M 365C

Fall 2013, Section 57465
Problem Set 3
Due Thu Sep 19

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-2, without reproving them.

Exercise 1

Show that the subset of $\mathbb{R}^2$ given by $E = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$ is open.

Answer of exercise 1

We need to show that every point $(x, y) \in E$ is an interior point. So, suppose $(x, y) \in E$. Then $x < y$. Let $\epsilon = \frac{1}{2}(y - x)$. Then $\epsilon > 0$. Suppose $(x', y') \in N_\epsilon((x, y))$. Then $|x' - x|^2 + |y' - y|^2 < \epsilon^2$. This implies that $|x' - x| < \epsilon$ and $|y' - y| < \epsilon$. But then it follows that $x' < x + \epsilon = x + \frac{1}{2}(y - x) = \frac{1}{2}(x + y)$, and $y' > y - \epsilon = y - \frac{1}{2}(y - x) = \frac{1}{2}(x + y)$. Thus $y' > x'$. In other words, $N_\epsilon((x, y)) \subset E$.

Exercise 2 (*Rudin 2.7*)

Let A_1, A_2, ... be subsets of a metric space X.

1. If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$, for any $n \in \mathbb{N}$.
2. If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \cup_{i=1}^\infty \bar{A}_i$. Show, by an example, that this inclusion can be proper, i.e. it may happen that $\bar{B} \neq \cup_{i=1}^\infty \bar{A}_i$.

Answer of exercise 2

First we show that more generally, for any collection $\{A_\alpha\}$ of subsets of X, and $B = \cup_\alpha A_\alpha$, we have $\bar{B} \supset \cup_\alpha \bar{A}_\alpha$. First, $A_\alpha \subset B$. Now suppose $x \in A'_\alpha$. Then for any neighborhood N of x, there exists some $y \neq x$ with $y \in N \cap A_\alpha$. But then $y \in N \cap B$ also. Hence $x \in B'$. So $A'_\alpha \subset B'$. This gives the desired $\bar{B} \supset \cup_\alpha \bar{A}_\alpha$.

Now, we show that for a finite collection $\{A_1, \ldots, A_n\}$ and $B = \cup_{i=1}^n A_i$ we have $\bar{B} \subset \cup_{i=1}^n \bar{A}_i$. All we need show is that if x is a limit point of B then x is a limit point of some A_i. To show this, suppose that x is not a limit point of any A_i. In this case there exists some ϵ_i for which $N_{\epsilon_i}(x)$ does not contain any point of A_i (other than x). But then taking $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, $N_\epsilon(x)$ does not contain any point of any A_i (other than x), thus it contains no point of B (other than x). This contradicts the assumption that x is a limit point of B.

These two assertions together establish the statements which were to be proven. The only thing left is to show that it may happen that $\bar{B} \neq \cup_{i=1}^\infty \bar{A}_i$. For this, take $A_i = \{1/i\} \subset \mathbb{R}$. Each $\bar{A}_i = A_i$, so $\cup_{i=1}^\infty \bar{A}_i = \cup_{i=1}^\infty A_i$. However, $\bar{B}$ contains 0 in addition to the union of the A_i.

Exercise 3 (*Rudin 2.8*)

Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Answer of exercise 3

Yes, if E is open, then every $x \in E$ is a limit point; this follows because every point is interior, i.e. $N_\epsilon(x) \subset E$, and $N_\epsilon(x)$ certainly contains points $y \neq x$.

No, if E is closed, not every point $x \in E$ need be a limit point; for example, if $E = \{(0,0)\}$ say, then E has no limit points.

Exercise 4

Show that the union of a finite number of compact sets is compact.

Answer of exercise 4

Suppose $K = \cup_{i=1}^n K_i$ with each K_i compact. Suppose given an open cover $\{G_\alpha\}$ of K. Since K_i is compact, there is a finite subset of $\{G_\alpha\}$ which covers K_i. Taking the union of these finite subsets gives a finite subset of $\{G_\alpha\}$ which covers the whole K. Thus every open cover has a finite subcover, as needed.

Exercise 5 (*Rudin 2.14*)

Show directly that the interval $(0,1) \subset \mathbb{R}$ is not compact, by giving an example of an open cover of $(0,1)$ which has no finite subcover.

Answer of exercise 5

Let $G_n = \{x \,|\, 1/n < x < 1\} \subset \mathbb{R}$. The set $\{G_n \,|\, n \in \mathbb{N}\}$ forms an open cover of $(0,1)$, since for any $x \in (0,1)$ and any $n > 1/x$, we have $x \in G_n$. However, for any finite subcollection $\{G_{n_k}\}$, letting $N = \max\{n_1, \ldots, n_k\}$, any $x < 1/N$ is not contained in any G_{n_k}, and thus $\{G_{n_k}\}$ cannot cover $(0,1)$.