M 365C
FALL 2013, SECTION 57465
PROBLEM SET 13
DUE TUE DEC 3

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-7, without reproving them.

Exercise 1 (*Rudin 7.12, modified*)

Suppose g and f_n are defined on $[0, \infty)$, are Riemann-integrable on $[a, b]$ whenever $0 \le a < b < \infty$, $|f_n(x)| \le g(x)$ for all x, $f_n \to f$ uniformly on every compact subset of $[0, \infty)$, and

$$\int_0^\infty g(x)\,\mathrm{d}x < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x = \int_0^\infty f(x)\,\mathrm{d}x.$$

(Here by $\int_0^\infty$ we mean the improper integral as defined in a previous homework problem.)

Exercise 2 (*Rudin 7.4*)

Consider

$$f(x) = \sum_{n=1}^\infty \frac{1}{1+n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Exercise 3 (*Rudin 7.20*)

If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n\,\mathrm{d}x = 0$$

for all $n \ge 0$, prove that $f(x) = 0$ for all $x \in [0, 1]$. (Hint: The integral of $f(x)$ times any polynomial is zero. Use the Weierstrass theorem to conclude that $\int_0^1 f(x)^2 \mathrm{d}x = 0$. Then use the result of an earlier homework problem.)

Exercise 4 (*Rudin 7.9*)

Let $\{f_n\}$ be a sequence of continuous functions which converge uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse true, i.e. if this equation holds for every such sequence, does it follow that $f_n \to f$ uniformly on E?

* Exercise 5 (*Rudin 7.25*)

Suppose $\phi(x, y)$ is a continuous bounded real function defined on the strip $x \in [0, 1]$, $y \in \mathbb{R}$, and c is any constant. Prove that the initial-value problem

$$y' = \phi(x, y), \qquad y(0) = c$$

admits a solution, i.e. that there exists a function $y(x)$ defined on $[0, 1]$ such that $y(0) = c$ and $y'(x) = \phi(x, y(x))$.

(See the long hint in Rudin which breaks this problem into six subparts.)