PRELIMINARY MATERIAL FOR M382D: DIFFERENTIAL TOPOLOGY

(These notes are swiped from a set written by Dan Freed when he taught this course; they should be basically applicable to my version of the course as well.)

Here is a brief list of topics you should know or review:

Linear algebra: abstract vector spaces and linear maps, dual space, direct sum, subspaces and quotient spaces, bases, change of basis, matrix computations, trace and determinant, bilinear forms, diagonalization of symmetric bilinear forms, inner product spaces.

Calculus: directional derivative, differential, (higher) partial derivatives, chain rule, second derivative as a symmetric bilinear form, Taylor's theorem, inverse and implicit function theorem, integration, change of variable formula, Fubini's theorem, vector calculus in 3 dimensions (div, grad, curl, Green and Stokes' theorems), fundamental theorem on ordinary differential equations.

You may or may not have studied all of these topics in your undergraduate years. Most of the topics listed here will come up at some point in the course.

Here are a few possible references (in addition to whatever texts you used as an undergraduate). The first three chapters of Spivak's *Calculus on Manifolds* contains most of the material on calculus and has the virtue of being short. *Advanced Calculus* by Loomis and Sternberg has all of the required material and much more. A more elementary text, with many applications, is *A Course in Mathematics for Students of Physics* by Bamberg and Sternberg. The text *Linear Algebra* by Hoffman and Kunze is a classic. There are many alternatives to these references.

Here we sketch some definitions and ideas from linear algebra and differential calculus. There are exercises interspersed in the text to help you review/learn the material. Some of them are open-ended; I encourage you to explore these with classmates. Don't worry if you don't complete all of the exercises; at least read through and think about each of them. These preliminary problems are *not* to hand in.

1. Linear Algebra

As stated above you should be comfortable with abstract vector spaces.

Exercise 1.

(a) *From algebra you know that if G is a group and H a normal subgroup, then the set of left cosets G/H is naturally a group. Similarly, show that if V is a vector space and W a vector subspace, then the set of cosets V/W is a vector space. It is called the quotient (vector) space. If V is finite dimensional, then what is the dimension of V/W in terms of that of V and W? Define a natural linear map $V \to V/W$. (Interpret 'natural' as 'without choice of basis.' There is a way to make this precise.)*

(b) *Suppose V_1 and V_2 are vector spaces. Define the direct sum $V_1 \oplus V_2$ to be the set $V_1 \times V_2$ of all pairs (v_1, v_2), $v_1 \in V_1$, $v_2 \in V_2$ with component-wise addition and scalar multiplication.*

Show that this is a vector space. What is its dimension, assuming V_1, V_2 are finite dimensional? Define natural linear maps $V_1 \to V_1 \oplus V_2$ and $V_1 \oplus V_2 \to V_1$. Suppose $W \subset V$ is a subspace. Is there a natural *isomorphism $V \to W \oplus V/W$?*

(c) *Let V, W be vector spaces. Let $\mathrm{Hom}(V, W)$ be the set of linear maps $V \to W$. Define a vector space structure on $\mathrm{Hom}(V, W)$. Assume V, W are finite dimensional. Given bases for V, W define a basis for $\mathrm{Hom}(V, W)$. What is $\dim \mathrm{Hom}(V, W)$?*

Exercise 2. *This problem gives practice with index notation and the summation convention, which states: an index which is repeated, once as a subscript and once as a superscript, is summed over. Note carefully the placement (superscript vs. subscript) of the indices in what follows. The actual name of the index (i or j or μ) is arbitrary, though as always a judicious choice of notation helps you and your readers.*

Let V be an n-dimensional vector space. Suppose $\{e_j\}$ and $\{f_i\}$ are two bases for V which are related by the equation

$$e_j = P^i_j f_i, \tag{1}$$

where P is an invertible matrix. My convention is that i is the row index and j the column index when we view P^i_j as the entry in a matrix.

