M 382D: Differential Topology
Spring 2015
Midterm
Due: Mon Mar 30

This is a take-home exam, due Monday March 30 at the beginning of class. *Work independently on this exam.* You may use Guillemin-Pollack, Warner, or our class notes, and you may freely ask me any question you like, but do not go searching in the library or on the web. You may use freely any result stated in class or in the lecture notes (even if we did not fully prove it, e.g. the Invariance of Domain theorem). Where our definitions differ from those in Guillemin-Pollack, you should use our definitions.

Problem 1. Regarding S^2 as the unit sphere in $\mathbb{A}^3$, consider the map $\gamma : (-1,1) \to S^2$ defined by

$$\gamma(t) = (t/2, \sqrt{1 - t^2/2}, -t/2)$$

1. Show that γ is an immersion.
2. Is the image of γ a submanifold of S^2? Why or why not?
3. Introduce a chart on a patch $U \subset S^2$ containing the point $\gamma(0)$. (One convenient choice would be to use spherical coordinates.) Recall that such a chart induces a basis for the tangent space $T_{\gamma(0)}S^2$. Express the derivative $\dot{\gamma}(0) \in T_{\gamma(0)}S^2$ in this basis. (Recall that $\dot{\gamma}(0)$ is shorthand for the vector $\mathrm{d}\gamma_0(\partial/\partial t)$.)

Solution to Problem 1. For each part, there are many approaches which work. Here I just give one sample.

1. Let $\iota : S^2 \to \mathbb{A}^3$ be the inclusion map and $f = \iota \circ \gamma$. Then $\mathrm{d}f_t = \mathrm{d}\iota_{\gamma(t)} \circ \mathrm{d}\gamma_t$, so to show $\mathrm{d}\gamma_t$ injective it is enough to show $\mathrm{d}f_t$ injective. But relative to the standard bases induced by the standard coordinates on $(-1,1)$ and $\mathbb{A}^3$, $\mathrm{d}f$ is represented by the Jacobian matrix of f, i.e.

   $$\mathrm{d}f_t = \begin{pmatrix} 1/2 & * & -1/2 \end{pmatrix}$$

   Here $*$ denotes the derivative of $\sqrt{1 - t^2/2}$, but its actual value is irrelevant. All that matters is that the matrix above is not identically zero, so $\mathrm{d}f$ has rank 1 and thus is injective on the 1-dimensional vector space $T_t(-1,1)$.

2. γ is injective (since $\gamma(t) = \gamma(t') \implies t/2 = t'/2 \implies t = t'$) and is an immersion (by the previous part), so to show $\gamma(-1,1)$ is a submanifold, we only need check that γ is a homeomorphism onto $\gamma(-1,1)$, i.e. that it has a continuous inverse. Here "continuous" means "continuous with respect to the subspace topology on $\gamma(-1,1)$." Note one convenient way of getting such a map is to take a map which is continuous on the whole S^2 and then just restrict it. In turn we could get such a map by beginning with one which is continuous on the whole $\mathbb{A}^3$ and restricting it to S^2.

So now finally consider

$$g : \mathbb{A}^3 \to \mathbb{A}^1$$

given by

$$g(x, y, z) = 2x.$$

Then $g(\gamma(t)) = t$, so g gives the desired continuous inverse to γ.

3. Let's choose the coordinates $(x, y, z) \to (x, z)$, which are good coordinates on the locus $\{y > 0\} \subset S^2$. Relative to these coordinates $\gamma(t) = (t/2, -t/2)$ and $\mathrm{d}\gamma$ is represented by the Jacobian matrix $(1/2, -1/2)$. Thus, $\mathrm{d}\gamma(\partial/\partial t) = \frac{1}{2}\partial/\partial x - \frac{1}{2}\partial/\partial z$.

Problem 2. Prove or disprove:

1. Let $f : M \to N$ be a smooth map and $q \in N$. Then $f^{-1}(q)$ is a submanifold of M.
2. Every smooth map $S^1 \times \mathbb{RP}^3 \to S^7$ is homotopic to a constant map.
3. Every smooth map $\mathbb{RP}^4 \to \mathbb{RP}^4$ is homotopic to a constant map.
4. If M is a smooth manifold of dimension m, and $k \leq m$, then there exists an embedding $\mathbb{R}^k \hookrightarrow M$.

Solution to Problem 2.

