① a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left.\begin{array}{l} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 \text{ free} \end{array}\right] \Longrightarrow \left.\begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 \text{ free} \end{array}\right] \Longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

So the solution set is $\vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

b) Yes, it is $\text{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$; it has basis $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ which has <u>dimension 1</u>

c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 4 & 5 & 6 & 6 \\ 7 & 8 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -3 & -6 & -6 \\ 0 & -6 & -12 & -12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left.\begin{array}{l} x_1 - x_3 = -1 \\ x_2 + 2x_3 = 2 \\ x_3 \text{ free} \end{array}\right] \Longrightarrow \left.\begin{array}{l} x_1 = -1 + x_3 \\ x_2 = 2 - 2x_3 \\ x_3 \text{ free} \end{array}\right] \Longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

So the solution set is $\vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

d) No, it is <u>not</u> a subspace: doesn't contain $\vec{0}$ $\quad (A\cdot\vec{0} \neq \vec{b})$

② a) Yes, Ran T is a subspace of W.

Its dimension is the rank of the standard matrix for T:

since that's a 7×3 matrix, its rank could be 0, 1, 2 or 3.

b) Yes, Nul T is a subspace of V.

Since $\dim \text{Ran } T + \dim \text{Nul } T = 3$ and $\dim \text{Ran } T$ could be 0, 1, 2, or 3, dim Nul T also could be 0, 1, 2 or 3.

③ a) Since $\vec{v}_2 = 3\vec{v}_1$, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not linearly independent.

(Could also use row reduction to see this.)

Therefore it is <u>not</u> a basis for $\mathbb{R}^3$.

b) Since $\vec{v}_2 = 3\vec{v}_1$ we can <u>throw away</u> $\vec{v}_2$:

$$S = \text{Span}\{\vec{v}_1, \vec{v}_3\}$$

and $\{\vec{v}_1, \vec{v}_3\}$ is lin. indep. (neither is a multiple of the other)

so $\{\vec{v}_1, \vec{v}_3\}$ is a <u>basis</u> for S.

Thus dim $S = 2$

c) Use Gram-Schmidt on the basis $\{\vec{y}_1, \vec{y}_2\}$ $\vec{y}_1 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$ $\vec{y}_2 = \begin{bmatrix} -1 \\ 7 \\ -10 \end{bmatrix}$

$$\vec{w}_1 = \vec{y}_1 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$$

$$\vec{w}_2 = \vec{y}_2 - \frac{\vec{y}_1 \cdot \vec{y}_2}{\|y_1\|^2} \vec{y}_1 = \begin{bmatrix} -1 \\ 7 \\ -10 \end{bmatrix} - \frac{70}{35}\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -10 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

so $\left\{\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right\}$ is an orthogonal basis

d) $$\vec{y} = \begin{bmatrix} -10 \\ 10 \\ 4 \end{bmatrix}$$

$$\text{proj}_S \vec{y} = \frac{\vec{y} \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \vec{w}_1 + \frac{\vec{y} \cdot \vec{w}_2}{\|\vec{w}_2\|^2} \vec{w}_2 = 0\vec{w}_1 + \frac{40}{10}\vec{w}_2 = 4\vec{w}_2$$

$$= \begin{bmatrix} -12 \\ 4 \\ 0 \end{bmatrix}$$

4 a) $\det(A-\lambda I) = \begin{vmatrix} -5-\lambda & -6 & 0 \\ 9/2 & 7-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -5-\lambda & -6 \\ 9/2 & 7-\lambda \end{vmatrix}$

$= (4-\lambda)\left[(-5-\lambda)(7-\lambda) - (-6)(9/2)\right] = (4-\lambda)(\lambda^2-2\lambda-8)$

$= (4-\lambda)(\lambda+2)(\lambda-4)$

So the eigenvalues are $\lambda=4$ (multiplicity 2), $\lambda=-2$

b)

$A-4I = \begin{bmatrix} -9 & -6 & 0 \\ 9/2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(A-4I)\vec{x}=\vec{0} \Rightarrow \left.\begin{array}{c} x_1+\frac{2}{3}x_2=0 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{array}\right] \Rightarrow \begin{array}{c} x_1 = -\frac{2}{3}x_2 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$

So $\vec{x} = x_2\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda=4$ eigenspace is $\text{Span}\left\{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$A+2I = \begin{bmatrix} -3 & -6 & 0 \\ 9/2 & 9 & 0 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} -3 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$(A+2I)\vec{x}=\vec{0} \Rightarrow \begin{array}{c} x_1+2x_2=0 \\ x_3=0 \\ x_2 \text{ free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}$