(a) *Suppose $\xi \in V$ is a vector. Then we can find real numbers ξ^j and $\tilde{\xi}^i$ such that $\xi = \xi^j e_j = \tilde{\xi}^i f_i$. Express $\tilde{\xi}^i$ in terms of the ξ^j by substituting (1) and using the uniqueness of the expansion of a vector in terms of a basis.*

(b) *Suppose $T : V \to V$ is a linear transformation. Relative to the basis $\{e_j\}$ it is expressed as the matrix A defined by $Te_j = A^i_j e_i$, and relative to the basis $\{f_i\}$ it is expressed as the matrix B defined by $Tf_i = B^j_i f_j$. What is the relationship between A and B?*

(c) *The dual space V^* is the vector space of all linear functionals $V \to \mathbb{R}$; it is also n dimensional. Every basis of V gives rise to a dual basis of V^*. For the basis $\{e_j\}$ of V the dual basis $\{e^i\}$ of V^* is defined by the equation*

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

(This equation defines the symbol δ^i_j.) The dual basis $\{f^j\}$ is defined similarly. Express f^j in terms of the e^i.

(d) *Suppose $\omega \in V^*$. Then we define its components relative to the basis $\{e^i\}$ by the equation $\omega = \omega_i e^i$ and its components relative to the basis $\{f^j\}$ by the equation $\omega = \tilde{\omega}_j f^j$. Express the ω_i in terms of the $\tilde{\omega}_j$.*

(e) *Compute the evaluation $\omega(\xi)$ in terms of the components in both pairs of dual bases. Check that the expressions agree under change of basis.*

2. Affine Spaces

We will distinguish between *points* and *vectors*. This distinction is present in ordinary calculus on flat space, and if anything is easier on manifolds, as we will see in the course. Points are elements of an *affine* space and vectors are elements of a *vector* space.

Definition 1. *Let V be a vector space. An* affine space A modeled on V *is a set A with a simply transitive action of V.*

The action is a map

$$\begin{aligned} A \times V &\longrightarrow A \\ (p, \xi) &\longmapsto p + \xi \end{aligned} \tag{2}$$

which satisfies $(p+\xi)+\xi' = p+(\xi+\xi')$ for all $p \in A$ and $\xi, \xi' \in V$. Simple transitivity asserts that for each $p \in A$ the map

$$\begin{aligned} V &\longrightarrow A \\ \xi &\longmapsto p + \xi \end{aligned} \tag{3}$$

is a bijection. Thus there is a well-defined "subtraction" map $A \times A \to V$ which sends a pair of points $p, q \in A$ to the unique $\xi \in V$ such that $q = p + \xi$.

Exercise 3. *Let V be a vector space and $\omega \in V^*$ a nonzero linear functional. Show that $\{\xi \in V : \omega(\xi) = 1\}$ is naturally an affine space, in this case an affine subspace of a vector space.*

Elements of an affine space A are called *points*; elements of the associated vector space V are called *vectors*. The vectors act as *displacements* or *translations* on points. Points cannot be added: what, approximating the Earth as flat, would be the sum of New York City and Boston? Rather, in an affine space we can subtract points to obtain a vector, and can take weighted averages of points: the center of mass of New York City and Boston is approximately Middletown, Connecticut.

Exercise 4. *Define the weighted averaging operation on an affine space A. Thus if $\lambda = (\lambda_1, \dots, \lambda_n)$ is an ordered n-tuple of scalers with $\lambda_1 + \cdots + \lambda_n = 1$, then define a map*

$$\begin{aligned} A^n &\longrightarrow A \\ (p_1, \dots, p_n) &\longmapsto \lambda_1 p_1 + \cdots + \lambda_n p_n \end{aligned} \tag{4}$$

which is the weighted average. Can you generalize to integrate an A-valued function defined on a measure space?

We will always take the vector space V to be defined over the field $\mathbb{R}$ or $\mathbb{C}$—usually the former. If V is finite dimensional, then there is a unique topology on V for which the vector addition and scalar multiplication are continuous. Any affine space over V then inherits this topology using (3). For any integer $n \geq 0$ there is a standard affine space

$$\mathbb{A}^n = \big\{(x^1, x^2, \dots, x^n) : x^\mu \in \mathbb{R}\big\} \tag{5}$$

which has associated vector space

$$\mathbb{R}^n = \big\{(\xi^1, \xi^2, \dots, \xi^n) : \xi^\mu \in \mathbb{R}\big\}. \tag{6}$$

The action is given by pointwise addition.

