

CHAPTER 6

Integration

1. The Definition

Our final chapter is the other half of calculus: the integral. The intuitive notion behind the integral is very simple. Specifically suppose that f is a function which is continuous on $[a, b]$ (we will see that it is not necessary to assume that f is continuous, but we will make this assumption for the sake of our intuition). Also assume for intuitional simplicity that f is positive on $[a, b]$. Finally assume that f is simple enough that we may easily draw its graph.

If, on the Cartesian plane, we draw the vertical lines $x = a$ and $x = b$, we see that there is a shape defined by the region surrounded by $x = a$, $x = b$, the graph of f and the x -axis. You should draw some examples of this situation. We have given one example in Figure 1. The point of integration is to find the area of this shape. Unless f is extremely simple (like a straight line or something), geometry does not give us a formula for this area. In fact, it's not even clear that we have precise definition for word 'area.'

Thus we need a new approach to finding (or even defining) this area. We begin by thinking of some objects of which we do know the area. The simplest is probably the rectangle: the area of a rectangle should certainly be the product of its length and width. Our approach to finding more complicated areas will actually be a very clever use of this simple observation.

Our strategy begins by estimating the area in question. Indeed, one naive way of estimating the area with rectangles is with a single rectangle. To do this, as shown in Figure 2, we place a rectangle with its bottom side on the x -axis, and its left and right sides on the line $x = a$ and $x = b$. We place the top of the rectangle at *some* value of the function on $[a, b]$. In the example shown in the figure, we chose the lowest value of f , but any choice would give an estimate.

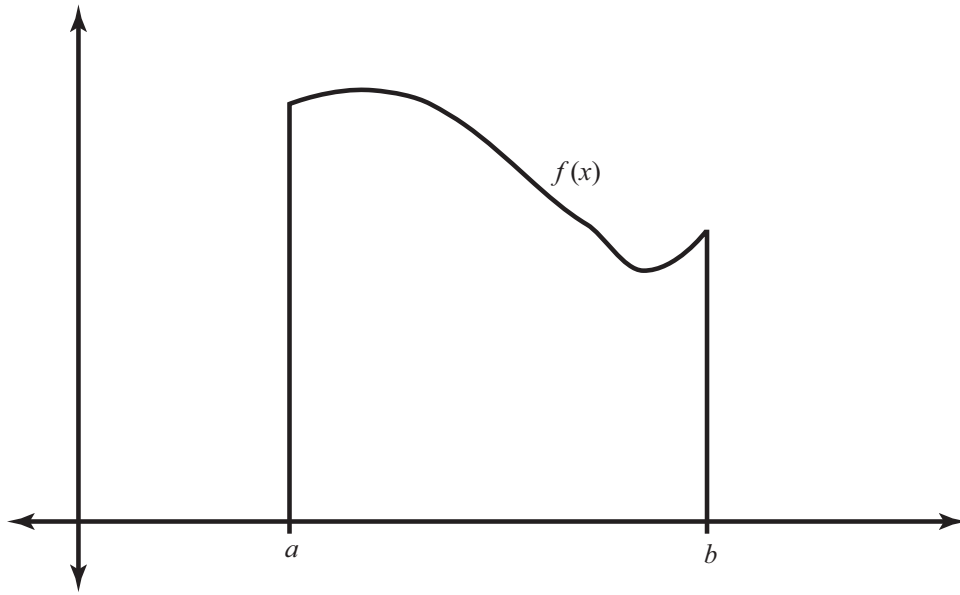


FIGURE 1. This is the shape of which we want to find the area.