1. False. e.g. take $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $f(x, y) = xy$; then $f^{-1}(0)$ is the union of the coordinate axes, which is not locally Euclidean around $(0, 0)$, since any neighborhood of $(0, 0)$ can be broken into four connected components upon removing one point.

2. True. Since $\dim(S^1 \times \mathbb{RP}^3) = 4 < 7$, Sard's theorem implies the image of such a map f misses at least one point $p \in S^7$. But there is a coordinate chart $x : S^7 \setminus \{p\} \to \mathbb{A}^7$ (by stereographic projection), which moreover is surjective onto $\mathbb{A}^7$. Thus we can homotope f to a constant map by

$$f_t(p) = x^{-1}((1 - t)x(p)).$$

3. False. The identity map has mod 2 degree 1, while the constant map has mod 2 degree 0, so the two cannot be homotopic.

4. True. Fix a coordinate chart (U, x) on M. $x(U)$ is open in $\mathbb{A}^m$ and thus contains some open ball $B \subset x(U)$. On the other hand $\mathbb{A}^m$ itself is diffeomorphic to the open unit ball, say via the map $x \mapsto \frac{x}{\sqrt{1+\|x\|^2}}$ which has inverse $x \mapsto \frac{x}{\sqrt{1-\|x\|^2}}$. Composing this with a rescaling, we get a diffeomorphism from $\mathbb{A}^m$ to B. Then composing with the inclusion into $x(U)$ we get a map from $\mathbb{A}^m$ to $x(U)$. Finally composing this with x^{-1} we get a map from $\mathbb{A}^m$ to M. All maps we used are embeddings, so their composition is also an embedding as desired.

Problem 3. Let $Q \subset \mathbb{A}^2$ be the unit circle. Consider the map $f : \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{A}^2$ given by

$$f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

1. Prove that f is not transverse to Q.
2. Exhibit a smooth homotopy $F : [0,1] \times \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{A}^2$ such that $F_0 = f$ and F_1 is transverse to Q.

Solution to Problem 3.

1. There are many approaches here. Here is one slick one which avoids any computation: f actually maps all of $\mathbb{A}^2 \setminus \{(0,0)\}$ to Q, so $f^{-1}(Q)$ is a 2-manifold, but if it were transverse to Q then $f^{-1}(Q)$ should have dimension $2-1=1$.
2. Again there are many approaches. The simplest way is to arrange that F_1 misses Q completely, e.g. by taking

$$F_t(x,y) = (1-t)f(x,y).$$

Problem 4. Suppose $U \subset \mathbb{A}^n$ is an open set and $f : U \to \mathbb{R}$ a smooth function. Define the graph of f as a subset of $\mathbb{A}^{n+1}$ and prove that it is a submanifold.

Solution to Problem 4. The graph of f is defined as

$$\Gamma_f = \{(x,y) : x \in U, y \in \mathbb{R}, f(x) = y\} \subset \mathbb{A}^{n+1}.$$

Consider the function $g : U \times \mathbb{A}^1 \to \mathbb{R}$ given by

$$g(x,y) = f(x) - y.$$

g is a submersion because

$$\mathrm{d}g = \begin{pmatrix} \partial_1 f \cdots \partial_n f & 1 \end{pmatrix}$$

with respect to the standard bases, and thus evidently has rank 1. Thus $\Gamma_f = g^{-1}(0)$ is a submanifold of $U \times \mathbb{R}$. This is not quite what we wanted: we wanted Γ_f to be a submanifold of the bigger space $\mathbb{A}^{n+1}$.

But note the general fact: if M is a submanifold of a manifold V and V is an open subset of a manifold N then M is also a submanifold of N. This is because the good charts on V which exhibit M as a submanifold work equally well as charts on N.

Using this, finally Γ_f is a submanifold of $\mathbb{A}^{n+1}$ as desired.

Problem 5. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, $n > 1$. Let $K \subset \mathbb{R}^n$ be compact and $\epsilon > 0$. Show that there exists a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{d}g_x \neq 0$ for all x and $\|f(x) - g(x)\| < \epsilon$ for $x \in K$. Also show that this is false for $n = 1$. (Hint: let M_n be the space of $n \times n$ matrices, and show that the map $F : \mathbb{R}^n \times M_n \to M_n$ given by $F(x,A) = \mathrm{d}f_x + A$ is a submersion. Thus there exists some A so that 0 is a regular value of F_A. Use this fact to construct the desired g. Where is the hypothesis $n > 1$ used?) (This is Exercise 8 on page 75 of Guillemin-Pollack.)