$\lambda=-2$ eigenspace is $\text{Span}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}$

c) Yes: $P=\begin{bmatrix}-2&-2&0\\1&3&0\\0&0&1\end{bmatrix}$ $\quad D=\begin{bmatrix}-2&0&0\\0&4&0\\0&0&4\end{bmatrix}$

(5) a) $\det(A-\lambda I)=\begin{vmatrix}1-\lambda&3\\3&1-\lambda\end{vmatrix}=(1-\lambda)^2-9$

$$=(\lambda^2-2\lambda+1)-9$$
$$=\lambda^2-2\lambda-8$$
$$=(\lambda-4)(\lambda+2)$$

Eigenvalues of A: $\lambda=4, \lambda=-2$

b) $\lambda=4$: $\quad A-4I=\begin{bmatrix}-3&3\\3&-3\end{bmatrix}\sim\begin{bmatrix}1&-1\\0&0\end{bmatrix}$

$$(A-4I)\vec{x}=\vec{0} \Rightarrow x_1-x_2=0 \Rightarrow x_1=x_2$$

eigenspace is $\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

$\lambda=-2$: $\quad A+2I=\begin{bmatrix}3&3\\3&3\end{bmatrix}\sim\begin{bmatrix}1&1\\0&0\end{bmatrix}$

$$(A+2I)\vec{x}=\vec{0} \Rightarrow x_1+x_2=0 \Rightarrow x_1=-x_2$$

eigenspace is $\text{Span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

So the desired <u>orthonormal</u> basis is $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

c) $Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}^T A \vec{x}$ $\qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

A has one positive and one negative eigenvalue so $\vec{0}$ is a saddle point

d) The maximum comes from the biggest eigenvector, here $\lambda = 4$:

$\begin{pmatrix} x \\ y \end{pmatrix} = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e. $x = 1/\sqrt{2},\ y = 1/\sqrt{2}$ or $x = -1/\sqrt{2},\ y = -1/\sqrt{2}$

(either answer would be OK, or both)

The minimum comes from the smallest eigenvector, here $\lambda = -2$:

$\begin{bmatrix} x \\ y \end{bmatrix} = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ i.e. $x = \frac{1}{\sqrt{2}},\ y = -\frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$

(either answer would be OK, or both)

6 a) Since the eigenvalues have $|\lambda| = \sqrt{5} > 1$, the origin is repelling

b) The $\vec{x}_n$ spiral out from the origin since the eigenvalues are complex. So, no.

c) The eigenvalues of A^2 are $\lambda = (2 \pm i)^2 = 3 \pm 4i$;

$$|\lambda| = \sqrt{20} > 1$$

so again, the origin is repelling

7 a) $T(a \sin t + b \cos t) = \begin{bmatrix} a \sin(0) + b \cos(0) \\ a \sin(\pi) + b \cos(\pi) \\ a \sin(2\pi) + b \cos(2\pi) \end{bmatrix} = \begin{bmatrix} b \\ -b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

So $\operatorname{Ran} T = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$; basis for Ran T is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

So $\dim \operatorname{Ran} T = 1$

b) $\dim V = 2$ $\quad \dim \operatorname{Ran} T + \dim \operatorname{Ker} T = 2$

$$1 + \dim \operatorname{Ker} T = 2$$

$$\dim \operatorname{Ker} T = 1$$

8. a) A is upper-triangular, so $\det(A)$ is the product of diagonal entries, i.e. $\det(A) = 1^5 = 1$

b) $$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

also upper-triangular, so $\det(A - \lambda I) = (1-\lambda)^5$

eigenvalues $\lambda = 1$ (multiplicity 5)

c) For $\lambda = 1$, $$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4 pivots $\Rightarrow \operatorname{rank}(A - I) = 4$

$$\dim \operatorname{Ker}(A-I) + \operatorname{rank}(A-I) = 5$$

$$\dim \operatorname{Ker}(A-I) + 4 = 5$$

$$\dim \operatorname{Ker}(A-I) = 1$$

So the 1-eigenspace has dimension 1

d) <u>No</u> (A has only 1 eigenspace which is 1-dimensional, and $1 < 5$)

e) <u>Yes</u> (B is symmetric)

9

a) FALSE (the columns have to be orthonormal)
b) TRUE (from text)
c) FALSE (also have to check $\vec{v}_2 \cdot \vec{v}_3 = 0$)
d) TRUE (can use Gram-Schmidt to get one)
e) TRUE (from HW)
f) TRUE ($\vec{x} = A^{-1}\vec{b}$)
g) TRUE (from HW)
h) FALSE (only the product of the eigenvalues is 1)
i) FALSE ($\|v\|^2 + \|w\|^2 = \|v+w\|^2$)