Exam solutions now posted on course web pages

Up to now: we've been studying $\mathbb{R}^n$
Now: a more general/abstract P&V on linear algebra

**Vector Spaces (Sec 4.1)**

A vector space $V$ is a set whose elements ("vectors") can be added to one another and can be multiplied by scalars (constants) obeying these axioms:

- If $\vec{x}, \vec{y}$ are in $V$ then $\vec{x} + \vec{y}$ is in $V$.
- If $\vec{x}$ is in $V$ then $c \cdot \vec{x}$ is in $V$ for any constant $c$.
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.
- $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
- There is a vector $\vec{0}$ in $V$ such that $\vec{0} + \vec{x} = \vec{x}$ for all $\vec{x}$ in $V$.
- $c(\vec{x} + \vec{y}) = c \cdot \vec{x} + c \cdot \vec{y}$.
- $(c + d) \cdot \vec{x} = c \cdot \vec{x} + d \cdot \vec{x}$.
- $c(d \cdot \vec{x}) = (cd) \cdot \vec{x}$.
- $1 \cdot \vec{x} = \vec{x}$.
- For every $\vec{x}$ in $V$ there is another vector $-\vec{x}$ in $V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$.

**Facts**

If $V$ is a vector space and $\vec{x} \in V$
then
- $(\cdot (-1)) \vec{x} = -\vec{x}$
- $c \cdot \vec{0} = \vec{0}$
- $0 \cdot \vec{x} = \vec{0}$
Ex \( V = \mathbb{R}^n \) is a vector space, for any \( n \).
(with our previous def. of \( \bar{x} + \bar{y} \) and \( c \bar{x} \))

Ex \( V = \{ \text{all doubly-infinite sequences of numbers} \} \)

- \( \bar{y} = (\ldots, y_3, y_2, y_1, y_0, y_1, y_2, y_3, \ldots) \) each \( y_i \) is a constant

- \( \bar{y} = (\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots) \)
  \( \bar{y} = (\ldots, 0, 0, 0, 0, 0, 0, \ldots) \)
  \( \bar{y} = (\ldots, 1, 1, 1, 1, 1, 1, \ldots) \)

Rule for addition: \( \bar{y} = (\ldots, y_{-1}, y_0, y_1, y_2, \ldots) = (y_k) \)
\( \bar{x} = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots) = (x_k) \)

Then we define \( \bar{x} + \bar{y} = (\ldots, x_{-1} + y_{-1}, x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots) = (x_k + y_k) \)
\( c \bar{x} = (\ldots, c x_{-1}, c x_0, c x_1, c x_2, \ldots) = (c x_k) \)

These rules obey all the vector space axioms.

Ex \( c (\bar{x} + \bar{y}) = c \bar{x} + c \bar{y} \)

- \( c (\ldots, x_{-1} + y_{-1}, x_0 + y_0, x_1 + y_1, \ldots) \)
- \( (\ldots, c x_{-1} + c y_{-1}, c x_0 + c y_0, c x_1 + c y_1, \ldots) \)
- \( (\ldots, c x_1, c x_0, c x_1, \ldots) + (\ldots, c y_1, c y_0, c y_1, \ldots) \)
\[ V = \mathbb{P}_n = \{ \text{all polynomials with real coefficients, of degree } \leq n \} \]

\[ \text{e.g. if } n=3, \quad f = x^3 - 3x^2 + 4x - 7 \in \mathbb{P}_3 \]
\[ g = 2x^2 - 3x + 9 \in \mathbb{P}_3 \]

Define addition, multi.

\[ \text{e.g. } \quad f + g = x^3 - x^2 + x + 2 \in \mathbb{P}_3 \]
\[ 2f = 2x^3 - 6x^2 + 8x - 14 \in \mathbb{P}_3 \]

We could check that \( V \) obeys all the vector space axioms.

(But I won't do it here.)

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Why not take \( \mathbb{P}'_n = \{ \text{all poly. of degree exactly } n \} \)?

Then if \( f, g \in \mathbb{P}'_n \), \( f + g \) might not be:\n
\[ \text{e.g. } \quad f = x^2 + 3 \in \mathbb{P}'_2 \]
\[ g = -x^2 - 7x + 1 \in \mathbb{P}'_2 \]
\[ f + g = 7x + 3 \notin \mathbb{P}'_2 \]

So \( \mathbb{P}'_n \) is not a vector space!

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\[ V = \mathcal{F} = \{ \text{all real-valued continuous functions of one variable} \} \]

\[ \text{e.g. } \quad f(x) = \sin x \in V \]
\[ f(x) = x^2 - \cos x + \sin \left( \frac{x^2 + 1}{17} \right) \in V \]

\underline{Additive law}: \( f \in V \) and \( g \in V \) define \( f + g \in V \)
by \( (f + g)(x) = f(x) + g(x) \)

\underline{Scalar mult.}: \( f \in V \) and constant \( c \) define \( cf \in V \)
by \( (cf)(x) = c \cdot f(x) \)
Checking the vector space axioms:

- One of the axioms is that there is a zero vector \( \vec{0} \in V \) such that \( \vec{x} + \vec{0} = \vec{x} \) for all \( \vec{x} \in V \).

