M 365C

Fall 2013, Section 57465
Problem Set 4
Due Thu Sep 26

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-2, without reproving them — except for the first exercise as noted.

Exercise 1 (*Rudin 2.12*)

Let $E = (\{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}) \subset \mathbb{R}$. Prove that E is compact directly from the definition of compactness (i.e., *without* using the theorem that says closed and bounded subsets of $\mathbb{R}$ are compact).

Answer of exercise 1

Suppose $\{G_\alpha\}$ is any open cover of E. Let $G_0 \in \{G_\alpha\}$ be an element with $0 \in G_0$ (this must exist, since $\{G_\alpha\}$ covers the whole of E.) Then 0 is an interior point of G_0 (since G_0 is open), hence there is $\epsilon > 0$ such that $N_\epsilon(0) \subset G_0$. Pick some $n > 1/\epsilon$. Then for all $m \geq n$ we have $1/m \in G_0$. Now let $G_1, \ldots, G_{n-1}$ be elements of $\{G_\alpha\}$ containing respectively 1, 1/2, ..., $1/(n-1)$. The collection $\{G_0, G_1, \ldots, G_{n-1}\}$ is an open cover of E, which gives the desired open subcover of the original $\{G_\alpha\}$. Thus E is compact.

Exercise 2 (*Rudin 2.19*)

1. If A and B are disjoint closed sets in some metric space X, prove that A and B are separated.
2. Prove the same for disjoint open sets.
3. Fix $p \in X$ and $\delta > 0$. Define $A = \{q \in X \mid d(p,q) < \delta\}$. Define $B = \{q \in X \mid d(p,q) > \delta\}$. Prove that A and B are separated.
4. Prove that every connected metric space with at least two points is uncountable. (Hint: use the previous part.)

Answer of exercise 2

1. We have $A = \bar{A}$ and $B = \bar{B}$, and A and B disjoint; so $A \cap \bar{B} = A \cap B = \emptyset$, and $\bar{A} \cap B = A \cap B = \emptyset$, as required.
2. Suppose $x \in A \cap \bar{B}$. Since $A \cap B = \emptyset$, x must be a limit point of B. There is a neighborhood N of x which is contained in A, since x is interior to A. But this neighborhood must also contain some point of B. This contradicts the fact that $A \cap B = \emptyset$.
3. A is a neighborhood, so A is open. Similarly, for any $q \in B$, letting $\epsilon < d(p,q) - \delta$ we can use the triangle inequality to show $N_\epsilon(q) \subset B$. Thus B is open. But A and B are obviously disjoint, so the previous part shows they are separated.

4. Suppose X is connected. Take any $p \in X$. For any $\delta > 0$, the sets A and B from the previous part are separated. But X is connected, so $A \cup B$ cannot be the whole of X. Thus there must be some point $q_\delta \in X$ such that $d(p, q_\delta) = \delta$. Now let $E = \{q_\delta \mid \delta \in \mathbb{R}\} \subset X$. The map $q_\delta \mapsto \delta$ is a bijective correspondence between E and $\mathbb{R}$. Thus E is uncountable, since $\mathbb{R}$ is uncountable; but E is a subset of X. Thus X must also be uncountable.

Exercise 3 (*Rudin 2.22, modified*)

Given a metric space X and a set $E \subset X$, we say E is *dense* in X if $\bar{E} = X$. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Answer of exercise 3

Say $p \in \mathbb{R}$. We have shown that between any two real numbers there is a rational number. Hence for any $\epsilon > 0$ there is a rational number between p and $p + \epsilon$. It follows that $N_\epsilon(p)$ contains a rational number, so p is a limit point of $\mathbb{Q}$.

* Exercise 4 (*Rudin 2.25, modified*)

Suppose that K is a compact metric space. Prove that K has a dense subset which is at most countable. (Hint: first show that for every $n \in \mathbb{N}$, there are finitely many neighborhoods of radius $1/n$ whose union covers K.)

Answer of exercise 4

Fix some $n \in \mathbb{N}$. Then $\{N_{1/n}(p) \,|\, p \in K\}$ is an open cover of K, which thus has a finite subcover; in other words, there exists a finite set $E_n \subset K$ such that for any $p \in K$, there exists some $q \in E_n$ such that $d(p, q) < 1/n$. Let $E = \cup_{n \in \mathbb{N}} E_n$; then E is a countable union of finite sets, hence E is at most countable. We would like to show E is dense in K. For this, it suffices to show that for any $p \in K$, there is some $q \in E$ with $d(p, q) < \epsilon$. But this follows from what we know aobut E_n: just pick some $n > 1/\epsilon$ and use the fact that $E_n \subset E$.

Exercise 5 (*Rudin 3.1*)

Suppose that $\{a_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that $\{|a_n|\}$ is also a convergent sequence in $\mathbb{R}$.

Answer of exercise 5

Suppose $a_n \to a$; we will prove $|a_n| \to |a|$. For this first note that $||x| - |y|| \le |x - y|$ (if x and y have the same sign then the two sides are simply equal, while if the signs are different then we may assume $x > 0$, $y = -z$ with $z > 0$, in which case the desired inequality is $|x - z| \le |x| + |z|$, which is the triangle inequality.) Now, fix some $\epsilon > 0$. Since $a_n \to a$, there is N such that $n > N \implies |a_n - a| < \epsilon$. But then it follows that $||a_n| - |a|| < \epsilon$ as well. Thus $|a_n| \to |a|$ as desired.