Solution to Problem 5. First the counterexample for $n = 1$: take $f(x) = x^2$, $K = [-2, 2]$ and $\epsilon = 1$. If $|f(x) - g(x)| < 1$ for all $x \in [-2, 2]$ then in particular

$$g(-2) > 3, \quad g(0) < 1, \quad g(2) > 3$$

Thus by the Mean Value Theorem there exist some $a \in (-2, 0)$ and $b \in (0, 2)$ with $g'(a) < 0$ and $g'(b) > 0$. Then by the Intermediate Value Theorem there must be some $c \in (a, b)$ with $g'(c) = 0$.

Now suppose $n > 1$. In this case consider $F(x, A)$ as given in the hint above. Fix any linear coordinate system on M_n (say the one given by taking the n^2 matrix entries in some fixed order). Then relative to these coordinates we have

$$\mathrm{d}F = \begin{pmatrix} * & 1_{n^2 \times n^2} \end{pmatrix}$$

and thus $\mathrm{d}F$ has rank n^2 (full rank), thus F is a submersion. It follows (by our transversality statements) that, for almost every A, the map $F_A(x) = \mathrm{d}f(x) + A$ is transverse to the point $0 \in M_n$. Fix one such A, with the additional property that $\|A\| < \epsilon / diam(K)$. Then every point in $F_A^{-1}(0)$ is a regular point of F_A. But since $n < n^2$ (using $n > 1$), F_A cannot have any regular points, thus $F_A^{-1}(0) = \emptyset$.

Now consider the map

$$g(x) = f(x) + Ax$$

which has

$$\mathrm{d}g = \mathrm{d}f + A = F_A$$

By the above, we have $\mathrm{d}g^{-1}(0) = \emptyset$; and

$$\|g(x) - f(x)\| = \|F_A x\| \leq \|A\| \|x\| \leq \epsilon$$

as desired.

Problem 6. Suppose given M, N smooth manifolds, and a smooth map $f : M \to N$. Suppose $q \in N$ is a regular value for f. Let $P = f^{-1}(q) \subset M$. Show that the normal bundle NP is a trivial bundle over P. (Recall the slogan that "any submanifold $P \subset M$ is given *locally* by k independent equations"; this problem says that if $P \subset M$ is given *globally* by k independent equations, then NP is trivial.) (One might view this problem as composed of two parts: one part is to give an isomorphism $(NP)_p \simeq \mathbb{R}^k$ for each $p \in P$, the other is to show that these isomorphisms fit together into a bundle isomorphism, i.e. to check that they vary smoothly with p. You need not write down every detail of the smoothness but please do not ignore it completely!)

Solution to Problem 6. For any $p \in P$ consider the map $\mathrm{d}f_p : T_pM \to T_qN$. Since q is a regular value this map is surjective; on the other hand we know that its kernel is T_pP. Thus $\mathrm{d}f_p$ descends to an *isomorphism* $\widetilde{\mathrm{d}f}_p : T_pM / T_pP \to T_qN$ i.e.

$$\widetilde{\mathrm{d}f}_p : N_pP \to T_qN$$

These fiberwise isomorphisms fit together into a single map

$$\widetilde{\mathrm{d}f} : NP \to P \times T_q N.$$

We need to check that $\widetilde{\mathrm{d}f}$ is *smooth*, so that it gives a bundle isomorphism. Informally speaking this smoothness follows from the fact that f_p depends smoothly on p; below, we will spell out all the details, just to be sure. Supposing for a moment that this has been done, choose any basis in $T_q N$ (once and for all, not depending on p); then we can identify $T_q N$ with $\mathbb{R}^k$ and thus obtain the desired bundle isomorphism $NP \to P \times \mathbb{R}^k$, completing the proof.

Now the gory details of smoothness. To check smoothness of $\widetilde{\mathrm{d}f}$ let us make some preliminary comments. In general, suppose we have two vector bundles A and A' over a base manifold P, and a map $\psi : A \to A'$ which takes fibers to fibers linearly, i.e. it comes from linear maps $\psi_p : A_p \to A'_p$ for each $p \in P$. Then to check smoothness of ψ at a point p, we fix local trivializations

$$\phi : A|_U \to U \times \mathbb{R}^n \qquad \phi' : A'|_U \to U \times \mathbb{R}^n$$

and consider the map

$$\phi' \circ \psi \circ \phi^{-1} : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$$

Since the local trivializations ϕ, ϕ' are diffeomorphisms, $\phi' \circ \psi \circ \phi^{-1}$ is smooth if and only if ψ is. But since it takes fibers to fibers linearly, and the fibers are just $\mathbb{R}^n$, this map is of the form

$$\phi' \circ \psi \circ \phi^{-1}(p, v) = (p, T(p)v)$$

for a map $T : U \to GL(n)$. Thus finally ψ is smooth if and only if T is smooth.

