M 365C

FALL 2013, SECTION 57465

MIDTERM 2 SAMPLES

True or False. Whichever way you think it goes, sketch a proof in a few lines. You may freely use any result we proved in class, or any result proved in Rudin. Throughout, let X and Y denote metric spaces.

1. If $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} x_n = x$, and $f : X \to Y$ is any function, then $\lim_{n\to\infty} f(x_n) = f(x)$.
2. Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both continuous, $f(a) > g(a)$ and $f(b) < g(b)$. Then there exists some $x \in [a, b]$ such that $f(x) = g(x)$.
3. If $f : [a, b] \to \mathbb{R}$ is differentiable at a and has $f'(a) > 0$, then there exists some $x \in (a, b)$ such that $f(x) > f(a)$.
4. If $\sum a_n$ converges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ converges.
5. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, the set $E = \{x \in \mathbb{R} \mid f(x)^3 > 2\}$ is open.
6. If $f : (0, 1) \to \mathbb{R}$ is bounded and continuous, then it is uniformly continuous.
7. Let $E = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$. Every function $f : E \to \mathbb{R}$ is continuous.
8. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $f(x) = x$ for all $x \in \mathbb{Q}$, then $f(x) = x$ for all $x \in \mathbb{R}$.
9. If $f : X \to Y$ is continuous, and $E \subset X$ is open, then $f(E) \subset Y$ is open.
10. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} f(a_n)$ is convergent.
11. Suppose given $f : \mathbb{R} \to \mathbb{R}$ such that there is no x with $f(x) = 0$. Define $g(x) = f(x)^2$. Suppose g is differentiable. Then f is also differentiable.
12. Suppose $f : \mathbb{R} \to \mathbb{R}$ has $f'(x) = 1$ for all $x \in \mathbb{R}$, and $f(0) = 0$. Then $f(x) = x$.
13. The function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^3$ is uniformly continuous.
14. Suppose given two functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$, such that $f(x) = g(x)$ except for countably many points x. Suppose f is Riemann integrable. Then g is also Riemann integrable and $\int_a^b f(x)\mathrm{d}x = \int_a^b g(x)\mathrm{d}x$.

Extra problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(3x)$. Prove carefully that g is continuous. Use only the definition of continuity.

1. **False**. For example, we could take $X = Y = \mathbb{R}$, $x_n = 1/n$, $x = 0$, and

$$f(x) = \begin{cases} 0 \text{ if } x \neq 0, \\ 1 \text{ if } x = 0 \end{cases}$$

2. **True**. Apply the Intermediate Value Theorem to $h(x) = f(x) - g(x)$: it has $h(a) < 0$, $h(b) > 0$, so at some $x \in [a, b]$ it must have $h(x) = 0$.

3. **True**. Since $f'(a) > 0$ we have $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} > 0$. Thus there exists some neighborhood N of a such that $x \in N \implies \frac{f(x)-f(a)}{x-a} > 0$. For any $x \in (a, b)$ with $x \in N$, we have $x > a$, and $\frac{f(x)-f(a)}{x-a} > 0$, so $f(x) - f(a) > 0$, i.e. $f(x) > f(a)$.

4. **False**. Consider the sequence $a_n = (-1)^n \frac{1}{n}$, $b_n = (-1)^n$. Then $\sum a_n$ converges but $\sum a_n b_n$ does not.

5. **True**. Since f is continuous, $h = f^3$ is also continuous. The set E is $h^{-1}((2, \infty))$, and $(2, \infty)$ is an open subset of $\mathbb{R}$. Thus E is the of the form $h^{-1}(U)$ where h is continuous and U open. Thus E is open.

6. **False**. This one is tricky. Consider the function $f(x) = \cos(1/x)$. This function is defined and continuous on $(0, 1)$. However, for any δ, there exist x, y with $|x - y| < \delta$ but $f(x) = -1$, $f(y) = 1$. (Take $x = \frac{1}{2\pi n}$, $y = \frac{1}{2\pi(n+1)}$, for large enough n.) Thus if we pick $\epsilon = 1$, there is no δ for which $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

7. **False**. For example, we could take

$$f(x) = \begin{cases} 0 \text{ if } x \neq 0, \\ 1 \text{ if } x = 0 \end{cases}$$

8. **True**. $\mathbb{Q}$ is dense in $\mathbb{R}$, so for any $x \in \mathbb{R}$ there is a sequence $\{x_n\}$ where all $x_n \in \mathbb{Q}$ and $x_n \to x$. Then $f(x_n) \to f(x)$. But $f(x_n) = x_n$, so this says $x_n \to f(x)$. By uniqueness of the limit, then $f(x) = x$.

9. **False**. Say $f : \mathbb{R} \to \mathbb{R}$ is a constant function. Then for any open subset $E \subset \mathbb{R}$, $f(E)$ consists of just a single point, so $f(E)$ is not open.

10. **False**. Say $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = 1$. Then suppose $\sum a_n$ is any convergent series. Then $b_n = f(a_n)$ is just the constant sequence $b_n = 1$. For $\sum b_n$ to be convergent we would need $b_n \to 0$, which the constant sequence $b_n = 1$ certainly doesn't satisfy.

11. **False**. Say

$$f(x) = \begin{cases} -1 \text{ if } x \leq 0, \\ 1 \text{ if } x > 0 \end{cases}$$

Then $f(x)$ is not continuous and hence not differentiable, but $f(x)^2$ is the constant function 1, which is differentiable.

12. **True**. Consider the function $g(x) = f(x) - x$. This function has $g'(x) = f'(x) - 1 = 1 - 1 = 0$. Thus g is a constant function, $g(x) = c$. So $f(x) = x + c$. Plugging in $x = 0$ we get $f(0) = c$. But we know $f(0) = 0$, so this says $c = 0$, i.e. $f(x) = x$.

13. **True**. Any continuous function on a compact set is uniformly continuous.

14. **False**. Say $f(x) = 0$ for all x. This is Riemann integrable. Say

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Then $g(x)$ differs from $f(x)$ only when $x \in \mathbb{Q}$. But $g(x)$ is not Riemann integrable (as we have shown in class).

Extra problem. Fix $p \in \mathbb{R}$ and $\epsilon > 0$. Since f is continuous at $3p$, there exists some δ' for which

$$|y - 3p| < \delta' \implies |f(y) - f(3p)| < \epsilon$$

Now, take $\delta = \delta'/3$, so $\delta' = 3\delta$. Now suppose x is arbitrary. Plugging in $y = 3x$ in the above implication gives

$$|3x - 3p| < 3\delta \implies |f(3x) - f(3p)| < \epsilon$$

i.e.

$$|x - p| < \delta \implies |g(x) - g(p)| < \epsilon$$

This shows that g is continuous at p. But p was arbitrary, so g is continuous.