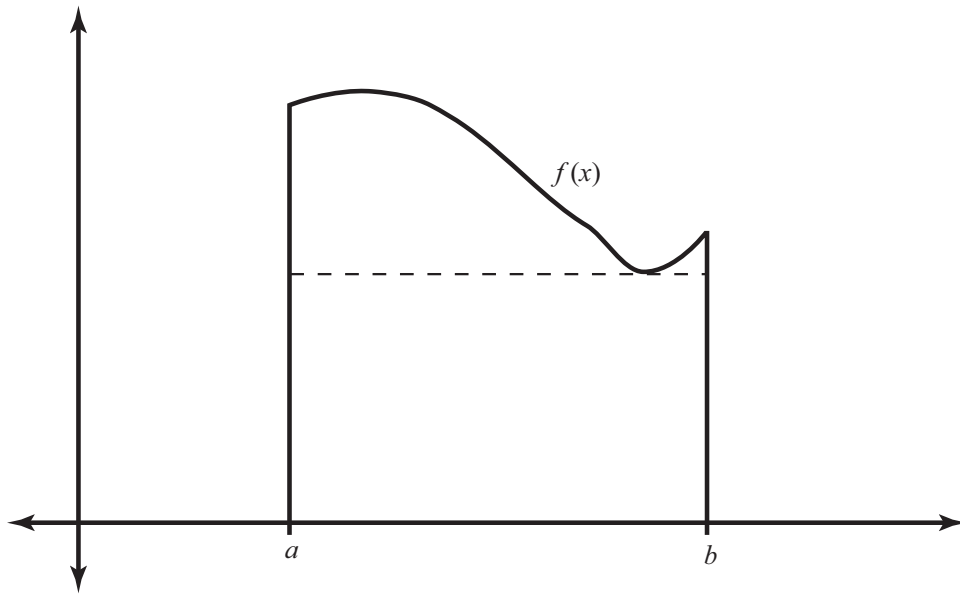


FIGURE 2. A naive way to estimate the area is by drawing a rectangle of similar size.

As you can tell from the figure, this estimate may not be all that precise: large parts of the shape lie outside of the rectangle and part of the rectangle lies outside of the shape. Thus we need to refine our

method of estimation: instead of using one rectangle, we will use two, as the top picture of Figure 3.

We split the interval $[a, b]$ into two pieces by choosing some point, x_1 , between a and b . We draw two rectangles: both of them have bottom side at the x -axis. One has left and right sides $x = a$ and $x = x_1$. Its top is at some value of f on $[a, x_1]$ (in this case the lowest). The other rectangle is similarly constructed between x_1 and b . To estimate the area of the shape in question, we simply add the areas of the two rectangles.

As we can see in the pictures, estimating in this way tends to work better than the first way. This should not be too surprising: we are using more information from the function than with the first estimation (i.e. two values rather than one). In the bottom picture of Figure 3, we use four rectangles: for this we have to choose 3 points between a and b and apply a similar construction. We expect that this estimation of the area will be even better than the second.

We now define the notion of integral precisely. An important step in our process was splitting the interval $[a, b]$ into multiple pieces so that we can use multiple rectangles. This leads to the first definition of this section.

Definition. A **partition** of an interval $[a, b]$ is a finite sequence (x_1, \dots, x_n) such that

$$a = x_0 < x_1 < \dots < x_n = b.$$

If P and Q are two partitions of $[a, b]$ we say that Q **refines** P if every point in $[a, b]$ which occurs in P also occurs in Q .

What is the intuitive purpose of the notion of a partition? What about of the notion of refinement?

As above, a partition leads to an estimate of the area in question. Indeed, a partition with n elements leads us to n rectangles: the bottom, left, and right sides of the rectangles are given and we need only choose the tops. Of course the top of the rectangle which lies between x_i and x_{i+1} should be a value of f between x_i and x_{i+1} . There are basically two systematic ways of choosing this value. We can use the highest value of f or we can choose the lowest value. Of course, not every function has a highest or lowest value on an interval, but we can use the next best thing.

