M 382D: Differential Topology
Spring 2015

Final

Due: Tue May 19

This is a take-home exam, due Tuesday May 19 by 5pm in my office. *Work independently on this exam.* You may use Guillemin-Pollack, Warner, or our class notes, and you may freely ask me any question you like, but do not go searching in the library or on the web. You may use freely any result stated in class or in the lecture notes (even if we did not fully prove it, e.g. the Invariance of Domain theorem). Where our definitions differ from those in Guillemin-Pollack, you should use our definitions.

Problem 1. Let $M = \mathbb{CP}^2$ with homogeneous coordinates (z^0, z^1, z^2). Let $N \subset M$ be the locus given by the equation $z^0 = 0$.

1. Show that N is a submanifold of M.
2. Fix orientations on N and M; explain carefully which orientations you take.
3. Compute the intersection number $I(N, N)$.

Problem 2. Prove each of the following statements true or false.

1. There exists $\omega \in \Omega^1(S^1 \times S^1)$ which is closed but not exact.
2. Suppose $\omega_1, \omega_2 \in \Omega(M)$ both closed. Then $\omega_1 \wedge \omega_2$ is exact.
3. Suppose $\varphi : M \to N$ is a local diffeomorphism and ω is a 1-form on N which is closed but not exact. Then $\varphi^* \omega$ is also closed but not exact.

Problem 3. Let $\xi_1, \xi_2, \xi_3 \in \mathfrak{X}(\mathbb{A}^2)$ be given by

$$\xi_1 = \frac{\partial}{\partial t^1} + \frac{\partial}{\partial t^2}, \quad \xi_2 = t^1 \frac{\partial}{\partial t^1}, \quad \xi_3 = (t^1 - t^2) \frac{\partial}{\partial t^2}$$

Find a coordinate system (x^1, x^2) on a patch of $\mathbb{A}^2$, for which two of the three vector fields ξ_i are the coordinate vector fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$.

Problem 4.

1. Give an example of a nonorientable smooth manifold M together with an orientable smooth submanifold $P \subset M$.
2. Show that if M and P are as in the previous part, and moreover P does not even have a *neighborhood* in M which is orientable, then there does not exist a smooth manifold N and smooth map $f : M \to N$, with $q \in N$ a regular value, such that $P = f^{-1}(q)$.

3. If we drop the phrase "a regular value" from the previous part, then things are different: show that if M is any manifold and $P \subset M$ any closed submanifold, there is a smooth function $f : M \to \mathbb{R}$ such that $P = f^{-1}(0)$.

Problem 5. Give an example of a compact manifold M with $\chi(M) = 0$ and a smooth map $f : M \to M$ whose Lefschetz number $L(f) \neq 0$.

Problem 6. Suppose M is a compact orientable manifold (without boundary) of dimension m, and $\omega \in \Omega^{m-1}(M)$. Show that $\mathrm{d}\omega$ vanishes at some point of M.