# M 365C

FALL 2013, SECTION 57465

MIDTERM 1

**True or False**. If true, sketch a proof in a few lines. If false, state a counterexample (in this case you do not have to prove that it is a counterexample.) You may use without proof anything that we proved in class or anything that is proved in Rudin chapters 1-3.

Throughout, let $X$ denote a metric space.

1. If $E \subset X$ is closed, then any subset of $E$ is also closed.

   **False.** For example, take $X = \mathbb{R}$; then $E = \mathbb{R}$ is closed, but the subset $(0,1) \subset E$ is not closed.

2. If $E \subset Y \subset X$, and $E$ is open when considered as a subset of the metric space $Y$, then $E$ is open when considered as a subset of the metric space $X$.

   **False.** For example, take $X = \mathbb{R}^2$, $Y = \{(x,0)\,|\,x \in \mathbb{R}\} \subset X$, and $E = \{(x,0)\,|\,0 < x < 1\} \subset Y \subset X$. Then $E$ is open when considered as a subset of $Y$ (this is just the fact that $(0,1)$ is an open subset of $\mathbb{R}$), but $E$ is not open when considered as a subset of $X$ (since any neighborhood of a point in $E$ will contain some points with $y \neq 0$.)

3. If $E \subset X$ is countable, then $\bar{E}$ is also countable.

   **False.** For example, take $X = \mathbb{R}$ and $E = \mathbb{Q}$. Then $E$ is countable, but $\bar{E} = \mathbb{R}$ (as shown in one of the homework assignments), which is not countable.

4. If $E \subset X$ is connected, then $\bar{E}$ is also connected.

   **True.** We will show the contrapositive: if $\bar{E}$ is disconnected, then $E$ is disconnected. Suppose $\bar{E}$ is disconnected; then $\bar{E} = A \cup B$ with $A$, $B$ nonempty and separated. Then $E = (A \cap E) \cup (B \cap E)$. Also $A \cap E$ and $B \cap E$ are separated: this follows from the fact that $\overline{A \cap E} = \overline{A} \cap \overline{E} \subset \overline{A}$, hence $\overline{A \cap E} \cap B = \emptyset$ (since $A$ and $B$ are separated), hence $\overline{A \cap E} \cap (B \cap E) = \emptyset$; similarly $\overline{B \cap E} \cap (A \cap E) = \emptyset$. This almost shows that $E$ is disconnected, but we still need to check that $A \cap E$ and $B \cap E$ are nonempty. For this, assume that $A \cap E = \emptyset$. Then $\overline{A \cap E} = \overline{A} \cap \overline{E} = \emptyset$ also. Then in particular $A \cap \overline{E} = \emptyset$. But we know $\overline{E} = A \cup B$. It follows that $\overline{E} = B$. This contradicts the fact that $A$, $B$ are separated and $A$ nonempty. Thus our assumption was false, so $A \cap E$ is nonempty; similarly $B \cap E$ is nonempty.

5. If $K_n \subset X$ is compact for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} K_n$ is compact.

   **False.** For example, say $X = \mathbb{R}$ and $K_n = \{n\} \subset \mathbb{R}$. Each $K_n$ contains a single point, hence in particular $K_n$ is a finite set, hence compact; but $\cup_{n=1}^{\infty} K_n = \mathbb{N}$ which is not bounded and hence not compact.

6. If $\{p_n\}$ is a sequence in $\mathbb{R}$, with $|p_n| \to 5$, then $\{p_n\}$ has a convergent subsequence.

   **True.** Since $|p_n| \to 5$, there exists some $N$ for which $n > N \implies ||p_n| - 5| < 1$, hence $|p_n| < 6$. Then let $M = \max\{|p_1|, |p_2|, \dots, |p_N|, 6\}$; for all $n$ we have $|p_n| \leq M$, so $\{p_n\}$ is a bounded sequence in $\mathbb{R}$, thus it has a convergent subsequence.

7. If $E \subset \mathbb{R}$ is compact, then $\{(x, y) \mid x \in E, y \in E\} \subset \mathbb{R}^2$ is compact.

   **True.** Let $F = \{(x, y) \mid x \in E, y \in E\}$. Since $F \subset \mathbb{R}^2$, to show it is compact, it suffices to show that it is closed and bounded. First we show $F$ is bounded. We know $E$ is compact, so $E$ is bounded, i.e. there is some $M$ for which $x \in E \implies |x| < M$. Then for $(x, y) \in F$ we have $\sqrt{|x|^2 + |y|^2} < |x| + |y| < 2M$. Thus $F$ is bounded. Next we show $F$ is closed. For this, suppose $(x, y)$ is a limit point of $F$. Then for every $\epsilon > 0$ there exists some $(x', y') \in F$ with $(x', y') \neq (x, y)$ and $\sqrt{|x' - x|^2 + |y' - y|^2} < \epsilon$; in particular $|x' - x| < \epsilon$. Thus either $x \in E$ or $x$ is a limit point of $E$, in which case again $x \in E$, since we know $E$ is compact and thus closed. So $x \in E$. Similarly $y \in E$. Thus $(x, y) \in F$, and so $F$ is closed.