Remark 1. *A vector space V may be regarded as an affine space (with associated vector space V), in which case we ignore the special role of $0 \in V$.*

Exercise 5. *Define the notion of an affine line in an affine space A. Notice that an affine line determines a one-dimensional subspace of the associated vector space V. (A one-dimensional subspace of a vector space is simply called a 'line'.) Generalize your definition to affine subspaces of higher dimension.*

Definition 2. *Let A, A' be affine spaces over V, V'. A map $\alpha : A \to A'$ is* affine *if there exists a linear map $L : V \to V'$ such that*

$$\alpha(p + \xi) = \alpha(p) + L(\xi) \tag{7}$$

for all $p \in A$, $\xi \in V$.

The *differential* $d\alpha : A \to \operatorname{Hom}(V, V')$ of the affine map (1) is the constant L; we define the differential of non-affine maps in the next section. We define an *affine subspace* to be the image of an injective affine map. (See Exercise 5.) Thus we have notions of affine lines, affine planes, etc. If A is a finite dimensional real affine space, then a collection of affine functions $x^1, x^2, \dots, x^n : A \to \mathbb{R}$ such that the differentials

$$dx^1, dx^2, \dots, dx^n \in V^* \tag{8}$$

form a basis is called an *affine coordinate system* on A. The ordered collection of these n functions defines an affine isomorphism $A \to \mathbb{A}^n$. The dual basis of V is denoted

$$\partial/\partial x^1,\ \partial/\partial x^2,\ \dots,\ \partial/\partial x^n \in V. \tag{9}$$

They can be viewed as translation-invariant=constant=parallel vector fields on A.

Exercise 6. *Show that the image of an affine line under an affine map is an affine line or a point. Does this property characterize affine maps?*

Exercise 7. *Let A be an affine space with underlying vector space V. Define $\operatorname{Aut}(A)$ as the group of invertible affine maps $A \to A$. Show that V is a normal subgroup of $\operatorname{Aut}(A)$. Identify the quotient group as a group of transformations—in fact, the group of all automorphisms—of some geometric object.*

3. Differential Calculus

The *differential* is a linear approximation to a nonlinear map. Let A, A' be affine spaces with underlying vector spaces V, V'. Let $U \subset A$ be an open set and $f : U \to A'$ a function. Fix $p \in U$. Then (if it exists) the differential of f at p is a linear map

$$df_p : V \longrightarrow V'. \tag{10}$$

For $\xi \in V$ small it approximates the nonlinear displacement $f(p+\xi) - f(p)$ due to f by the linear displacement $df_p(\xi)$. Turning that around, it approximates the non-affine map f near p,

$$p + \xi \longmapsto f(p+\xi), \tag{11}$$

by the affine map

$$p + \xi \longmapsto f(p) + df_p(\xi). \tag{12}$$

The map (12) is the first-order Taylor approximation, which is a reasonable approximation if f is differentiable. Recall that the zeroth-order Taylor approximation is the constant map

$$p + \xi \longmapsto f(p), \tag{13}$$

which is a good approximation near p if f is continuous at p.

Exercise 8. *Give a definition of the differential in terms of ϵ, δ. First recall the definition of continuity of f at p: for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\xi\| < \delta$, then $\|f(p+\xi) - f(p)\| < \epsilon$. This quantifies the idea that the zeroth-order approximation* (13) *is a good one. Now give a similar ϵ-δ definition of differentiability and the differential. Check that your definition works for real-valued functions of a single variable.*

