M 382D: Differential Topology
Spring 2015
Midterm
Due: Mon Mar 30

This is a take-home exam, due Monday March 30 at the beginning of class. *Work independently on this exam.* You may use Guillemin-Pollack, Warner, or our class notes, and you may freely ask me any question you like, but do not go searching in the library or on the web. You may use freely any result stated in class or in the lecture notes (even if we did not fully prove it, e.g. the Invariance of Domain theorem). Where our definitions differ from those in Guillemin-Pollack, you should use our definitions.

Problem 1. Regarding S^2 as the unit sphere in $\mathbb{A}^3$, consider the map $\gamma : (-1,1) \to S^2$ defined by

$$\gamma(t) = (t/2, \sqrt{1 - t^2/2}, -t/2)$$

1. Show that γ is an immersion.
2. Is the image of γ a submanifold of S^2? Why or why not?
3. Introduce a chart on a patch $U \subset S^2$ containing the point $\gamma(0)$. (One convenient choice would be to use spherical coordinates.) Recall that such a chart induces a basis for the tangent space $T_{\gamma(0)}S^2$. Express the derivative $\dot{\gamma}(0) \in T_{\gamma(0)}S^2$ in this basis. (Recall that $\dot{\gamma}(0)$ is shorthand for the vector $d\gamma_0(\partial/\partial t)$.)

Problem 2. Prove or disprove:

1. Let $f : M \to N$ be a smooth map and $q \in N$. Then $f^{-1}(q)$ is a submanifold of M.
2. Every smooth map $S^1 \times \mathbb{RP}^3 \to S^7$ is homotopic to a constant map.
3. Every smooth map $\mathbb{RP}^4 \to \mathbb{RP}^4$ is homotopic to a constant map.
4. If M is a smooth manifold of dimension m, and $k \leq m$, then there exists an embedding $\mathbb{R}^k \hookrightarrow M$.

Problem 3. Let $Q \subset \mathbb{A}^2$ be the unit circle. Consider the map $f : \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{A}^2$ given by

$$f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

1. Prove that f is not transverse to Q.
2. Exhibit a smooth homotopy $F : [0,1] \times \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{A}^2$ such that $F_0 = f$ and F_1 is transverse to Q.

Problem 4. Suppose $U \subset \mathbb{A}^n$ is an open set and $f : U \to \mathbb{R}$ a smooth function. Define the graph of f as a subset of $\mathbb{A}^{n+1}$ and prove that it is a submanifold.

Problem 5. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, $n > 1$. Let $K \subset \mathbb{R}^n$ be compact and $\epsilon > 0$. Show that there exists a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{d}g_x \neq 0$ for all x and $\|f(x) - g(x)\| < \epsilon$ for $x \in K$. Also show that this is false for $n = 1$. (Hint: let M_n be the space of $n \times n$ matrices, and show that the map $F : \mathbb{R}^n \times M_n \to M_n$ given by $F(x, A) = \mathrm{d}f_x + A$ is a submersion. Thus there exists some A so that 0 is a regular value of F_A. Use this fact to construct the desired g. Where is the hypothesis $n > 1$ used?) (This is Exercise 8 on page 75 of Guillemin-Pollack.)

Problem 6. Suppose given M, N smooth manifolds, and a smooth map $f : M \to N$. Suppose $q \in N$ is a regular value for f. Let $P = f^{-1}(q) \subset M$. Show that the normal bundle NP is a trivial bundle over P. (Recall the slogan that "any submanifold $P \subset M$ is given *locally* by k independent equations"; this problem says that if $P \subset M$ is given *globally* by k independent equations, then NP is trivial.) (One might view this problem as composed of two parts: one part is to give an isomorphism $(NP)_p \simeq \mathbb{R}^k$ for each $p \in P$, the other is to show that these isomorphisms fit together into a bundle isomorphism, i.e. to check that they vary smoothly with p. You need not write down every detail of the smoothness but please do not ignore it completely!)