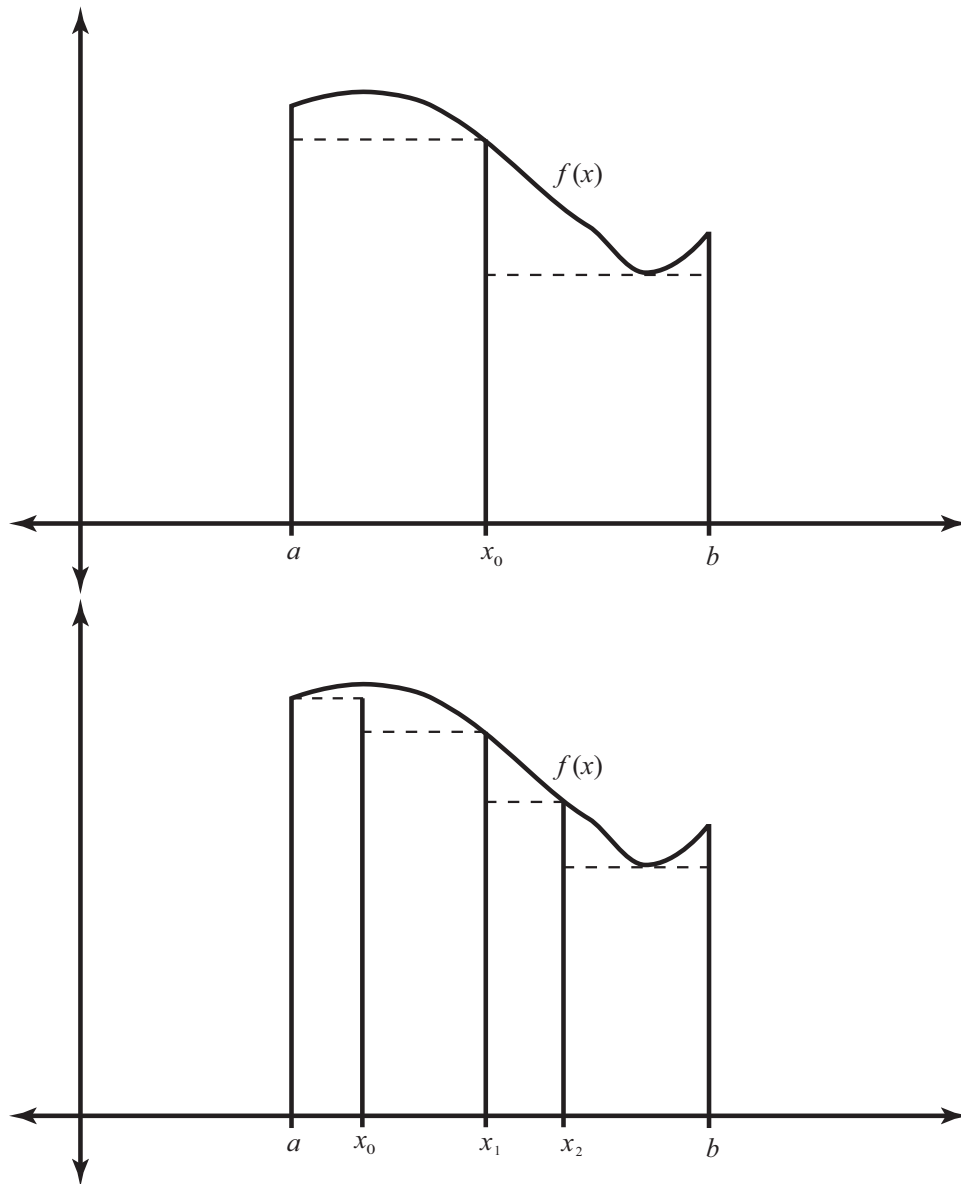


FIGURE 3. A somewhat better estimate is achieved by drawing multiple rectangles

Definition. Let $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a, b]$ and let f be a function whose domain contains $[a, b]$ which is bounded on $[a, b]$.

For $1 \leq i \leq n$ we set

$$\begin{aligned} M_i(f, P) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \\ &= \sup f([x_{i-1}, x_i]) \\ m_i(f, P) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \\ &= \inf f([x_{i-1}, x_i]) \\ \Delta_i &= x_i - x_{i-1}. \end{aligned}$$

Thus $m_i(f, P)$ is the ‘lowest’ value of f on the interval $[x_{i-1}, x_i]$ and $M_i(f, P)$ is the ‘highest.’ Notice that $m_i(f, P)$ and $M_i(f, P)$ exist because f is bounded on $[a, b]$. Δ_i is the length of the interval (and so the width of the resulting rectangle). These notions thus give us to systematic ways of estimating the area, given a partition.

