CHAPTER 1

Introduction

1. Goals

The purpose of this course is three-fold:

(1) to provide an introduction to the basic definitions and theorems of calculus and real analysis.
(2) to provide an introduction to writing and discovering proofs of mathematical theorems. These proofs will go beyond the mechanical proofs found in your Discrete Mathematics course.
(3) and most importantly to let you experience the joy of mathematics: the joy of personal discovery.

2. Proofs

Hopefully all of you have seen some proofs before. A proof is the name that mathematicians give to an explanation that leaves no doubt. The level of detail in this explanation depends on the audience for the proof. Mathematicians often skip steps in proofs and rely on the reader to fill in the missing steps. This can have the advantage of focusing the reader on the new or crucial ideas in the proof but can easily lead to frustration if the reader is unable to fill in the missing steps. More seriously these missing steps can easily conceal mistakes: many mistakes in proofs, particularly at the undergraduate level, begin with "it is obvious that".

In this course we will try to avoid missing any steps in our proofs. Each statement should follow from a previous one by a simple property of arithmetic, by a definition, or by a previous theorem, and this justification should be clearly stated in plain language. Writing clear proofs is a skill in itself. Often the shortest proof is not the clearest.

There is no mechanical process to produce a proof but there are some basic guidelines you should follow. The most basic is that every object that appears should be defined; when a variable, function, or set appears we should be able to look back and find a statement defining that object:

(1) Let $\epsilon > 0$ be arbitrary.

(2) Let $f(x) = 2x + 1$.
(3) Let $A = \{x \in \mathbb{R} : x^{13} - 27x^{12} + 16x^2 - 4 = 0\}$.
(4) By the definition of continuity there exists a $\delta > 0$ such that...

Always watch out for hidden assumptions. In a proof, you may want to say "Let $x \in A$ be arbitrary," but this does not work if $A = \varnothing$ (where $\varnothing$ denotes the empty set). A common error in real analysis is to write $\lim_{n\to\infty} a_n$ or $\lim_{x\to a} f(x)$ without first checking whether the limit exists (often the hardest part).

A key factor in determining how a proof should be written is the intended audience. For this course, your intended audience is another student in the class who is clueless as to how to prove the theorem, but who knows all the definition and the results of the course covered up to that point.

3. Logic

For the most part, we will avoid using overly technical logical notation in our definitions and statements. Instead we will use their plain English equivalents and we suggest you do the same in your proofs. Beyond switching to the contrapositive and negating a definition, formal logical manipulation is rarely helpful in proving statements in real analysis.

On the other hand, you should be familiar with the basic logical operators and we will give a short review here. If P and Q are **propositions**, i.e., statements that are either true or false, then you should understand what is meant by

(1) not P
(2) P or Q (the mathematical use of "or" is not exclusive so that "P or Q" is considered true if both P and Q are true).
(3) P and Q
(4) if P then Q (or "P implies Q")
(5) P if and only if Q (sometimes written P is equivalent to Q)

Similarly if $P(x)$ is a **predicate**, that is a statement that becomes a proposition when an object such as a real number is inserted for x, then you should understand

(1) for all x, $P(x)$ is true
(2) there exists an x such that $P(x)$ is true

Simple examples of such a $P(x)$ are "$x > 0$" or "x^2 is an integer." These statements are true for some values of x and not for others.

You should also be familiar with the formulae for negating the various operators and quantifiers.

Most of our theorems will have the form of implications: "if P then Q." P is called the **hypothesis** and Q the **conclusion**.

Definition. The **contrapositive** of the implication "if P then Q" is the implication "if not Q then not P."

The contrapositive is logically equivalent to the original implication. This means that one is valid (true) if and only if the other is valid. Sometimes it is easier or better to pass to the contrapositive formulation when proving a theorem.

Definition. The **converse** of the implication if P then Q is the implication if Q then P.

The converse is *not* logically equivalent to the original implication and this fact is the underlying source of error in many undergraduate proofs.

Definition. A statement that is always true is called a **tautology**. A statement that is always false is called a **contradiction**.

To show that an argument is not valid it suffices to find one situation in which the hypotheses are true but the conclusion is false. This type of situation is called a **counterexample**.

One technique of proof is by contradiction. To prove "P implies Q" we might assume that P is true and Q is false and obtain a contradiction. Whenever you use contradiction, it is usually a good idea to see if you can rephrase your proof in a way that does not use contradiction. Often times, contradiction is not necessary and avoiding its use can lead to cleaner (and more understandable) proofs.

You will notice that we mostly spoke in great generality in the preceding review. If at any point you found yourself confused in regards to the meaning of a definition or the purpose of a concept, you should create some examples of the situation being described. Be as concrete as you can. In fact, as we will emphasize later, you should use this strategy any time you are confused when reading these notes (or any mathematics textbook).