In \( V = \mathbb{F} \), \( \vec{0} \) is the zero function: \( f_0(t) = 0 \) for all \( t \).

Indeed, if \( f \in V \) is any vector (function)

then \( f + f_0 = f \)  
\[
\begin{align*}
(f + f_0)(t) &= f(t) + f_0(t) \\
&= f(t) + 0 \\
&= f(t)
\end{align*}
\]

- We could also (should?) check that \( V \) satisfies the rest of the vector space axioms, e.g.

\( c(d \vec{x}) = (cd) \vec{x} \)

which here becomes \( c(d f) = (cd) f \)

(I won't check them all now...)

Even though "vector" now means any element of \( V \) — not necessarily a column of numbers — we still use some intuition from previous chapters...

But remember that \( \vec{x}, \vec{y} \) are elements of \( V \) now!
Subspaces

Say V is a vector space.

A subspace H of V is a subset of V with 3 properties:

1. The vector \( \overrightarrow{0} \in V \) is contained in H.
2. If \( \overrightarrow{u}, \overrightarrow{v} \) are in H then \( \overrightarrow{u} + \overrightarrow{v} \) is also in H. ("closed under addition")
3. If \( \overrightarrow{u} \) is in H and \( c \) is any constant then \( c \overrightarrow{u} \) is in H. ("closed under multiplication")

Ex

\( V = \mathbb{R}^3 \)

\[ H = \text{xy-plane} = \{ \text{vectors of the form } \left[ \begin{array}{c} x \\ y \\ 0 \end{array} \right] \} \]

But not

\[ H' = \{ \text{vectors of the form } \left[ \begin{array}{c} x \\ y \\ 1 \end{array} \right] \} \]

because

\[
\left[ \begin{array}{c} x \\ y \\ 1 \end{array} \right] + \left[ \begin{array}{c} x' \\ y' \\ 1 \end{array} \right] = \left[ \begin{array}{c} x + x' \\ y + y' \\ 2 \end{array} \right]
\]

so \( H' \) is not closed under addition

\( H' \) is not a subspace!

Ex

For any vector space V, the subset \( H = \{ \overrightarrow{0} \} \) is a subspace.

(Why? \( \overrightarrow{0} + \overrightarrow{0} = \overrightarrow{0} \) and \( c \cdot \overrightarrow{0} = \overrightarrow{0} \))
Ex Define \( H' = \left\{ \text{all vectors in } \mathbb{R}^3 \text{ which have } 0 \text{ as at least one of their entries} \right\} \)

\( H' \) is not a subspace of \( \mathbb{R}^3 \):

\[ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \]

Ex Say \( V = \mathbb{F} \)

Then: \( H = \left\{ \text{all polynomial functions} \right\} \) is a subspace of \( V \).

\[ \begin{aligned} &\text{Because:} \\ &\quad \text{the zero function is a polynomial} \\ &\quad \text{the sum of 2 poly. is a poly.} \\ &\quad \text{a scalar multiple of a poly. is a poly.} \end{aligned} \]

And: \( H = \left\{ \text{all periodic functions with period 1} \right\} \) is a subspace of \( V \).

\[ \begin{aligned} &\text{Because:} \\ &\quad \text{zero } f^n \text{ is periodic} \\ &\quad \text{sum of 2 periodic } f^n \text{ is periodic} \\ &\quad \text{a scalar multiple of periodic } f^n \text{ is periodic} \end{aligned} \]

Ex If \( \vec{v}_1 \) and \( \vec{v}_2 \) are elements of a vector space \( V \)

Define \( \text{Span} \{ \vec{v}_1, \vec{v}_2 \} \) to be the set of all lin. comb. of \( \vec{v}_1 \) and \( \vec{v}_2 \)

i.e. all vectors of the form \( x_1 \vec{v}_1 + x_2 \vec{v}_2 \) where \( x_1, x_2 \) are constants
Then \( H = \text{Span}\{\vec{v}_1, \vec{v}_2\} \) is a subspace of \( V \).

Why? \( \vec{0} \in \text{Span}\{\vec{v}_1, \vec{v}_2\} \)

\[ (x_1\vec{v}_1 + x_2\vec{v}_2) + (x'_1\vec{v}_1 + x'_2\vec{v}_2) = (x_1 + x'_1)\vec{v}_1 + (x_2 + x'_2)\vec{v}_2 \in H \]
so \( H \) is closed under addition

\[ c(x_1\vec{v}_1 + x_2\vec{v}_2) = (cx_1)\vec{v}_1 + (cx_2)\vec{v}_2 \in H \]
so \( H \) is closed under scalar mult.

Fact If \( \vec{v}_1, \ldots, \vec{v}_p \) are vectors in \( V \)
then \( \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} \) is a subspace of \( V \).

(Why? Just like the above example)