Definition. Let P be a partition of the interval $[a, b]$ and let f be a function which is bounded on $[a, b]$. We define the **upper Riemann sum** of f with respect to P , denoted $U(f, P)$, by

$$U(f, P) = \sum_{i=1}^n M_i(f, P) \Delta_i$$

and we define the **lower Riemann sum** of f with respect to P , denoted $L(f, P)$, by

$$L(f, P) = \sum_{i=1}^n m_i(f, P) \Delta_i.$$

Check for yourself that both of these definitions provide estimates as we have described. Check that (given P) the upper Riemann sum is the highest estimate of the type we have described and the lower Riemann sum is the lowest. Given that our definitions have been a bit involved, we should also try some examples.

6.1. Let $f(x) = x$ and $g(x) = x^2$. Let $n \in \mathbb{N}$ and let P_n be the partition of $[0, 1]$ given by $(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1)$ be a partition of $[0, 1]$. Draw a picture illustrating $L(f, P_4)$, $U(f, P_4)$, $L(g, P_4)$, and $U(g, P_4)$. Find expressions for $L(f, P_n)$, $U(f, P_n)$, $L(g, P_n)$, and $U(g, P_n)$.

6.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and let P be a partition of $[a, b]$. Let $m = \inf f([a, b])$ and $M = \sup f([a, b])$.

(1) Prove

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a).$$

(2) Prove that if Q refines P then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Hint: For the second part, first prove that it suffices to show this when Q has one more element than P .

Thus we see that as our estimates get more and more refined, the lower sums go up. Thus we might expect that the best estimate that a lower sum will give us is the highest one. Likewise, we expect that the best estimates that the upper sums can give us is the lowest one. This leads us to the following definition (once again the highest lower sum and the lowest upper sum may not exist).

Definition. Let f be a function which is bounded on the interval $[a, b]$. We define the **upper Riemann integral** of f , denoted $U(f, [a, b])$, by

$$U(f, [a, b]) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

We define the **lower Riemann integral** of f , denoted $L(f, [a, b])$, by

$$L(f, [a, b]) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

(Why do these exist?) We say f is **Riemann integrable** on $[a, b]$ if $L(f, [a, b]) = U(f, [a, b])$. In this case we call the common value of $U(f, [a, b])$ and $L(f, [a, b])$ the **(definite) Riemann integral of f** over the interval $[a, b]$ which we denote by $\int_a^b f$ or $\int_a^b f(x)dx$.

6.3. Show that for f and g as in Problem 6.1 for all $n \in \mathbb{N}$, $U(f, P_n) \neq U(f, [0, 1])$, $L(f, P_n) \neq L(f, [0, 1])$ and similarly for g .

We begin by giving a non-example: that is, we show that not every bounded function is integrable.

6.4. Let f be given by

$$f(x) = \begin{cases} 0 & x \in [0, 1] \setminus \mathbb{Q} \\ 1 & x \in [0, 1] \cap \mathbb{Q}. \end{cases}$$

Show that f is not integrable.

2. Integrable Functions

Of course we hope that many functions are integrable as this will allow us to compute a lot of different areas. In this section, we will show that all continuous functions are in fact integrable, because the definition of integrability is a bit elaborate, it is necessary to do some preliminary work.

6.5. Let f be a function which is bounded on $[a, b]$ and let P and Q be partitions of $[a, b]$. Prove that $L(f, P) \leq U(f, Q)$.

Hint: Consider the partition $R = P \cup Q$.

6.6. Let f be a function which is bounded on $[a, b]$. Show that $L(f) \leq U(f)$.

6.7. Let $f(x) = x$. Let P_n be the partition of $[0, 1]$ given by $P_n = (0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1)$.

(1) Find $L(f, P_n)$ and $U(f, P_n)$.

(2) Find $U(f, P_n) - L(f, P_n)$.

(3) Show f is integrable on $[0, 1]$ and find $\int_0^1 f$.

The next result gives a very useful characterization of integrability.

