M 365C

Fall 2013, Section 57465
Problem Set 12
Due Thu Nov 21

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-6, without reproving them.

Exercise 1

Prove Rudin's Theorem 7.9: Suppose $\{f_n\}$ is a sequence of functions, $f_n : E \to \mathbb{R}$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$, i.e. $f_n \to f$ pointwise on E. Put $M_n = \sup\{|f_n(x) - f(x)| : x \in E\}$. Then $f_n \to f$ uniformly on E if and only if $\lim_{n\to\infty} M_n = 0$.

Exercise 2 (*Rudin 7.1*)

Suppose $f_n : E \to \mathbb{R}$ is a sequence of functions. We say that $\{f_n\}$ is *uniformly bounded* on E if there exists some $M \in \mathbb{R}$ such that for all $x \in E$ and all $n \in \mathbb{N}$ we have $|f_n(x)| < M$.

Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Exercise 3 (*Rudin 7.2*)

If $\{f_n\}$ and $\{g_n\}$ are sequences of functions mapping $E \to \mathbb{R}$, and converging uniformly on E, prove that $\{f_n + g_n\}$ converges uniformly on E. If in addition each f_n is bounded and each g_n is bounded, prove that $\{f_n g_n\}$ converges uniformly on E.

Exercise 4 (*Rudin 7.3*)

Construct sequences $\{f_n\}$, $\{g_n\}$ of functions mapping $X \to \mathbb{R}$ (with X some metric space), such that $\{f_n\}$ and $\{g_n\}$ both converge uniformly on X, but $\{f_n g_n\}$ does not converge uniformly on X.