$. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are two vectors in V.

(a) What are the possibilities for the dimension of $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

(b) What are the possibilities for the dimension of $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$?

Problem 7.

Consider the two vector spaces

$$V = \mathbb{R}^4$$

$$W = \mathbb{P}_6 = \{\text{polynomials } p(t) \text{ with real coefficients, of degree less than or equal to } 6\}.$$

Let T be the linear transformation from W to V given by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ 0 \end{bmatrix}.$$

(a) If $p(t) = t^2 - 3t + 1$, compute $T(p)$.

(b) Find a basis for $\operatorname{Ker} T$. What is the dimension of $\operatorname{Ker} T$?

(c) What is the dimension of $\operatorname{Ran} T$?

Problem 8.

True or False. If a statement is sometimes true and sometimes false, write “false”. You do not have to justify your answers.

(a) The row space of a 4×7 matrix can have any dimension between 0 and 7.

(b) If T is a linear transformation from a vector space V to a vector space W, then $T(\mathbf{0}) = \mathbf{0}$.

(c) Any subspace of $\mathbb{R}^3$ is geometrically a line or a plane.

(d) If $\mathbf{v}$ is an eigenvector of A and $\mathbf{v}$ is an eigenvector of B, then $\mathbf{v}$ is an eigenvector of $A + B$.

(e) If $\mathbf{v}$ is an eigenvector of A and $\mathbf{w}$ is an eigenvector of A, then $\mathbf{v}+\mathbf{w}$ is an eigenvector of A.

(f) If H_1 and H_2 are both subspaces of V, then their union is also a subspace of V.

(g) If H is a subspace of V, $\mathbf{h}$ is a vector in H, and $\mathbf{v}$ is a vector in V, then $\mathbf{h} + \mathbf{v}$ is a vector in V.

(h) If A is an invertible matrix, then 0 is an eigenvalue of A.

(i) If V is a vector space and $\mathcal{B}$ is a basis for V, then any subset of $\mathcal{B}$ is also a basis for V.

(j) Any 2-dimensional vector space contains infinitely many vectors.

(k) If A is a 4×8 matrix with rank 3, then $\dim \operatorname{Nul} A = 5$.

(l) If T is a linear transformation from an 8-dimensional space to a 4-dimensional space, and $\dim \operatorname{Ran} T = 3$, then $\dim \ker T = 5$.