M 365C

FALL 2013, SECTION 57465
PROBLEM SET 11
DUE THU NOV 14

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-6, without reproving them.

Exercise 1

Show directly from the definition of the Riemann integral that

$$\int_0^1 x \,\mathrm{d}x = \frac{1}{2}.$$

Answer of exercise 1

Let P_n be the partition of $[0,1]$ into n equal intervals of length $1/n$. Then on the i-th interval, the infimum of the function $f(x) = x$ is $m_i = \frac{i-1}{n}$, while the supremum is $M_i = \frac{i}{n}$.

Then

$$L(P_n, f) = \sum_{i=1}^{n} \frac{1}{n}\frac{i-1}{n} = \frac{1}{n^2}\sum_{i=1}^{n}(i-1) = \frac{1}{n^2}\frac{n(n-1)}{2} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)$$

and

$$U(P_n, f) = \sum_{i=1}^{n} \frac{1}{n}\frac{i}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n^2}\right)$$

Thus for any n we have

$$\frac{1}{2}\left(1 - \frac{1}{n^2}\right) \le \underline{\int_0^1} x\mathrm{d}x \le \overline{\int_0^1} x\mathrm{d}x \le \frac{1}{2}\left(1 + \frac{1}{n^2}\right)$$

It follows that both the lower and upper integrals lie in the interval of size $\frac{1}{n^2}$ around $\frac{1}{2}$, for any n. Thus both lower and upper integrals equal $\frac{1}{2}$. It follows that the integral is $\frac{1}{2}$ as desired.

Exercise 2 (*Rudin 6.8*)

Fix $a \in \mathbb{R}$. Suppose $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on $[a, b]$ for every $b > a$. We define

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\mathrm{d}x$$

if the limit on the right side exists. In that case we say that the integral on the left side converges.

Now take $a = 1$, assume that $f(x) \geq 0$ for all $x \in [1, \infty)$, and also assume that f is monotonically decreasing. Prove that

$$\int_1^\infty f(x)\mathrm{d}x$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges. (This is the "integral test" for convergence of series.)

Answer of exercise 2

Let P_k be the partition of $[1, k+1]$ into k equal pieces of length 1. Since f is monotonically decreasing, $U(P_k, f) = \sum_{n=1}^{k} f(n)$, while $L(P_k, f) = \sum_{n=2}^{k} f(n)$. Thus we have

$$\sum_{n=2}^{k} f(n) \leq \int_1^k f(x)\mathrm{d}x \leq \sum_{n=1}^{k} f(n).$$

If the sum converges, i.e. $\sum_{n=1}^{\infty} f(n) = M$, then for all t we have $\int_1^t f(x)\mathrm{d}x \leq M$; thus $\int_1^t f(x)\mathrm{d}x$ is monotone increasing with t and bounded above; thus its limit as $t \to \infty$ exists.

If the sum diverges, then $\sum_{n=2}^{k} f(n) \leq \int_1^k f(x)\mathrm{d}x$ is unbounded above as $k \to \infty$; it follows that $\int_1^t f(x)\mathrm{d}x$ is also unbounded above as $t \to \infty$; then its limit as $t \to \infty$ cannot exist.