Remark 2. *In Exercise 8 we use a* norm *on V and V'. A norm on V is a function $\|-\| : V \to \mathbb{R}$ such that for all $\xi, \xi_1, \xi_2 \in V$, we have $\|\xi\| \geq 0$ with equality iff $\xi = 0$; $\|\lambda\xi\| = |\lambda| \, \|\xi\|$ for all $\lambda \in \mathbb{R}$; and $\|\xi_1 + \xi_2\| \leq \|\xi_1\| + \|\xi_2\|$. Here are some examples of norms on $\mathbb{R}^n$:*

$$\begin{aligned} \|(\xi^1, \dots, \xi^n)\| &= \sqrt{(\xi^1)^2 + \dots + (\xi^n)^2} \\ \|(\xi^1, \dots, \xi^n)\| &= \sqrt[p]{(\xi^1)^p + \dots + (\xi^n)^p}, \qquad p \geq 1 \\ \|(\xi^1, \dots, \xi^n)\| &= \max(\xi^1, \dots, \xi^n) \end{aligned} \tag{14}$$

A norm on V induces a topology on V: for the balls $\{\xi \in V : \|\xi\| < r\}$ $(r > 0)$ and their affine translates form a basis of open sets. If V is finite dimensional than the norm topology on V is independent of the choice of norm, and so the definition of continuity and of the differential in Exercise 8 do not depend on the choice of norm.

Exercise 9.

(a) *If $\xi \in V$ is a vector, define the* directional derivative *in terms of the differential by*

$$\xi f(p) = df_p(\xi) \in V'. \tag{15}$$

(b) *If $A = \mathbb{A}^n$ is the standard affine space, or if we choose a coordinate system on A to give an isomorphism with $\mathbb{A}^n$, then define the* partial derivatives *of a differentiable function as*

$$\frac{\partial f}{\partial x^i}(p) = df_p(\partial/\partial x^i) \tag{16}$$

where $\partial/\partial x^i$ are the vector fields in (9). *Check that this corresponds to the definition in terms of difference quotients. Notice the consistency of notation between* (15) *and* (16)*: set $\xi = \partial/\partial x^i$.*

We can also approach this in the reverse order. Namely, given $f : U \to A'$ as above, $p \in U$ and $\xi \in V$ we first define the directional derivative

$$\xi f(p) = \frac{d}{dt}\Big|_{t=0} f(p + t\xi) \in V', \tag{17}$$

which is the usual derivative in one-variable calculus of the function f restricted to the parametrized affine line $t \mapsto p + t\xi$ through p with velocity ξ. Then a basic theorem asserts that if all directional derivatives exist in a neighborhood of p and are continuous at p, then the directional derivatives at p depend linearly on $\xi \in V$ and f is differentiable at p with differential defined by (15).

Exercise 10. *Prove this theorem or look it up. You'll need the definition in Exercise 8.*

Exercise 11. *Now that the differential has been defined, check the statements after Definition 2. Namely, verify that the differential of an affine function is what it is claimed to be.*

Exercise 12 (*Important!*)**.** *Relate the differential defined here with the matrix of partial derivatives you learned about in vector calculus. For this assume you have affine coordinates, or equivalently a map between standard affine spaces.*

Exercise 13. *Let x, y, z be the standard coordinates on $\mathbb{A}^3$.*

(a) *Compute the differential of the function $f(x, y, z) = xe^y - y^3 \cos z - 5xyz^2$. Express your answer in terms of the differentials of the coordinate functions, i.e., in terms of dx, dy, dz.*
(b) *Evaluate the directional derivative of f at $(1, 2, -2)$ in the direction $(-1, 3, 2)$.*
(c) *Write the differential of f at $(1, 2, -2)$ with respect to the basis $\{(1, 0, 1), (1, 1, 0), (0, -1, 2)\}$ of $\mathbb{R}^3$.*

Exercise 14. *Suppose A is an n dimensional affine space with associated vector space V. Let $\gamma : (t_0, t_1) \to A$ be a smooth curve, where $(t_0, t_1) \in \mathbb{R}$ is an open set. Define the tangent vector $\dot\gamma(t) \in V$ as a limit of difference quotients. Given an affine coordinate system $x : A \to \mathbb{A}^n$ we can write the composition of γ and x as n functions $x^i(t)$, $i = 1, \dots, n$. Then the ordinary derivatives $\dot x^i(t)$ give the coordinates of $\dot\gamma(t)$ in the basis of V induced from the coordinate system x. Check directly that if we change to a new affine coordinate system y, that the new coordinates of the tangent vector define the same abstract vector $\dot\gamma(t)$. What is the relationship between $\dot\gamma(t)$ and $d\gamma_t$?*

