Theorem.
Let f be a real-valued function. Let $D \subset \mathbb{R}$ be the domain of f, and $p \in D$. Then,

f is continuous at p

if and only if

for all sequences $(x_n) \subset D$ such that $x_n \to p$, we have $f(x_n) \to f(p)$.

Proof.
First, we prove the forward direction: assume that f is continuous at p, and suppose given some sequence $(x_n) \subset D$, such that $x_n \to p$. We would like to show that $f(x_n) \to f(p)$.

Fix some arbitrary $\epsilon > 0$. Since f is continuous at p, there exists a $\delta > 0$ such that

$$(x \in D, |x - p| < \delta) \implies |f(x) - f(p)| < \epsilon.$$

Also, since $x_n \to p$, there exists an $N \in \mathbb{N}$ such that

$$n \geq N \implies |x_n - p| < \delta.$$

Combining these two (and the fact that $x_n \in D$ from above), we have that

$$n \geq N \implies |f(x_n) - f(p)| < \epsilon.$$

So $f(x_n) \to f(p)$.

Next, we prove the backward direction. For this we switch to its contrapositive. So, assume that f is *not* continuous at p. We would like to show that there exists some sequence $(x_n) \subset D$, such that $x_n \to p$, and $f(x_n) \not\to f(p)$.

Since f is not continuous at p, there exists some $\epsilon > 0$ such that, for all $\delta > 0$, there exists an $x \in D$ with $|x - p| < \delta$ and $|f(x) - f(p)| \geq \epsilon$. Fix this ϵ. Then for any $n \in \mathbb{N}$, taking $\delta = 1/n$, it follows that there exists an $x_n \in D$ with $|x_n - p| < 1/n$ and $|f(x_n) - f(p)| \geq \epsilon$. This defines our sequence $(x_n) \subset D$.

Since $|x_n - p| < 1/n$, we have $p - 1/n \leq x_n \leq p + 1/n$; and $p + 1/n \to p$, $p - 1/n \to p$, so applying the "Squeeze Theorem" (problem 3.19) gives $x_n \to p$.

But since $|f(x_n) - f(p)| \geq \epsilon$ for all n, $f(x_n) \not\to f(p)$ (problem 3.10).

So we have shown that (x_n) has all the desired properties.