6.8. Let f be a function which is bounded on $[a, b]$. Show that f is integrable on $[a, b]$ if and only if for all $\epsilon > 0$ there exists a partition $P = (x_0, \dots, x_n)$ of the interval $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i(f, P) - m_i(f, P)) \Delta_i < \epsilon.$$

6.9. Let f be a function which is increasing and bounded on $[a, b]$. Prove that f is integrable on $[a, b]$.

Hint: For a monotone function we know explicitly what $M_i(f, P)$ and $m_i(f, P)$ are.

6.10. Let f be a function which is continuous on $[a, b]$. Prove that f is integrable on $[a, b]$.

Hint: Use Problem 4.57 to conclude that f is uniformly continuous on $[a, b]$. Let $\epsilon > 0$ be arbitrary and choose $\delta > 0$ so that if $x, y \in [a, b]$

with $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Let P be any partition with each $\Delta_i x < \delta$ and use Problem 6.8.

3. Properties of Integrals

For every concept we have introduced, we have seen that there is something like ‘Limit Laws,’ which are useful in computing examples. As you probably already know, integrals are no exception.

6.11. Let f and g be functions which are integrable on $[a, b]$. Let $c \in \mathbb{R}$ be an arbitrary constant. Prove that

- (1) $c \cdot f$ is integrable on $[a, b]$ and

$$\int_a^b c \cdot f = c \int_a^b f.$$

- (2) $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Hint: Show

$$M_i(f + g, P) \leq M_i(f, P) + M_i(g, P)$$

and hence conclude that $U(f + g, P) \leq U(f, P) + U(g, P)$. Similarly, show that $L(f + g, P) \geq L(f, P) + L(g, P)$. Use Problem 6.8 to conclude integrability. Then prove the equation.

6.12. Let f and g be functions which are integrable on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f \leq \int_a^b g$.

6.13. Let $a < c < b$ and let f be a function.

- (1) Assume f is integrable on $[a, b]$. Prove that f is integrable on $[a, c]$ and $[c, b]$.
- (2) Assume that f is integrable on $[a, c]$ and on $[c, b]$. Prove that f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

It is actually convenient for notational reasons for us to consider something like backwards integrals. In other words, for $a < b$, the concept we have been studying so far might be called the “integral from a to b .” The “integral from b to a ” is just the opposite.

Definition. If f is a function which is integrable on $[a, b]$ we define $\int_b^a f = -\int_a^b f$. We also define $\int_a^a f = 0$.

6.14. Let f be integrable on an interval containing a , b and c . Prove that, no matter the order of a , b , and c , we have

$$\int_a^b f = \int_a^c f + \int_c^b f .$$

6.15. Let f be a function which is integrable on $[a, b]$. Prove that

- (1) $|f|$ is integrable on $[a, b]$
- (2) $|\int_a^b f| \leq \int_a^b |f|$.

Hint: For the first part, prove that $M_i(|f|, P) - m_i(|f|, P) \leq M_i(f, P) - m_i(f, P)$.

6.16. Prove that if f and g are integrable on $[a, b]$ then $f \cdot g$ is integrable on $[a, b]$.

Hint: Since f and g are integrable we may define

$$\begin{aligned} M_f &= \sup\{|f(x)| : a \leq x \leq b\}, \\ M_g &= \sup\{|g(x)| : a \leq x \leq b\}. \end{aligned}$$

Use the fact that

$$f(x) \cdot g(x) - f(y) \cdot g(y) = f(x) \cdot (g(x) - g(y)) + (f(x) - f(y)) \cdot g(y)$$

to conclude

$$\begin{aligned} M_i(f \cdot g, P) - m_i(f \cdot g, P) \\ \leq L_f \cdot (M_i(g, P) - m_i(g, P)) + (M_i(f, P) - m_i(f, P)) \cdot M_g. \end{aligned}$$

Now use Problem 6.8

Thus the product of integral functions is integrable. Integrals, however, do not behave so well with respect to multiplication. In other words, we have seen the the product of sequences converges to the product of the limits of the two sequences and likewise for limits of functions. The analogous statement, however is not true for integrals.