Exercise 15. *Suppose $U \subset \mathbb{A}^n$ is a connected open set and $f : U \to \mathbb{A}^m$ is a smooth function whose differential df_p vanishes for all $p \in U$. Prove that f is constant.*

Exercise 16. *Let U be a connected open subset of an affine space A and $f : U \to A'$ a smooth map to an affine space A'. Prove that f extends to an affine map $A \to A'$ if and only if the differential $df : U \to \mathrm{Hom}(V, V')$ is constant. Here V, V' are the vector spaces associated to the affine spaces A, A'.*

The *chain rule* computes the differential of the composite of two maps. Thus if A, A', A'' are affine spaces with associated vector spaces V, V', V'', and $U \subset A$, $U' \subset A'$ open sets, and $f : U \to A'$, $g : U' \to A''$ differentiable functions such that the composition $g \circ f : U \to A''$ is defined, then $g \circ f$ is differentiable and

$$d(g \circ f)_p = dg_{f(p)} \circ df_p, \qquad p \in U. \tag{18}$$

Exercise 17. *Continuing with the notation just established, suppose that $A'' = A$ and the compositions $f \circ g$ and $g \circ f$ are defined and equal to the identity map. Prove that for each $p \in U$ the differential df_p is an invertible map.*

Higher directional derivatives are defined as iterations of (17). Thus if $f : U \to A'$ for $U \subset A$ open, and $p \in U$, then for $\xi_1, \xi_2 \in V$ we define

$$(\xi_1 \xi_2 f)(p) = \xi_1 \big(\xi_2 f\big)(p), \tag{19}$$

where we need to assume that the directional derivative $\xi_2 f$ exists in a neighborhood of p: the outer derivative in (19) is the directional derivative of the function $\xi_2 f : U \to V'$ at p in the direction ξ_1.

A basic theorem asserts that if all second directional derivatives exist and are continuous in a neighborhood of p, then for any $\xi_1, \xi_2 \in V$ we have

$$(\xi_1 \xi_2 f)(p) = (\xi_2 \xi_1 f)(p). \tag{20}$$

In other words, mixed partials commute.

Exercise 18. *Show that relative to an affine coordinate system*

$$\frac{\partial^2 f}{dx^j dx^i} = \frac{\partial^2 f}{dx^i dx^j} \tag{21}$$

for all i, j.

If all second directional derivatives exist and are continuous at p define the second differential $d^2 f_p$ by the formula

$$(d^2 f_p)(\xi_1, \xi_2) = (\xi_1 \xi_2 f)(p). \tag{22}$$

The second differential is a *symmetric* bilinear form

$$d^2 f : V \times V \to V'. \tag{23}$$

Exercise 19. *Relate* (23) *to the Hessian square matrix of second partial derivatives you learned in vector calculus in case the codomain of f is $A' = \mathbb{R}$. Review the diagonalization of a symmetric bilinear form. How does that apply to the second partial? Review the statements about the local behavior of f at p if $df_p = 0$ and d^2f_p is nondegenerate. Do this first for functions of one variable ($A = \mathbb{R}$).*

Exercise 20. *Now suppose $\xi_1, \xi_2 : U \to V$ are vector* fields *which may not be constant. Is* (20) *still true? If not, what can you say about the difference between the two sides of the equation. For example, ostensibly it is a second derivative of f. Is that indeed true? (When ξ and η are constant,* (20) *asserts that the difference is not a second derivative of f but rather a constant: zero!) Can you give a meaning to $\xi_1\xi_2 - \xi_2\xi_1$?*

Finally, please look up the inverse and implicit function theorems. We will discuss then in lectures, but best if you have thought about the statements in advance.