M 382D: Differential Topology
Spring 2015

Exercise Set 1
Due: Wed Jan 28

There will be weekly homework assignments due each Wednesday at the *beginning* of class. Please work the problems neatly and staple your pages together. There is no need to copy over the problem or hand in the problem sheet. Please number the problems. Feel free to discuss the problems and work together, but you must write up your own solutions.

You are very welcome to discuss the problems (and the class generally) with me during my office hours, or to make an appointment with me for another time.

Exercise 1. Consider the n-sphere $S^n \subset \mathbb{A}^{n+1}$ defined by

$$S^n := \left\{(x^1, \ldots, x^{n+1}) \in \mathbb{A}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 = 1\right\}$$

Cover S^n with two coordinate charts (called *stereographic projections*) as follows. Let

$$\begin{aligned} n &= (0, \ldots, 0, +1) \\ s &= (0, \ldots, 0, -1) \end{aligned}$$

be the north and south pole. Identify $\mathbb{A}^n$ with its image under

$$\begin{aligned} \mathbb{A}^n &\longrightarrow \mathbb{A}^{n+1} \\ (x^1, \ldots, x^n) &\longmapsto (x^1, \ldots, x^n, 0) \end{aligned}$$

For $p \in S^n$ let $x(p)$ be the intersection of the line through p and n with $\mathbb{A}^n$, and let $y(p)$ be the intersection of the line through p and s with $\mathbb{A}^n$. On which subsets of S^n are x and y well defined coordinate maps? What is the transition function between the two coordinate systems?

Exercise 2. Prove that the union of the two coordinate axes in $\mathbb{R}^2$ is not a manifold. (Hint: what happens to a neighborhood of the origin when the origin is removed?)

Exercise 3. Suppose X, Y are smooth manifolds. Then:

1. Describe a natural structure of smooth manifold on $X \times Y$.
2. If $f : X \to X'$ and $g : Y \to Y'$ are smooth, show that the *product map*

   $$f \times g : X \times Y \to X' \times Y'$$

   defined by

   $$(f \times g)(x, y) = (f(x), g(y))$$

   is smooth.
3. Show that the *projection map* $X \times Y \to X$ is smooth.

4. Suppose $U \subset X$ is an open subset. Describe a natural structure of smooth manifold on U.

5. Give an example showing that if $Z \subset X$ is a subset which is not required to be open, then Z may not even be a topological manifold.

Exercise 4. Prove the following.

1. Suppose $U \subset \mathbb{A}^n$ is a connected open set and $f : U \to \mathbb{A}^m$ is a smooth function whose differential df_x vanishes for all $x \in U$. Prove that f is constant.

2. Let $U, V \subset \mathbb{A}^n$ be open subsets and $f : U \to \mathbb{A}^n$ and $g : V \to \mathbb{A}^n$ be smooth maps such that the compositions $f \circ g$ and $g \circ f$ are defined and equal to the identity map. Prove that for each $x \in U$ the differential df_x is an invertible map.

3. Let U be a connected open subset of an affine space A and $f : U \to B$ a smooth map to an affine space B. Prove that f extends to an affine map $A \to B$ if and only if the differential $df : U \to Hom(V, W)$ is constant. Here V, W are the vector spaces associated to the affine spaces A, B and $Hom(V, W)$ is the vector space of linear maps from V to W.

Exercise 5. Let X denote the set of affine lines in $\mathbb{A}^2$. Topologize X and show that it is a topological manifold. What is $\dim X$? Is X connected? Is X compact? Is X simply connected? Can you recognize X as a familiar topological manifold: do you know a familiar topological manifold which is homeomorphic to X?