To apply this to our map $\widetilde{\mathrm{d}f}$, we need local trivializations of the two bundles NP and $P \times T_q N$ over P. For NP, first recall that in general, local trivializations of a quotient bundle E/F are obtained as follows: take a patch U on which there exists a basis $\{e_i\}_{i=1,\dots,m}$ of sections of E, such that $\{e_i\}_{i=k+1,\dots,m}$ are a basis for $F \subset E$; then we get a map

$$\phi : (E/F)|_U \to U \times \mathbb{R}^k$$

by

$$\phi\left(\left[\sum_{i=1}^{k} c^i e_i\right] \in (E/F)_p\right) = \left(p, (c^1, \dots, c^k)\right).$$

In our situation, we have the quotient bundle $NP = TM|_P / TP$. Fix a point $p \in P$. Since P is a submanifold there exists a chart (U, x) on M, containing p, such that

$$x(P \cap U) = x(U) \cap \{x^1 = \cdots = x^k = 0\}.$$

This chart induces a local basis $\{\frac{\partial}{\partial x^i}\}_{i=1,\dots,m}$ of sections of TM, which we can restrict to sections of $TM|_P$; the subset $\{\frac{\partial}{\partial x^i}\}_{i=k+1,\dots,m}$ give a local basis of sections of TP. So, the above general story about quotient bundles now says we have a local trivialization of NP given by

$$\phi\left(\left[\sum_{i=1}^{k} c^i \frac{\partial}{\partial x^i}\right] \in NP_p\right) = \left(p, (c^1, \ldots, c^k)\right).$$

For $P \times T_q N$ we get a trivialization just by fixing a basis in the fixed vector space $T_q N$; we'll use the basis $\{\frac{\partial}{\partial y^j}\}_{j=1}^{k}$ induced by some chart (V, y) on N with $q \in V$, which gives the trivialization

$$\phi'\left(\sum_{i=1}^{k} b^i \frac{\partial}{\partial y^i} \in p \times T_q N\right) = (p, (b^1, \ldots, b^k)).$$

(In this way we could even give a *global* trivialization of $P \times T_q N$, and this was useful above; but in principle that fact does not matter at this precise moment, since right now our interest is just in checking smoothness, which is a purely local question.)

Now we want to compute $\widetilde{\mathrm{d}f}$ relative to these trivializations. We write $f : M \to N$ concretely in terms of its component functions $f^j = y^j \circ f$. Then we have

$$\widetilde{\mathrm{d}f}_p\left(\left[\sum_{i=1}^{k} c^i \frac{\partial}{\partial x^i}\right]\right) = \mathrm{d}f_p\left(\sum_{i=1}^{k} c^i \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} c^i \frac{\partial f^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}.$$

Thus

$$\phi' \circ \widetilde{\mathrm{d}f} \circ \phi^{-1}(p, \vec{c}) = (p, T(p)\vec{c})$$

where $T(p)$ is the $k \times k$ matrix with entries

$$T(p)^j_i = \frac{\partial f^j}{\partial x^i}(p)$$

So finally, the smoothness of $\widetilde{\mathrm{d}f}$ follows from the fact that these depend smoothly on p. In other words, it is just the smoothness of f, as we hoped.

Remark. We could also have further "rectified" the situation by taking distinguished charts (U, x) and (V, y), with $p \in U$ and $q \in V$, such that $y \circ f \circ x^{-1}$ is the standard projection $\mathbb{R}^m \to \mathbb{R}^k$ on the first k coordinates. In this case we would just have $f^j = x^j$ above, and so the matrix $T(p)^j_i$ would be the identity matrix, which of course is also smooth as a function of p. (In a sense this approach would be more natural, since the way we prove P is a submanifold in the first place is to use these adapted charts anyway; the only reason I didn't adopt this approach is that I wanted to see the Jacobian of f appear explicitly, and to justify the slogan that "smoothness of $\widetilde{\mathrm{d}f}$ comes from smoothness of f".)