6.17. Give an example of an interval $[a, b]$ and two functions f and g which are integrable on $[a, b]$ such that

$$\left[\int_a^b f(x)dx \right] \left[\int_a^b g(x)dx \right] \neq \int_a^b f(x)g(x)dx.$$

Given the complication resulting from the previous behavior, you are probably aware of some techniques people have developed to deal with integrals of products. We will state and prove them in a later section.

6.18. Given an example of a function f which is integrable on $[0, 1]$, but not continuous.

6.19. Show that the function $f(x) = \sqrt{1-x^2}$ is integrable on $[0, 1]$.

Thus

$$\int_0^1 \sqrt{1-x^2}dx$$

exists as a real number. But we also see that it is one fourth the area of the unit circle, that is it equals $\pi/4$. Multiplying by 4, we conclude that π exists as a real number.

4. Fundamental Theorems of Calculus

We saw that even computing $\int_0^1 xdx$ from the definition of integral took some work. Of course nobody actually computes integrals this way in calculus. Instead they use the Fundamental Theorem of Calculus. Loosely speaking, this theorem says that the derivative and the integral are opposites (or probably more appropriately inverses). As its name suggests, it is considered the crucial result in understanding calculus. It is commonly broken into two different theorems and we will also prove it this way.

6.20. Prove the first Fundamental Theorem of Calculus:

Let f be a function which is integrable on $[a, b]$. Define a function $F : [a, b] \rightarrow \mathbb{R}$ by

$$F(x) = \int_a^x f = \int_a^x f(t)dt$$

F is uniformly continuous on $[a, b]$, and if f is continuous at $c \in (a, b)$ then $F'(c) = f(c)$.

Hint: For uniform continuity show that if $a \leq x \leq y \leq b$ then

$$F(y) - F(x) = \int_x^y f.$$

Now estimate $\int_x^y f$ from above and below in terms of $y - x$. For $F'(c) = f(c)$ show that

$$\frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x - c} \int_c^x [f - f(c)].$$

Use the fact that f is continuous at c to conclude that the right hand side can be made arbitrarily small by taking x sufficiently close to c .

The second part of the the Fundamental Theorem is the one that it often more convenient to use.

6.21. Prove the second Fundamental Theorem of Calculus:

If f is a function which is continuous on $[a, b]$ and differentiable on (a, b) , then $\int_a^b f'(x)dx = f(b) - f(a)$.

Hint: If $P = (x_0, x_1, \dots, x_n)$ is any partition of $[a, b]$ then, by Problem 5.14, for each $1 \leq i \leq n$ we can find $t_i \in (x_{i-1}, x_i)$ so that

$$f(b) - f(a) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = \sum_{i=1}^n f'(t_i) \Delta_i.$$

Show $L(f', P) \leq f(b) - f(a) \leq U(f', P)$.

6.22. Suppose that $k \in \mathbb{N}$. Compute

$$\lim_{n \rightarrow \infty} \frac{1}{n^{k+1}} \sum_{i=1}^n i^k.$$

5. Integration Rules

Once we have the fundamental theorems of calculus we can prove two of the important rules of integration; integration by parts, and integration by substitution (or u -substitution). These turn out to be reinterpretations of the product rule and of the chain rule, respectively.

6.23. Prove that if f and g are functions that are differentiable on $[a, b]$ and both f' and g' are integrable on $[a, b]$ then

$$\int_a^b f(x) \cdot g'(x) dx = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f'(x) \cdot g(x) dx.$$

6.24. Suppose that u is differentiable on $[c, d]$ and u' is continuous on $[c, d]$. Let $a = u(c)$ and $b = u(d)$. Suppose that f is continuous on $u([c, d])$. Prove that

$$\int_a^b f = \int_c^d (f \circ u) \cdot u'.$$

Hint: Let $F(x) = \int_a^x f$. Use Problem 6.20 to show that F is differentiable and use Problem 5.9 to show that $F \circ u$ is differentiable. Now apply Problem 6.21.

Notice we don't need to assume that $u([c, d]) = [a, b]$.